Preface

Welcome to the seventh volume of *Research in Collegiate Mathematics Education (RCME VII)*. The present volume, like previous volumes in this series, reflects the importance of research in mathematics education at the collegiate level. The editors in this series encourage communication between mathematicians and mathematics educators and, as pointed out by the International Commission on Mathematics Instruction (ICMI), much more work is needed in concert by these two groups. As is true in any research field, over time researchers in collegiate mathematics education have developed a variety of theoretical approaches. As a natural part of the development of ways to investigate the learning and teaching of mathematics, researchers have constructed a specialized terminology that mathematicians may find challenging. In fact, the editors of the first volume of this series wrote about this (Dubinsky, Schoenfeld, & Kaput, 1994):

> As Alan Schoenfeld's opening chapter makes clear, mathematicians who are not familiar with educational research are likely to be in for some surprises. The field is not what you might think it is, and its methods are not what you might expect. This should come as no great shock. Compare people's preconceptions of what a mathematician does with the reality of being a mathematical researcher. The same applies here. (pp. vii-viii)

Overview of this Volume

Nine papers constitute this volume. The first two examine problems students experience when converting a representation from one particular system of representations to another. One paper is on the role of fractional and decimal representations in relation to determining the rationality of a number and the second is about moving among numeric, algebraic, and graphic representation in the context of the definite integral. The next three papers investigate student learning about proof. The first one looks at this issue in terms of professors' points of view about proof and its teaching while the second examines two students' experiences with proving. The third paper in this group focuses on students making and testing conjectures about infinite processes. In the next two papers, the focus is instructor knowledge for teaching calculus. The first examines how graduate teaching assistants gain knowledge of student thinking and the second explores the perceptions and practices of professors teaching calculus. The final two papers in the volume address the nature of "conception" in mathematics. The first of these proposes a model that leads to a definition of "conception" and the last paper reports on the availability of strategies for problem solving in connection with the idea of "conception."

Conversions Among Representations

There are two papers related to theoretical approaches with representations: one from Zazkis and Sirotic, and the other from Camacho, Depool and Santos. Zaskis and Sirotic investigated prospective teachers' understanding of irrational number representations by surveying 46 future teachers and through semi-structured interviews with 16 volunteers from that group. From a theoretical point of view of representations, they assume two things. First, they take into account the fundamental fact that there is no direct access to mathematical objects but only to their representations, and secondly that any representation captures only partially the mathematical object. Zazkis and Sirotic use the notions of *transparent* and *opaque* representation, with respect to a certain property of the represented object, as if the property can be "seen" (i.e., a maximum or minimum in a graph) or is "hidden" (i.e., a maximum or minimum in the algebraic representation of a function). Zazkis and Sirotic show us some ways that students' identifications of irrational numbers may be correct, but that their mathematical reasoning about the representations may not be appropriate. The results of the study support, in general, the importance of promoting articulation about representations among students. In particular, Zazkis and Sirotic provide us with pedagogical elements to improve the teaching of rational and irrational number concepts.

In a semester long study conducted by Camacho, Depool and Santos, the researchers observed the problem-solving efforts of 31 first-year engineering students enrolled in a reform calculus course, interviewing a subset of 6 students, to explore students' learning of the definite integral. The instructional approach embraced the "Rule of Four," including algebraic, numeric, graphic, and technologically rich representations (see, e.g., Schwarz, 1989). Camacho, Depool and Santos' perspective is based on Duval's (1993) theoretical approach for learning through both distinguishing between mathematical concepts and negotiating among multiple representations of a concept. Particularly, the authors were aware of the difficulties that students have when passing from one representation to another in a mathematical task related to definite integrals. Therefore, they designed and had students work on a number of non-routine problems related to geometrical representations and algebraic representations. Some of these problems asked students to decide whether given statements were true or false. The results of the study suggest that, regardless of the representations or technology used, students tended to give primacy to algebraic over graphic and numeric representations. The results also indicate that students were able to analyze data given in a graphic form though the authors further state that students experienced serious difficulties in constructing examples and counter-examples that could help them in the resolution process of tasks. From a general point of view, the results obtained by the authors – in a technology rich setting – are similar to observations by Eisenberg and Dreyfus (1991) and Vinner (1989) about the challenges of learning to use visualization in mathematics.

Teaching and Learning Proof

In relation to exploring and proving conjectures, we have three papers in this volume: one written by Alcock, another by Alcock and Weber, and the last by Brown, McDonald and Weller. Alcock's paper is related to the teaching and learning of mathematical proof. From discussions with five mathematicians who taught courses about mathematical reasoning, Alcock distilled four important aspects of student experience when learning proof: *instantiation*, *structural thinking*, *creative thinking*, and *critical thinking*. In the first part of Alcock's report, she shares various comments from professors about how these aspects are exhibited. In the second part, Alcock provides evidence from professors' explanations about the types of strategies used in the mathematics classroom to describe the four types of thinking among students. The professors play an important role in demonstrating the use of examples and counter-examples. In her analysis, through the voices of the professors, Alcock provides examples of professors who were directing their approach to promote structural thinking, while others directed their approach towards instantiation and creative thinking. In the third part of the article, Alcock asserts that the four modes of thinking can be viewed as interdependent and that those who are successful at producing correct proofs will switch flexibly from one to another mode of thinking according to changing goals during the proving process. Alcock concludes with some suggestions for teaching proof, including a balancing of attention among the four modes of thinking.

The paper by Alcock and Weber reports some of the problems students have in the process of proving mathematical statements. They analyze two interviews from two different students, Brad and Carla, around the verification or falsification of four mathematical statements. In two of the situations it is necessary to provide a counter-example, in one a proof by contradiction is efficacious, and in the fourth it is possible to construct a direct proof. Examination of the techniques used by the two students suggest they had two distinct approaches: (1) a *syntactic* approach to proving, focused on logical syntax, without much attention to variety in the representation of mathematical concepts, and (2) a *referential* approach to proving, focused on the use of representations of mathematical concepts. Literature has provided us with mathematicians who can be classified under this theoretical approach (Hadamard, 1945/1975): Hermite was a mathematician who used a syntactic approach, and Poincaré was a mathematician who used a referential approach. Alcock and Weber show us the importance of the two approaches in the learning of mathematics.

Brown, McDonald, and Weller address one of the major problem areas for mathematics learning at university level: the intersection between the concept of proof and the concept of infinity. The investigation carried out by Brown, McDonald, and Weller included 12 students learning set theory. The principal focus was: Prove or disprove: $\bigcup_{k=1}^{n} P(1, 2, \ldots, k) = P(\mathbb{N})$, where $\mathbb{N}$ denotes the set of natural numbers and P "the power set of" (the set of all subsets of the given set). This task was difficult for students and, for them, involved both potential and actual infinity; these ideas are a major challenge documented by authors in the didactics of mathematics and by historians of mathematics. Grounded in Action, Process, Object, Schema (APOS) theory, the researchers conducted their study in two phases. First, they interviewed five students using an individual semi-structured protocol; and one year later they talked with seven students, using small groups and a expanded version of the original semi-structured interview. The design of the questions in the second phase, and the subsequent answers given by students, presented interesting information about the students' difficulties and how they addressed them. The authors have suggested a teaching approach that takes into account APOS theory-based instruction to help undergraduates in the construction of an understanding of actual infinity. One of the implications of their analysis is that even if the results obtained with their study are consistent with the Basic Metaphor of Infinity offered by Lakoff and Núñez (2000), the results go beyond what is suggested by Lakoff and Núñez; new structures are needed to interpret the Brown, McDonald and Weller results.

Knowledge for Teaching

Two papers in this volume address aspects of the knowledge needed for teaching calculus. In the first, Kung interviewed current and former teaching assistants (TAs) about student thinking in calculus and how the TAs learned about it. The author connects the ideas of college instructor learning about student thinking to the literature on the knowledge needed for teaching in grades K-12 and to research on college calculus learning. As Kung points out, the first teaching experiences of many future college faculty occur when they are TAs, during graduate school. And, like the induction of K-12 teachers, the early experiences of TAs may be quite formative. A main result of his interviews with TAs about their on-the-job learning around student thinking was that the participants reported learning different things about student thinking, depending on the setting. Kung discusses this duality between types of knowledge about student thinking and types of interactions participants had with students (e.g., watching students work in groups or grading students' written work), then offers a framework to use in understanding and developing TAs' instructional practice. Where Kung's paper explores the complexities of the knowledge needed for teaching by a careful examination of the experiences of novices (TAs), Sofronas and DeFranco report on beliefs about calculus instruction held by more experienced instructors: college mathematics professors.

Sofronas and DeFranco suggest that the efforts made by ICMI to develop communication between mathematicians and mathematics educators has had little impact. Taking into account research on pedagogical content knowledge and the Knowledge Base for Teaching (KBT) at the K-12 level, through interviews with seven mathematicians teaching calculus, Sofronas and DeFranco propose a KBT model for collegiate mathematics instruction. As noted by a participant in their work, it is true that "most empirical research on the effectiveness of collaborative learning was concerned with a small scale [groups]" (see, Dillenbourg, 1999), so if we think that collaborative learning is important, we need to continue doing and communicating more research on the subject (see, e.g., Star & Smith III, 2006; Hitt, 2007). One of the results of the Sofronas and DeFranco study was that the instructional methods of the mathematician participants were grounded in their own learning styles without taking into consideration the knowledge base in pedagogy. It is interesting to see that for mathematicians, the history of mathematics is influential for some in their teaching, and some of the participants also thought that a knowledge of this history permitted them to have a better approach to teaching mathematics. The papers by Kung and Sofronas and DeFranco illustrate the complexity of the KBT in higher education.

Mathematical Conception

The final two papers in the volume explore the structure of mathematical conception. The theoretical foundations of Balacheff and Gaudin's paper are the works of the French tradition, mainly using the theoretical approaches and epistemological language of authors like Bachelard (1938), Brousseau (1997), and Vergnaud (1991). For some time the term *conception* was used in the didactics of mathematics in a vague way without settling on a characterization of the term so that researchers in mathematics education and professors of mathematics might share it (Artigue, 1991). One of the first attempts at characterization was provided by Duroux in connection with epistemological obstacles (as cited by Brousseau, 1997, pp. 99–100) and developed further by Vergnaud. Balacheff and Gaudin's work includes another fundamental element not considered by Verngaud, which they have named *control structure*. Indeed, in other theoretical approaches this notion of control structure has already been suggested, for example, as control processes in problem-solving (Schoenfeld, 1985), or as an epistemic frame (Perkins & Simmons, 1988), or as a verification process (Margolinas, 1989, 1993). Balacheff and Gaudin's extension of Vergnaud's theoretical approach adds a control structure to result in a conception C consisting of a quadruplet (P, R, L, Σ) in which: P is a set of problems, R is a set of operators, L is a representation system, and Σ is a control structure. Using this approach, Balacheff and Gaudin provide examples of conceptions related to function such as the Conception of Table: $C_T = (P_T, R_T, \text{Table}, \Sigma_T)$. Their analysis about C_T is made from a historical point of view, affirming that C_T is essentially formed from empirical foundations. Like the mathematicians in Sofronas and DeFranco's study, Balacheff and Guadin turn to the historical genesis of concept. They also conducted an experiment to understand the complexity of notions that students associate with the concept of function. Their experiment was carried out in a technological environment using the geometry software Cabri with 12th grade students. Balacheff and Gaudin illustrate their theoretical framework with two conceptions: Curve-Algebraic and Algebraic-Graph.

The volume concludes with the paper by Mesa. Mesa's theoretical frame is provided in an early version of Balacheff's work on what a conception is (see above and the article of Balacheff and Gaudin in this volume). According to Balacheff and Gaudin's theoretical approach, explicit attention to control structures is very important in problem-solving. Mesa takes into consideration the structure of control related to the conception of Initial Value Problems (IVPs) and makes a compelling argument that IVPs are a kind of problem that permits the development of a control structure. Related to the development of this control structure, Mesa analyzes calculus textbooks' treatment of IVPs to examine the richness of strategy discourse and explicitness of control structures offered in the texts. Mesa's results portray the textbooks' focus on solution verification as a primary strategy and illuminate the limited nature of strategy discourse about control structures supported by the textbooks in her study.

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