CHAPTER 1

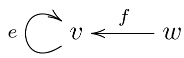
Directed graphs and Cuntz-Krieger families

A *directed graph* $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 and functions $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 are called *edges*. For each edge e, $s(e)$ is the *source* of e and $r(e)$ the *range* of e; if $s(e) = v$ and $r(e) = w$, then we also say that v *emits* e and that w *receives* e, or that e is an edge from v to w. All the graphs in these notes are directed, so we sometimes get lazy and call them graphs. If there is more than one graph around, we might write r_E and s_E to emphasise that we are talking about the range and source maps for E.

We usually draw a graph by placing the vertices in a plane, and drawing a directed line from $s(e)$ to $r(e)$ for each edge $e \in E^1$. If necessary, we label the edge by its name.

EXAMPLE 1.1. If $E^0 = \{v, w\}$, $E^1 = \{e, f\}$, $r(e) = s(e) = v$, $s(f) = w$ and $r(f) = v$, then we could draw

(1.1)



An edge which begins and ends at the same vertex v, like the edge e in Example 1.1, is called a *loop based at* v[1]. A vertex which does not receive any edges, like the vertex w in Example 1.1, is called a *source*. (Using the word "source" in two ways doesn't seem to cause confusion.) A vertex which emits no edges is called a *sink*.

Conversely, every drawing like (1.1) determines a graph.

EXAMPLE 1.2. The drawing

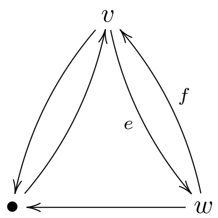
represents a graph E in which $E^0 = \{v\}$, $E^1 = \{e, f\}$ and e and f are both loops based at v. Notice that we are allowing multiple edges between the same pair of vertices; graph theorists often don't allow this.

Drawings are a useful aid when trying to follow arguments about graphs. However, there are many ways to draw the same graph, so it is important to remember that two directed graphs E and F are the same (formally, *isomorphic*) if and only if there are bijections $\phi^0 : E^0 \to F^0$ and $\phi^1 : E^1 \to F^1$ such that $r_F \circ \phi^1 = \phi^0 \circ r_E$ and $s_F \circ \phi^1 = \phi^0 \circ s_E$.

When it doesn't matter what an edge is called, we don't bother to label it in a drawing; when it doesn't matter what a vertex is called, we denote it by a $\bullet$.

[1] This is standard graph-theory terminology. Unfortunately the word "loop" is used in the graph-algebra literature to mean a closed path.

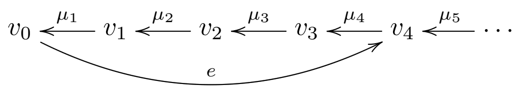
EXAMPLE 1.3. The drawing



represents a graph E with three vertices, two of which are called v and w, and five edges, two of which are called e and f. We do this to simplify notation when we are only going to refer to e and f.

These first three examples are all *finite graphs* in which both E^0 and E^1 are finite sets. In general, we allow either or both to be infinite.

EXAMPLE 1.4. The drawing



represents a graph E in which $E^0 = \{v_n : n \geq 0\}$ is infinite, and E^1 is the union of a singleton set $\{e\}$ and an infinite set $\{\mu_i : i \geq 1\}$.

EXAMPLE 1.5. The drawing

$$\cdots \longrightarrow v_3 \longrightarrow v_2 \longrightarrow v_1 \xrightarrow{(\infty)} v_0 \longrightarrow v_{-1} \longrightarrow v_{-2} \longrightarrow \cdots$$

represents a graph E with $E^0 = \{v_i : i \in \mathbb{Z}\}$, one edge from v_{i+1} to v_i for each $i \neq 0$, and infinitely many edges from v_1 to v_0.

For reasons which we will discuss in Chapter 5, graphs in which some vertices receive infinitely many edges pose extra problems for us. So we shall consider mainly the *row-finite graphs* in which each vertex receives at most finitely many edges, that is, in which $r^{-1}(v)$ is a finite set for every $v \in E^0$. The graph in Example 1.4 is row-finite, but that in Example 1.5 is not because v_0 receives infinitely many edges.

REMARK 1.6. The word "row-finite" refers to the corresponding property of the *vertex matrix* A_E of the graph E, which is the $E^0 \times E^0$ matrix defined by

$$A_E(v, w) = \#\{e \in E^1 : r(e) = v,\ s(e) = w\}.$$

(A_E is sometimes called the *adjacency matrix* of E.) The graph E is row-finite if and only if each row $\{A_E(v, w) : w \in E^0\}$ of A_E has finite sum.

We now seek to represent a directed graph by operators on Hilbert space: the vertices will be represented by orthogonal projections and the edges by partial isometries. Formally, let E be a row-finite directed graph and $\mathcal{H}$ a Hilbert space. A *Cuntz-Krieger E-family* $\{S, P\}$ *on* $\mathcal{H}$ consists of a set $\{P_v : v \in E^0\}$ of mutually orthogonal projections on $\mathcal{H}$ and a set $\{S_e : e \in E^1\}$ of partial isometries on $\mathcal{H}$, such that

(CK1) $S_e^* S_e = P_{s(e)}$ for all $e \in E^1$; and

(CK2) $P_v = \sum_{\{e \in E^1 : r(e) = v\}} S_e S_e^*$ whenever v is not a source.

Conditions (CK1) and (CK2) are called the *Cuntz-Krieger relations*, and Condition (CK2) in particular is often called the *Cuntz-Krieger relation at v*.

Saying that the projections P_v are mutually orthogonal means that the ranges $P_v\mathcal{H}$ are mutually orthogonal subspaces of $\mathcal{H}$. The relation (CK1) says that S_e is a partial isometry with initial space $P_{s(e)}\mathcal{H}$ (Proposition A.4); relation (CK2) implies that the range projection $S_eS_e^*$ of S_e is dominated by $P_{r(e)}$, and hence that $S_e\mathcal{H} \subset P_{r(e)}\mathcal{H}$ (Proposition A.1). Thus S_e is an isometry of $P_{s(e)}\mathcal{H}$ onto a closed subspace of $P_{r(e)}\mathcal{H}$; expressing this algebraically gives the relation

$$S_e = P_{r(e)}S_e = S_eP_{s(e)}, \tag{1.2}$$

which is used all the time in manipulations with Cuntz-Krieger families. Relation (CK2) also implies that the partial isometries S_e associated to the edges e with $r(e) = v$ have mutually orthogonal ranges (see Corollary A.3) with span $P_v\mathcal{H}$, so

$$P_v\mathcal{H} = \bigoplus\nolimits_{\{e\in E^1\,:\,r(e)=v\}} S_e\mathcal{H},$$

in the sense that the map $(h_e) \mapsto \sum_e h_e$ is an isomorphism of the direct sum onto $P_v\mathcal{H}$.

Remark 1.7. Since the initial and range projections of the S_e are all contained in $\mathcal{H}_e := \overline{\operatorname{span}}\big\{\bigcup_{v\in E^0} P_v\mathcal{H}\big\}$, we may as well assume that $\mathcal{H} = \mathcal{H}_e$, in which case we say the family is *non-degenerate*. If $\{S, P\}$ is a non-degenerate Cuntz-Krieger E-family, then the mutual orthogonality of the P_v implies that $\mathcal{H} = \bigoplus_{v\in E^0} P_v\mathcal{H}$, in the sense that the obvious map $(h_v) \mapsto \sum_v h_v$ of

$$\bigoplus\nolimits_{v\in E^0} P_v\mathcal{H} := \big\{(h_v) \in \prod\nolimits_{v\in E^0} P_v\mathcal{H} : \sum\nolimits_v \|h_v\|^2 < \infty\big\}$$

into $\mathcal{H}$ is an isomorphism (that $\sum_v \|h_v\|^2 < \infty$ implies that the sum $\sum_v h_v$ converges in norm in $\mathcal{H}$).

Remark 1.8. Since the orthogonal projections on closed subspaces of $\mathcal{H}$ are the bounded operators P satisfying $P^2 = P = P^*$ and the partial isometries are the bounded operators S satisfying $S = SS^*S$, we can talk about projections and partial isometries in any C^*-algebra B (see Appendix A.1). A Cuntz-Krieger E-family in B then consists of projections $\{P_v \in B : v \in E^0\}$ satisfying $P_vP_w = 0$ for $v \neq w$ (so that $\{P_v\}$ is a mutually orthogonal family of projections) and partial isometries $\{S_e \in B : e \in E^1\}$ satisfying (CK1) and (CK2).

Example 1.9. Consider the directed graph:

We have $S_e^*S_e = P_v = S_f^*S_f$, $P_v = S_eS_e^* + S_fS_f^*$. Take $\mathcal{H} = \ell^2(\mathbb{N}) = \overline{\operatorname{span}}\{e_n : n \geq 0\}$, P_v to be the identity operator 1, $S_e(e_n) = e_{2n}$ and $S_f(e_n) = e_{2n+1}$. Then $\{S, P\}$ is a Cuntz-Krieger family for this graph.

In any Cuntz-Krieger family $\{T, Q\}$ for this graph with Q_v non-zero, $Q_v\mathcal{H}$ must be infinite-dimensional. To see this, note that T_e is an isometry of $Q_v\mathcal{H}$ onto $T_e\mathcal{H}$, so $\dim Q_v\mathcal{H} = \dim T_e\mathcal{H}$. Similarly, $\dim Q_v\mathcal{H} = \dim T_f\mathcal{H}$. Thus $Q_v\mathcal{H} = T_e\mathcal{H} \oplus T_f\mathcal{H}$ implies

$$\dim Q_v\mathcal{H} = \dim T_e\mathcal{H} + \dim T_f\mathcal{H} = 2\dim Q_v\mathcal{H},$$

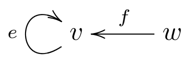
so $\dim Q_v\mathcal{H}$ can only be 0 or ∞.

In general there is no problem finding Cuntz-Krieger E-families with every P_v and every S_e non-zero: take $\mathcal{H}_v$ to be a separable infinite-dimensional Hilbert space for each $v \in E^0$, set $\mathcal{H} = \bigoplus_v \mathcal{H}_v$, take P_v to be the projection of $\mathcal{H}$ on $\mathcal{H}_v$, decompose $\mathcal{H}_v$ as a direct sum $\mathcal{H}_v = \bigoplus_{r(e)=v} \mathcal{H}_{v,e}$ of infinite-dimensional subspaces, and take S_e to be a unitary isomorphism of $\mathcal{H}_{s(e)}$ onto $\mathcal{H}_{r(e),e}$, viewed as a partial isometry on $\mathcal{H}$ with initial space $\mathcal{H}_{s(e)}$.

Example 1.9 shows why we need to take the spaces $\mathcal{H}_v$ to be infinite-dimensional. This is not always necessary, though:

Example 1.10. Consider the graph E which consists of a single vertex v and a single loop e based at v. Then the Cuntz-Krieger relations say that $S_e^* S_e = P_v = S_e S_e^*$, so that S_e is a unitary operator on $P_v\mathcal{H}$ (and is 0 on $(P_v\mathcal{H})^\perp$). There is no other restriction on S_e: if U is a unitary operator on $\mathcal{H}$, then we can take $P_v = 1$ and $S_e = U$. So there is no restriction on $\dim \mathcal{H}$; we could even take $\mathcal{H} = \mathbb{C}$ and U to be multiplication by $e^{i\theta}$, for example.

Example 1.11. For the graph



we can define a Cuntz-Krieger family on $\mathcal{H} = \ell^2$ by

$$P_v(x_0, x_1, x_2, \cdots) = (0, x_1, x_2, \cdots), \quad P_w(x_0, x_1, x_2, \cdots) = (x_0, 0, 0, \cdots),$$
$$S_f(x_0, x_1, x_2, \cdots) = (0, x_0, 0, \cdots), \quad \text{and} \quad S_e(x_0, x_1, x_2, \cdots) = (0, 0, x_1, x_2, \cdots).$$

It is important here that $P_v\mathcal{H}$ is infinite-dimensional: in any Cuntz-Krieger family for this graph, the Cuntz-Krieger relation at v implies that

$$\dim(P_v\mathcal{H}) = \dim(S_f\mathcal{H}) + \dim(S_e\mathcal{H}) = \dim(P_w\mathcal{H}) + \dim(P_v\mathcal{H}),$$

so if P_w and P_v are both non-zero, $P_v\mathcal{H}$ must be infinite-dimensional. It is worth observing now that the crucial factor in this argument is the presence of the edge f entering the loop.

We will be interested in the C^*-algebras $C^*(S, P)$ generated by Cuntz-Krieger families $\{S, P\}$, so we now investigate the $*$-algebraic consequences of the Cuntz-Krieger relations.

Proposition 1.12. *Suppose that E is a row-finite graph and $\{S, P\}$ is a Cuntz-Krieger E-family in a C^*-algebra B. Then*

(a) *the projections $\{S_e S_e^* : e \in E^1\}$ are mutually orthogonal;*
(b) $S_e^* S_f \neq 0 \Longrightarrow e = f$;
(c) $S_e S_f \neq 0 \Longrightarrow s(e) = r(f)$;
(d) $S_e S_f^* \neq 0 \Longrightarrow s(e) = s(f)$.

Proof. For part (a), suppose first that $r(e) = r(f)$. Then the Cuntz-Krieger relation at $r(e)$ implies that $P_{r(e)}$ is the sum of $S_e S_e^*$, $S_f S_f^*$ and other projections, which because $P_{r(e)}$ is a projection implies that $S_e S_e^*$ and $S_f S_f^*$ are mutually orthogonal (see Corollary A.3). On the other hand, if $r(e) \neq r(f)$, then (1.2) implies that

$$(S_e S_e^*)(S_f S_f^*) = (S_e S_e^* P_{r(e)})(P_{r(f)} S_f S_f^*) = (S_e S_e^*)0(S_f S_f^*) = 0.$$

Part (b) follows from (a), since $S_e^*S_f = S_e^*(S_eS_e^*)(S_fS_f^*)S_f = 0$ when $e \neq f$. For (c), we just note that part (a) implies that $S_eS_f = (S_eP_{s(e)})(P_{r(f)}S_f)$ vanishes unless $s(e) = r(f)$, and a similar argument gives (d). $\square$

Part (c) of Proposition 1.12 is particularly crucial: it says that S_eS_f is zero unless the pair ef is a path of length 2 in the graph E. More generally, a *path of length* n in a directed graph E is a sequence $\mu = \mu_1\mu_2\cdots\mu_n$ of edges in E such that $s(\mu_i) = r(\mu_{i+1})$ for $1 \leq i \leq n-1$. We write $|\mu| := n$ for the length of μ, and regard vertices as paths of length 0; we denote by E^n the set of paths of length n, and write $E^* := \bigcup_{n\geq 0} E^n$. (Now our notation for the sets of vertices and edges should make more sense.) We extend the range and source maps to E^* by setting $r(\mu) = r(\mu_1)$ and $s(\mu) = s(\mu_{|\mu|})$ for $|\mu| > 1$, and $r(v) = v = s(v)$ for $v \in E^0$. If μ and ν are paths with $s(\mu) = r(\nu)$, we write $\mu\nu$ for the path $\mu_1\cdots\mu_{|\mu|}\nu_1\cdots\nu_{|\nu|}$.

Remark 1.13. It may seem slightly odd to define the source of $\mu = \mu_1\mu_2\cdots\mu_n$ to be $s(\mu_n)$ rather than $s(\mu_1)$. However, this is forced on us by Proposition 1.12(c): the conventions of composition, which is the multiplication in $B(\mathcal{H})$, say that in the product RT we perform T first. This means that if we want juxtaposition of edges in a path μ to be consistent with juxtaposition of the corresponding partial isometries S_{μ_i} in $B(\mathcal{H})$, then we need to traverse μ_{i+1} before μ_i.

For $\mu \in \prod_{i=1}^n E^1$, we define $S_\mu := S_{\mu_1}S_{\mu_2}\cdots S_{\mu_n}$, and for $v \in E^0$, we define $S_v := P_v$. Proposition 1.12(c) says that $S_\mu = 0$ unless μ is a path; if μ is a path, then

$$
\begin{aligned}
S_\mu^*S_\mu &= (S_{\mu_1}S_{\mu_2}\cdots S_{\mu_n})^*S_{\mu_1}S_{\mu_2}\cdots S_{\mu_n} \\
&= S_{\mu_n}^*\cdots S_{\mu_2}^*(S_{\mu_1}^*S_{\mu_1})S_{\mu_2}\cdots S_{\mu_n} \\
&= S_{\mu_n}^*\cdots S_{\mu_2}^*P_{s(\mu_1)}S_{\mu_2}\cdots S_{\mu_n} \\
&= S_{\mu_n}^*\cdots S_{\mu_2}^*P_{r(\mu_2)}S_{\mu_2}\cdots S_{\mu_n} \\
&= S_{\mu_n}^*\cdots S_{\mu_3}(S_{\mu_2}^*S_{\mu_2})S_{\mu_3}^*\cdots S_{\mu_n} \\
&\ \vdots \\
&= P_{s(\mu_n)} = P_{s(\mu)}.
\end{aligned} \tag{1.3}
$$

Thus for $\mu \in E^*$, S_μ is a partial isometry with initial projection $P_{s(\mu)}$, and since $P_{r(\mu)}S_\mu S_\mu^* = S_\mu S_\mu^*$, the range of S_μ is a subspace of $P_{r(\mu)}\mathcal{H}$.

Proposition 1.12 extends to the partial isometries S_μ as follows:

Corollary 1.14. *Suppose that E is a row-finite graph and $\{S, P\}$ is a Cuntz-Krieger E-family in a C^*-algebra B. Let $\mu, \nu \in E^*$. Then*

(a) *if $|\mu| = |\nu|$ and $\mu \neq \nu$, then $(S_\mu S_\mu^*)(S_\nu S_\nu^*) = 0$;*

(b) $S_\mu^*S_\nu = \begin{cases} S_{\mu'}^* & \text{if } \mu = \nu\mu' \text{ for some } \mu' \in E^* \\ S_{\nu'} & \text{if } \nu = \mu\nu' \text{ for some } \nu' \in E^* \\ 0 & \text{otherwise;} \end{cases}$

(c) *if $S_\mu S_\nu \neq 0$, then $\mu\nu$ is a path in E and $S_\mu S_\nu = S_{\mu\nu}$;*

(d) *if $S_\mu S_\nu^* \neq 0$, then $s(\mu) = s(\nu)$.*

Proof. For (a), let i be the smallest integer such that $\mu_i \neq \nu_i$. Then, applying (1.3) to $\mu_1\mu_2\cdots\mu_{i-1}$ gives

$$\begin{aligned} S_\mu^* S_\nu &= (S_{\mu_1}S_{\mu_2}\cdots S_{\mu_n})^* S_{\nu_1}S_{\nu_2}\cdots S_{\nu_n} \\ &= S_{\mu_n}^*\cdots S_{\mu_i}^*(S_{\mu_{i-1}}^*\cdots S_{\mu_1}^*)(S_{\mu_1}S_{\mu_2}\cdots S_{\mu_{i-1}})S_{\nu_i}\cdots S_{\nu_n} \\ &= S_{\mu_n}^*\cdots S_{\mu_i}^* P_{s(\mu_{i-1})} S_{\nu_i}\cdots S_{\nu_n} \\ &= S_{\mu_n}^*\cdots S_{\mu_i}^* P_{r(\mu_i)} S_{\nu_i}\cdots S_{\nu_n} \\ &= S_{\mu_n}^*\cdots S_{\mu_i}^* S_{\nu_i}\cdots S_{\nu_n}, \end{aligned}$$

which vanishes by Proposition 1.12(b), giving (a).

For part (b), assume first that $n := |\mu| \leq |\nu|$, and factor $\nu = \alpha\nu'$ with $|\alpha| = n$. Then

$$S_\mu^* S_\nu = S_\mu^*(S_\alpha S_{\nu'}) = (S_\mu^* S_\alpha)S_{\nu'}.$$

If $\mu = \alpha$, then (1.3) implies that

$$S_\mu^* S_\nu = P_{s(\mu)}S_{\nu'} = P_{r(\nu')}S_{\nu'} = S_{\nu'}.$$

If $\mu \neq \alpha$, then part (a) implies that $S_\mu^* S_\nu = (S_\mu^* S_\alpha)S_{\nu'} = 0$. This gives (b) when $|\mu| \leq |\nu|$. When $|\mu| > |\nu|$, we can either run a similar argument factoring $\mu = \beta\mu'$, or take adjoints and apply what we have just proved.

Parts (c) and (d) follow from the corresponding parts of Proposition 1.12. □

Corollary 1.15. *Suppose that E is a row-finite graph and $\{S, P\}$ is a Cuntz-Krieger E-family in a C^*-algebra B. For μ, ν, α, $\beta \in E^*$, we have*

$$(1.4) \qquad (S_\mu S_\nu^*)(S_\alpha S_\beta^*) = \begin{cases} S_{\mu\alpha'}S_\beta^* & \text{if } \alpha = \nu\alpha' \\ S_\mu S_{\beta\nu'}^* & \text{if } \nu = \alpha\nu' \\ 0 & \text{otherwise.} \end{cases}$$

In particular, it follows that every non-zero finite product of the partial isometries S_e and S_f^* has the form $S_\mu S_\nu^*$ for some $\mu, \nu \in E^*$ with $s(\mu) = s(\nu)$.

Proof. The formula follows from part (b) of Corollary 1.14. To see the last statement, we suppose that W is a non-zero word — that is, a product of finitely many S_e and S_f^*. Any adjacent S_e's can be combined into a single term S_μ, and since W is non-zero, μ must be a path. Similarly, any adjacent S_f^*'s can be combined into an S_ν^* for some $\nu \in E^*$. Thus W is a product of terms of the form $S_\mu S_\nu^*$ for $\mu, \nu \in E^*$. (Since $E^0 \subset E^*$, we can write S_ν^*, for example, as $S_{s(\nu)}S_\nu^* = P_{s(\nu)}S_\nu^*$.) The formula (1.4) implies that we can combine this product into one term of the same form. □

Corollary 1.16. *If $\{S, P\}$ is a Cuntz-Krieger E-family for a row-finite graph E, then*

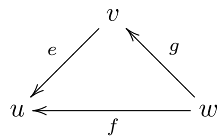
$$C^*(S,P) = \overline{\operatorname{span}}\{S_\mu S_\nu^* : \mu, \nu \in E^*,\ s(\mu) = s(\nu)\}.$$

Proof. The formula (1.4) implies that

$$\operatorname{span}\{S_\mu S_\nu^* : \mu, \nu \in E^*,\ s(\mu) = s(\nu)\}$$

is a subalgebra of $C^*(S,P)$, and since $(S_\mu S_\nu^*)^* = S_\nu S_\mu^*$, it is a $*$-subalgebra. Thus its closure is a C^*-subalgebra of $C^*(S,P)$, and since it contains the generators $S_e = S_e S_{s(e)}^*$ and $P_v = S_v S_v^*$, it is all of $C^*(S,P)$. □

EXAMPLE 1.17. Let $\{S, P\}$ be a Cuntz-Krieger family for the following directed graph E:



When $s(\mu) = s(\nu)$ we have $S_\mu S_\nu^* = S_\mu P_{s(\mu)} S_\nu^*$; unless $s(\mu) = w$, we can apply the Cuntz-Krieger relation at $s(\mu)$, and keep doing this until the paths begin at w. For example,

$$\begin{aligned} P_u &= S_e S_e^* + S_f S_f^* = S_e P_v S_e^* + S_f S_f^* \\ &= S_e \left(S_g S_g^* \right) S_e^* + S_f S_f^* \\ &= S_{eg} S_{eg}^* + S_f S_f^*. \end{aligned}$$

Thus

$$\begin{aligned} C^*(S, P) &= \overline{\operatorname{span}}\{S_\mu S_\nu^* : \mu, \nu \in E^*,\ s(\mu) = s(\nu) = w\} \\ &= \operatorname{span}\left\{S_\mu S_\nu^* : \mu, \nu \in \{w, f, g, eg\}\right\}. \end{aligned}$$

Since w is a source, two paths μ, ν with $s(\mu) = w = s(\nu)$ cannot satisfy $\nu = \mu\nu'$ unless $\mu = \nu$. Hence

$$(S_\mu S_\nu^*)(S_\alpha S_\beta^*) = \begin{cases} S_\mu S_\beta^* & \text{if } \alpha = \nu \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\{S_\mu S_\nu^* : \mu, \nu \in \{w, f, g, eg\}\}$ is a set of matrix units which spans $C^*(S, P)$. So we have by Proposition A.5 that if one is non-zero, they all are, and $C^*(S, P)$ is isomorphic to $M_4(\mathbb{C})$.

A path μ in a directed graph E is a *cycle*[2] if $|\mu| \geq 1$, $r(\mu) = s(\mu)$ and $s(\mu_i) \neq s(\mu_j)$ for $i \neq j$. In the graph-theory literature, people sometimes insist that $|\mu| > 1$, but this would make our statements much more complicated. The crucial feature of the graph in Example 1.17 is that there are no cycles.

PROPOSITION 1.18. *Suppose E is a finite directed graph with no cycles, and $w_1, \cdots, w_n$ are the sources in E. Then for every Cuntz-Krieger E-family $\{S, P\}$ in which each P_v is non-zero we have*

$$C^*(S, P) \cong \bigoplus_{i=1}^n M_{|s^{-1}(w_i)|}(\mathbb{C}),$$

where $s^{-1}(w_i) = \{\mu \in E^* : s(\mu) = w_i\}$.

PROOF. As in Example 1.17, finitely many applications of the Cuntz-Krieger relations show that

$$C^*(S, P) = \operatorname{span}\{S_\mu S_\nu^* : s(\mu) = s(\nu) = w_i \text{ for some } i\}$$

and $A_i := \operatorname{span}\{S_\mu S_\nu^* : s(\mu) = s(\nu) = w_i\}$ is isomorphic to $M_{|s^{-1}(w_i)|}(\mathbb{C})$. When $\mu \in s^{-1}(w_i)$ and $\alpha \in s^{-1}(w_j)$ for some $j \neq i$, μ cannot extend α and vice versa. Thus $A_i A_j = 0$, and $C^*(S, P) \cong \bigoplus_{i=1}^n A_i$ by Proposition A.7. □

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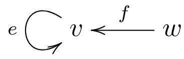
[2]In the graph-algebra literature, cycles are called *simple loops*.

EXAMPLE 1.19. Consider a Cuntz-Krieger family $\{S,P\}$ for the following directed graph E:



The Cuntz-Krieger relations say that $S_e^*S_e = P_{s(e)} = P_v$, $S_f^*S_f = P_w$ and $P_v = S_eS_e^* + S_fS_f^*$. The element $P_v + P_w$ is an identity for $C^*(S,P)$. The element $S_e + S_f$ satisfies

$$(S_e + S_f)^*(S_e + S_f) = S_e^*S_e + S_f^*S_e + S_e^*S_f + S_f^*S_f = P_v + 0 + 0 + P_w,$$

and hence is an isometry in $C^*(S,P)$. Since

$$(S_e + S_f)(S_e + S_f)^* = S_eS_e^* + S_fS_e^* + S_eS_f^* + S_fS_f^* = P_v,$$

we can recover P_v, $P_w = (S_e+S_f)^*(S_e+S_f) - P_v$, $S_e = (S_e+S_f)P_v$ and $S_f = (S_e + S_f)P_w$ from the single element S_e+S_f. Thus $C^*(S,P)$ is generated by the isometry S_e+S_f. Conversely, if V is an isometry, then $P_w = 1-VV^*$, $P_v = VV^*$, $S_e = VP_v$, $S_f = VP_w$ defines a Cuntz-Krieger E-family such that $C^*(S,P) = C^*(V)$.

Coburn's theorem [**93**, Theorem 3.5.18] says that all C^*-algebras generated by one non-unitary isometry are isomorphic, and in particular isomorphic to the *Toeplitz algebra* $\mathcal{T}$ generated by the unilateral shift. The isometry $S_e + S_f$ is non-unitary precisely when $P_w \neq 0$, so we deduce that all Cuntz-Krieger E-families with $P_w \neq 0$ generate C^*-algebras isomorphic to $\mathcal{T}$.

Proposition 1.18 and Example 1.19 suggest that, provided two Cuntz-Krieger families are non-trivial in the sense that appropriate vertex projections P_v are nonzero, the Cuntz-Krieger families generate isomorphic C^*-algebras. This is indeed a general phenomenon. To study it, we introduce a C^*-algebra which is universal for C^*-algebras generated by Cuntz-Krieger E-families, and analyse the representations of this C^*-algebra.

To build the universal C^*-algebra generated by a Cuntz-Krieger E-family, we mimic the behaviour of the spanning set $\{S_\mu S_\nu^*\}$. In the next proposition, the symbols $d_{\mu,\nu}$ are purely formal, all but finitely many coefficients $z_{\mu,\nu}$ in each sum are 0, and the vector space operations on the formal sums are defined by

$$a\big(\sum w_{\mu,\nu}d_{\mu,\nu}\big) + b\big(\sum z_{\mu,\nu}d_{\mu,\nu}\big) = \sum(aw_{\mu,\nu} + bz_{\mu,\nu})d_{\mu,\nu}.$$

The elements $d_{\alpha,\beta}$ obtained by setting $z_{\alpha,\beta} = 1$ and $z_{\mu,\nu} = 0$ otherwise then form a basis for V.

PROPOSITION 1.20. *Let E be a row-finite directed graph. Then the vector space V of formal linear combinations*

$$V = \big\{\sum z_{\mu,\nu}d_{\mu,\nu} : \mu,\nu \in E^*, s(\mu) = s(\nu)\big\}$$

is a $*$-algebra with $(d_{\mu,\nu})^* = d_{\nu,\mu}$ and

$$d_{\mu,\nu}d_{\alpha,\beta} = \begin{cases} d_{\mu\alpha',\beta} & \text{if } \alpha = \nu\alpha' \\ d_{\mu,\beta\nu'} & \text{if } \nu = \alpha\nu' \\ 0 & \text{otherwise.} \end{cases}$$

To prove this, one has to check that the product is associative and compatible with the $*$-operation. This is tedious but routine.

For every Cuntz-Krieger family $\{S,P\}$ on $\mathcal{H}$, the operators $\{S_\mu S_\nu^*\}$ satisfy the relations imposed on the $\{d_{\mu,\nu}\}$, and hence there is a $*$-representation $\pi_{S,P}$ of V

on $\mathcal{H}$ such that $\pi_{S,P}(d_{\mu,\nu}) = S_\mu S_\nu^*$. Since the norm of a projection P satisfies $\|P\|^2 = \|P^*P\| = \|P\|$, every non-zero projection has norm 1, and thus for every non-zero partial isometry W, we have $\|W\|^2 = \|W^*W\| = 1$. Thus

$$\left\|\pi_{S,P}\left(\sum z_{\mu,\nu} d_{\mu,\nu}\right)\right\| \leq \sum |z_{\mu,\nu}| \, \|S_\mu S_\nu^*\| \leq \sum |z_{\mu,\nu}|.$$

It follows that

$$\|a\|_1 := \sup\{\|\pi_{S,P}(a)\| : \{S,P\} \text{ is a Cuntz-Krieger } E\text{-family}\}$$

is finite for every v in V, and $\|\cdot\|_1$ is an algebra seminorm satisfying $\|a^*a\|_1 = \|a\|_1^2$. Let I be the $*$-ideal $\{u \in V : \|u\|_1 = 0\}$. Then $V_0 = V/I$ is a $*$-algebra, and the quotient norm $\|\cdot\|_0$ defined by $\|v+I\|_0 = \inf\{\|u+j\|_1 : j \in I\}$ is a C^*-norm, so the completion $\overline{V_0}$ is a C^*-algebra. Each $\pi_{S,P}$ is $\|\cdot\|_0$-continuous, and hence extends uniquely to a representation of $\overline{V_0}$.

We have now outlined the main steps in the proof of:

PROPOSITION 1.21. *For any row-finite directed graph E, there is a C^*-algebra $C^*(E)$ generated by a Cuntz-Krieger E-family $\{s,p\}$ such that for every Cuntz-Krieger E-family $\{T,Q\}$ in a C^*-algebra B, there is a homomorphism $\pi_{T,Q}$ of $C^*(E)$ into B satisfying $\pi_{T,Q}(s_e) = T_e$ for every $e \in E^1$ and $\pi_{T,Q}(p_v) = Q_v$ for every $v \in E^0$.*

PROOF. Take $C^*(E) = \overline{V_0}$, and check that $s_e := d_{e,s(e)}$, $p_v := d_{v,v}$ form a Cuntz-Krieger E-family which generates V_0. To get $\pi_{T,Q}$, choose a faithful representation $\rho : B \to B(\mathcal{H})$, and take $\pi_{T,Q} = \rho^{-1} \circ \pi_{\rho(T),\rho(Q)}$. □

The C^*-algebra $C^*(E)$ is called the *C^*-algebra of the graph E* or the *Cuntz-Krieger algebra of E*, and is generically described as a *graph algebra*. In these notes, $\{s,p\}$ will always be the universal family which generates $C^*(E)$; in general, we will try to use lower-case letters for a Cuntz-Krieger family only when we think the family has a universal property.

Those whose native languages have definite and indefinite articles may have noticed that we have been making implicit uniqueness assertions about $C^*(E)$ and $\{s,p\}$. We take the view that the next corollary justifies this, and that it is okay to talk about "the" graph algebra $C^*(E)$ and "the" generating family $\{s,p\}$, provided we remember when it matters that they are only unique up to isomorphism in the following precise sense.

COROLLARY 1.22. *Suppose E is a row-finite directed graph, and C is a C^*-algebra generated by a Cuntz-Krieger E-family $\{w,r\}$ such that for every Cuntz-Krieger E-family $\{T,Q\}$ in a C^*-algebra B, there is a homomorphism $\rho_{T,Q}$ of C into B satisfying $\rho_{T,Q}(w_e) = T_e$ for every $e \in E^1$ and $\rho_{T,Q}(r_v) = Q_v$ for every $v \in E^0$. Then there is an isomorphism ϕ of $C^*(E)$ onto C such that $\phi(s_e) = w_e$ for every $e \in E^1$ and $\phi(p_v) = r_v$ for every $v \in E^0$.*

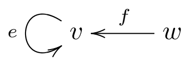
PROOF. We take $\phi := \pi_{w,r}$. It is onto because the range of $\pi_{w,r}$ is a C^*-algebra containing $\{w_e, r_v\}$, hence is all of C. Since $\rho_{s,p} \circ \pi_{w,r}$ is the identity on $\{s,p\}$, it is the identity on all of $C^*(E)$. Thus

$$\phi(a) = 0 \Longrightarrow \pi_{w,r}(a) = 0 \Longrightarrow a = \rho_{s,p}(\pi_{w,r}(a)) = 0,$$

and ϕ is injective. □

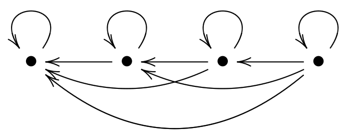
EXAMPLE 1.23. For the graph E which consists of a single loop at a single vertex v, Cuntz-Krieger E-families $\{S, P\}$ are determined by the single operator S_e, which is a unitary operator of $P_v\mathcal{H}$ onto $P_v\mathcal{H}$. The operator P_v is an identity for $C^*(S,P)$, and S_e is a unitary element of $C^*(S,P)$. So $(C^*(E), s_e)$ is universal for C^*-algebras generated by a unitary element: if U is a unitary element of a C^*-algebra B, then there is a homomorphism $\pi_U : C^*(E) \to B$ such that $\pi(s_e) = U$. We know from spectral theory that if $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and $\iota : \mathbb{T} \to \mathbb{C}$ is the function $\iota(z) = z$, then $(C(\mathbb{T}), \iota)$ has this same universal property. So Corollary 1.22 gives an isomorphism ϕ of $C^*(E)$ onto $C(\mathbb{T})$ such that $\phi(s_e) = \iota$.

EXAMPLE 1.24. In Example 1.19 we considered Cuntz-Krieger families $\{S, P\}$ for the following directed graph E:



We saw there that $S_e + S_f$ is an isometry which generates $C^*(S,P)$, and that every isometry on Hilbert space gives a Cuntz-Krieger E-family. Thus $(C^*(E), s_e + s_f)$ is the universal C^*-algebra (A, a) generated by an isometry a. In the representation-theoretic analysis of (A, a), Coburn's theorem becomes the assertion that, if π is a representation of A and $\pi(a)$ is non-unitary, then π is faithful on A (see [**1**], for example).

EXAMPLE 1.25. For a more exotic example, consider the following graph E:



Hong and Szymański prove in [**56**, Theorem 4.4] that $C^*(E)$ is isomorphic to the non-commutative sphere $C(S_q^7)$ of Vaksman and Soibelman, by checking that $C(S_q^7)$ has the universal property which characterises $C^*(E)$ and applying Corollary 1.22. In [**56**] and [**58**], they show that a broad range of non-commutative spheres, projective spaces and lens spaces are isomorphic to the C^*-algebras of suitable directed graphs.

REMARK 1.26. We have insisted that our graphs are countable, and hence all our graph algebras are separable. However, we have not really used this hypothesis, and one can talk about graph algebras of uncountable graphs. Katsura has recently shown that there is a graph E with uncountably many vertices whose C^*-algebra is prime but not primitive [**77**, Proposition 13.4]. Of course one can take this two ways: as evidence that uncountable graphs are interesting, or as evidence that they should be avoided.