CHAPTER 1

Hyperplane sections of ℓ_p-balls

1.1. Lecture 1

Historically, the Fourier analytic approach to sections of convex bodies originates from the study of hyperplane sections of the unit cube $Q_n = [-1/2, 1/2]^n$ in $\mathbb{R}^n$. Looking at the three dimensional cube, one may think that the maximal hyperplane section of the cube is the one perpendicular to the main diagonal:

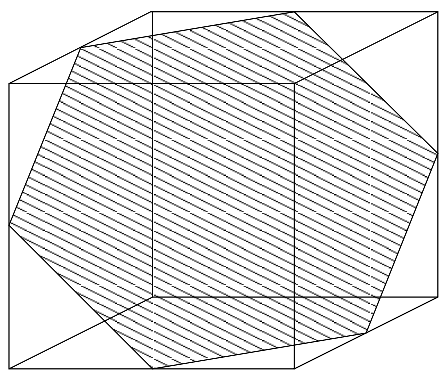

Figure 1. Section orthogonal to the main diagonal.

However, a quick computation shows that the area of this section is $\frac{3\sqrt{3}}{4}$, which is smaller than $\sqrt{2}$, the area of the section perpendicular to the vector $(1, 1, 0)$:

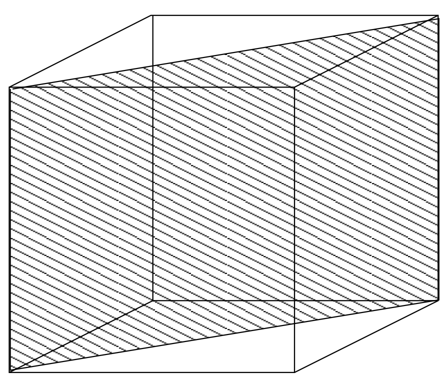

Figure 2. Section orthogonal to the vector $(1, 1, 0)$.

For large dimensions, the same effect can be seen from the following formula, which dates back to Laplace (1812) [**La**]:

$$\mathrm{Vol}_{n-1}\left(Q_n \cap (\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})^{\perp}\right) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin(r/\sqrt{n})}{r/\sqrt{n}}\right)^n dr.$$

Here and in the sequel, for $\xi \in S^{n-1}$, we denote the central hyperplane section orthogonal to ξ by $\xi^{\perp}$, i.e

$$\xi^{\perp} = \{x \in \mathbb{R}^n : (x, \xi) = 0\}.$$

In order to see that for large dimensions the section orthogonal to the main diagonal is not maximal, one can reproduce an argument appearing also in the classical central limit theorem in probability, as follows:

$$\frac{\sin x}{x} \sim 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots$$

and therefore as $n \to \infty$

$$\mathrm{Vol}_{n-1}\left(Q_n \cap (\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})^{\perp}\right) \sim \frac{1}{\pi}\int_{-\sqrt{n}}^{\sqrt{n}}\left(1 - \frac{r^2}{6n}\right)^n dr \sim$$

$$\sim \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-\frac{r^2}{6}}\, dr = \sqrt{\frac{6}{\pi}}.$$

But

$$\mathrm{Vol}_{n-1}\left(Q_n \cap (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots, 0)^{\perp}\right) = \sqrt{2} > \sqrt{\frac{6}{\pi}}.$$

Surprisingly enough, the problem of finding the maximal section of the cube remained open till 1986, when K. Ball [**Ba1**] proved that in all dimensions the maximum is indeed $\sqrt{2}$. The proof was based on the general formula for central hyperplane sections of the cube going back to Polya (1913) [**Pol**]:

$$\mathrm{Vol}_{n-1}(Q_n \cap \xi^{\perp}) = \frac{1}{\pi}\int_{-\infty}^{\infty}\prod_{k=1}^{n}\frac{\sin(r\xi_k)}{r\xi_k} dr. \tag{1.1}$$

Here we assume that $\sin(r\xi_k)/r\xi_k = 1$ if $\xi_k = 0$.

To prove this formula, we use an argument also appearing in [**NP**]. Denote by

$$2Q_n = B_{\infty}^n = \{x \in \mathbb{R}^n : \|x\|_{\infty} = \max_{1 \le k \le n} |x_k| \le 1\}$$

the unit ball of the space ℓ_{∞}^n. Let

$$A_{\xi}(t) = \mathrm{Vol}_{n-1}(B_{\infty}^n \cap \{\xi^{\perp} + t\xi\})$$

be the $(n-1)$-dimensional volume of the hyperplane section of the ball B_{∞}^n perpendicular to $\xi \in S^{n-1}$ and at distance t from the origin.

THEOREM 1.1. *For every $t \in \mathbb{R}$ and $\xi \in S^{n-1}$ we have*

$$A_\xi(t) = \frac{2^n}{\pi}\int_0^\infty \cos(tr)\prod_{k=1}^n \frac{\sin(r\xi_k)}{r\xi_k}dr.$$

PROOF. Note that $A_\xi(t) = \int_{(x,\xi)=t}\chi(\|x\|_\infty)dx$, where χ is the indicator function of the interval $[-1,1]$. Taking the Fourier transform in the variable t and using the Fubini theorem we get

$$\widehat{A_\xi}(r) = \int_{\mathbb{R}} A_\xi(t)e^{-itr}dt = \int_{\mathbb{R}} e^{-itr}\int_{(x,\xi)=t}\chi(\|x\|_\infty)dx\ dt =$$

$$= \int_{\mathbb{R}^n}\chi(\|x\|_\infty)e^{-ir(x,\xi)}dx.$$

Clearly, $\chi(\|x\|_\infty) = \chi(|x_1|)\cdots\chi(|x_n|)$, therefore

$$\widehat{A_\xi}(r) = \prod_{k=1}^n\int_{-1}^1 e^{-irx_k\xi_k}dx_k = 2^n\prod_{k=1}^n\frac{\sin(r\xi_k)}{r\xi_k}.$$

Since the function $A_\xi(t)$ is even, we get

$$2\pi A_\xi(t) = (\widehat{A_\xi})^\wedge(t) = 2^n\int_{\mathbb{R}} e^{-itr}\prod_{k=1}^n\frac{\sin(r\xi_k)}{r\xi_k}dr.$$

□

We now give an idea of the proof of K. Ball's theorem on the maximal section of the cube.

THEOREM 1.2. *(K. Ball* [**Ba1**] *) For every $\xi \in S^{n-1}$*

$$\mathrm{Vol}_{n-1}(Q_n\cap\xi^\perp) \le \sqrt{2},$$

with equality for $\xi = (\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,\dots,0)$.

SKETCH OF PROOF.

Case 1. Suppose $|\xi_k| > \frac{1}{\sqrt{2}}$, for some k. Obviously, the volume of the projection of $Q_n\cap\xi^\perp$ to the plane $x_k = 0$ does not exceed the volume of the projection of the whole cube, which is 1. On the other hand the volume of this projection is $|\xi_k|$ (which is the corresponding cosine) multiplied by the volume of $Q_n\cap\xi^\perp$. So,

$$1 \ge |\xi_k|\ \mathrm{Vol}_{n-1}(Q_n\cap\xi^\perp).$$

The assumption on ξ_k yields the result.

Case 2. $|\xi_k| \le \frac{1}{\sqrt{2}}$ for all k. Applying Hölder's inequality to formula (1.1) gives

$$\mathrm{Vol}_{n-1}(Q_n\cap\xi^\perp) \le \int_{-\infty}^\infty\prod_{k=1}^n\left|\frac{\sin(\pi r\xi_k)}{\pi r\xi_k}\right|dr \le$$

$$\le \prod_{k=1}^{n} \left(\int_{-\infty}^{\infty} \left| \frac{\sin(\pi r \xi_k)}{\pi r \xi_k} \right|^{\frac{1}{\xi_k^2}} dr \right)^{\xi_k^2} =$$

$$= \prod_{k=1}^{n} \left(\frac{1}{\xi_k} \right)^{\xi_k^2} \left(\int_{-\infty}^{\infty} \left| \frac{\sin(\pi x)}{\pi x} \right|^{\frac{1}{\xi_k^2}} dx \right)^{\xi_k^2}.$$

The main ingredient of the proof is Ball's integral inequality, which says that for all $s \ge 2$,

$$\int_{-\infty}^{\infty} \left| \frac{\sin(\pi x)}{\pi x} \right|^{s} dx \le \sqrt{\frac{2}{s}}.$$

The proof of this inequality is quite difficult and can be found in the original paper of Ball or in the paper [**NP**] (the latter proof was reproduced in [**K9**, p. 145]). Applying Ball's integral inequality with $s = \frac{1}{\xi_k^2}$ (note that $s \ge 2$ because of the assumption $\xi_k \le \frac{1}{\sqrt{2}}$), we get

$$\mathrm{Vol}_{n-1}(Q_n \cap \xi^{\perp}) \le \prod_{k=1}^{n} \left(\sqrt{2\xi_k^2} \right)^{\xi_k^2} \left(\frac{1}{\xi_k} \right)^{\xi_k^2} = \prod_{k=1}^{n} (\sqrt{2})^{\xi_k^2} = \sqrt{2}.$$

□

Let us now consider the unit balls B_q^n of the spaces ℓ_q^n, $0 < q < \infty$, defined by

$$B_q^n = \{x \in \mathbb{R}^n : \|x\|_q = (|x_1|^q + \cdots + |x_n|^q)^{1/q} \le 1\}.$$

Meyer and Pajor [**MeyP**] discovered an analog of formula (1.1) for central hyperplane sections of the balls B_q^n. The original result of Meyer and Pajor was proved in the case $1 \le q \le 2$ using probabilistic methods, and later the formula was extended to all $0 < q < \infty$ in [**K4**] by Fourier methods.

THEOREM 1.3. *For every $\xi \in S^{n-1}$,*

$$\mathrm{Vol}_{n-1}(B_q^n \cap \xi^{\perp}) = \frac{q}{\pi(n-1)\Gamma(\frac{n-1}{q})} \int_0^{\infty} \prod_{k=1}^{n} \gamma_q(t\xi_k) dt, \tag{1.2}$$

where $\gamma_q(t) = (e^{-|z|^q})^\wedge(t)$, $t \in \mathbb{R}$.

Theorem 1.3 will be proved in Lecture 2 as a part of a more general result. Now let us show an application of this theorem to the problem of finding the extremal central sections of ℓ_q^n-balls, $0 < q \le 2$. We start with some properties of the functions γ_q (for more about these functions, see [**K9**, Section 2.8]).

LEMMA 1.4. *For $0 < q \le 2$, the function γ_q is positive on $[0, \infty)$, and the function $\log(\gamma_q(\sqrt{x}))$ is convex on $[0, \infty)$.*

Proof. A function f is called *completely monotonic* on $[0,\infty)$ if it is infinitely differentiable on $(0,\infty)$ and for all $k \in \mathbb{N}\cup\{0\}$ and $x>0$

$$(-1)^k f^{(k)}(x) \geq 0.$$

The celebrated Bernstein's theorem (see [**Fel**, Ch.18, Section 4]) asserts that every completely monotonic continuous at zero function is the Laplace transform of a finite measure on $[0,\infty)$, i.e. there exists a finite Borel measure on $[0,\infty)$ such that

$$f(x) = \int_0^\infty e^{-tx} d\mu(t), \qquad \forall x \geq 0.$$

One can check that e^{-z^α} is completely monotonic for $0<\alpha\leq 1$. Therefore for every $0<q\leq 2$ there exists a finite measure $\mu_{q/2}$ so that

$$e^{-z^{q/2}} = \int_0^\infty e^{-tz} d\mu_{q/2}(t), \qquad \forall z \geq 0,$$

and

$$e^{-|z|^q} = \int_0^\infty e^{-tz^2} d\mu_{q/2}(t), \qquad \forall z \in \mathbb{R}.$$

Taking the Fourier transform of both sides by z (the function $e^{-|z|^q}$ is integrable so we can apply the Fubini theorem), we get

$$\gamma_q(\xi) = \sqrt{\pi}\int_0^\infty t^{-1/2} e^{-\xi^2/4t} d\mu_{q/2}(t),$$

which in particular implies that $\gamma_q(\xi)>0$. Using the Cauchy-Schwartz inequality we get that for all ξ_1, $\xi_2>0$

$$\gamma_q^2\left(\sqrt{\frac{\xi_1+\xi_2}{2}}\right) = \pi\left(\int_0^\infty t^{-1/2} e^{-\xi_1/8t} e^{-\xi_2/8t} d\mu_{q/2}(t)\right)^2 \leq$$

$$\leq \pi \int_0^\infty t^{-1/2} e^{-\xi_1/4t} d\mu_{q/2}(t) \int_0^\infty t^{-1/2} e^{-\xi_2/4t} d\mu_{q/2}(t) = \gamma_q(\sqrt{\xi_1})\cdot\gamma_q(\sqrt{\xi_2}).$$

□

We are ready to prove the result on the extremal sections.

Theorem 1.5. *For* $0<q\leq 2$ *and any* $\xi \in S^{n-1}$

$$\mathrm{Vol}_{n-1}(B_q^n \cap (\frac{1}{\sqrt{n}},\dots,\frac{1}{\sqrt{n}})^\perp) \leq \mathrm{Vol}_{n-1}(B_q^n \cap \xi^\perp)$$

$$\leq \mathrm{Vol}_{n-1}(B_q^n \cap (1,0,\dots,0)^\perp).$$

Proof. For all $0<\xi_1<\eta_1<\eta_2<\xi_2$ such that $\xi_1^2+\xi_2^2=\eta_1^2+\eta_2^2$ and all $t>0$, the convexity result from Lemma 1.4 implies

$$\gamma_q(t\xi_1)\cdot\gamma(t\xi_2) \geq \gamma_q(t\eta_1)\cdot\gamma_q(t\eta_2).$$

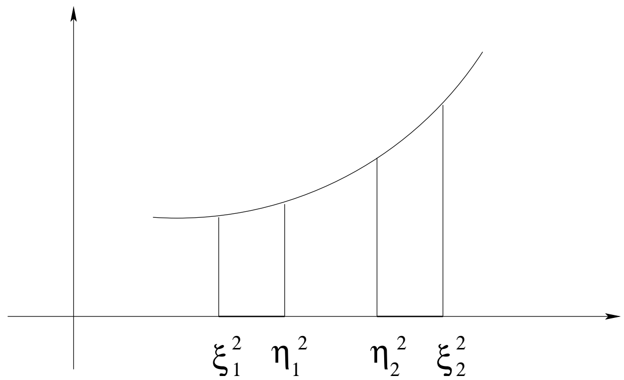

Figure 3.

Combined with the fact that γ_q is a non-negative function, the latter shows that the integral in formula (1.2) is minimal when all the coordinates of the vector ξ are equal, and it is maximal when one of the coordinates is equal to 1 and the others are equal to zero.

$\square$

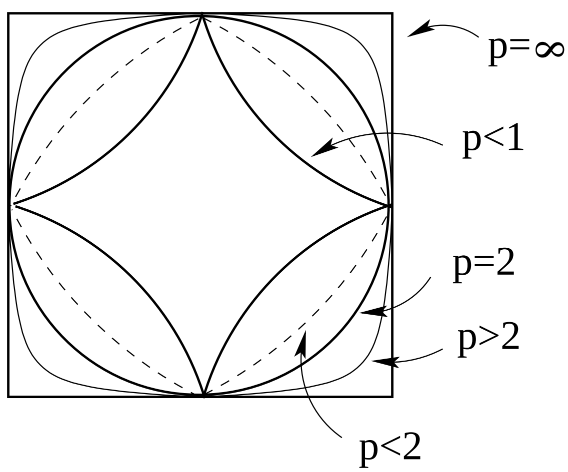

Figure 4. ℓ_p^2-balls

To conclude this section let us give an overview of known results on the extremal sections of ℓ_q^n-balls.

- $q = \infty$
  max: $\xi = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \ldots, 0)$, Ball [**Ba1**]
  min: $\xi = (1, 0, \ldots, 0)$, Hadwiger [**Ha**], Hensley [**He**], Vaaler [**V**]
- $0 < q \leq 2$
  max: $\xi = (1, 0, \ldots, 0)$, Meyer-Pajor [**MeyP**] $(1 \leq q \leq 2)$,
  Caetano [**C**], Barthe [**Bar**] $(0 < q < 1)$.

min: $\xi = (\frac{1}{\sqrt{n}}, \cdots, \frac{1}{\sqrt{n}})$, [**K4**]

- $q > 2$

  min: $\xi = (1, 0, \ldots, 0)$, Meyer-Pajor [**MeyP**]

  max: open, the answer depends on n, q – Oleszkiewicz [**O**].

Complex versions of these results were obtained by Oleszkiewicz and Pelczynski [**OP**] for $q = \infty$, and in [**KZ**] for $0 < q < 2$.

CHAPTER 2

Volume and the Fourier transform

2.1. Lecture 2

The Fourier analytic approach to sections of convex bodies is based on certain general formulas relating the volume of sections to the Fourier transform of powers of the Minkowski functional of the body. In this lecture we are going to discuss these formulas and prove some of them. In fact, we are going to show that formulas (1.1) and (1.2) from the previous section are particular cases of a general formula that applies to all star bodies. Let us start with a calculation that will lead us to this general formula.

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of complex-valued infinitely differentiable rapidly decreasing functions on $\mathbb{R}^n$. Elements of this class will be called *test functions*.

Denote by $\mathcal{S}'(\mathbb{R}^n)$ the class of *distributions*, i.e. continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$.

If f is a locally integrable function with power growth at infinity, then f acts on test functions by integration:

$$\langle f, \phi \rangle = \int_{\mathbb{R}^n} f(x)\phi(x)dx.$$

The Fourier transform of a distribution f is a distribution $\hat{f} \in \mathcal{S}'(\mathbb{R}^n)$ defined by

$$\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle, \qquad \forall \phi \in \mathcal{S}(\mathbb{R}^n).$$

It is easy to see ([**K9**, p. 14]) that, for $0 < p < n$, the functions $\|x\|_\infty^{-p}$ and $\|x\|_q^{-p}$ are locally integrable on $\mathbb{R}^n$. Let us compute their Fourier transforms in the sense of distributions.

Lemma 2.1. *([**K6**]) If $0 < p < n$, the Fourier transforms of the distributions $\|x\|_\infty^{-p}$, $\|x\|_q^{-p}$, $q > 0$ are locally integrable functions*

$$(\|x\|_\infty^{-p})^\wedge(\xi) = 2^n p \int_0^\infty t^{n-p-1} \prod_{k=1}^n \frac{\sin(t\xi_k)}{t\xi_k} dt,$$

$$(\|x\|_q^{-p})^\wedge(\xi) = \frac{q}{\Gamma(p/q)} \int_0^\infty t^{n-p-1} \prod_{k=1}^n \gamma_q(t\xi_k) dt.$$

Proof. We will prove only the second formula, the case of ℓ_∞^n can be treated in the same spirit. It is easy to check that

$$\|x\|_q^{-p} = (|x_1|^q + \cdots + |x_n|^q)^{-p/q} = \frac{q}{\Gamma(p/q)} \int_0^\infty y^{p-1} e^{-y^q(|x_1|^q+\cdots+|x_n|^q)} dy.$$

Since $(f(tx))^\wedge(\xi) = t^{-n}\hat{f}(\xi/t)$, for a fixed y the Fourier transform of the function

$$x \to e^{-y^q(|x_1|^q+\cdots+|x_n|^q)}, \qquad x \in \mathbb{R}^n$$

is equal to the function

$$\xi \to y^{-n} \prod_{k=1}^n \gamma_q(\xi_k/y).$$

Therefore, for any even test function ϕ, we have

$$\langle(\|x\|_q^{-p})^\wedge, \phi\rangle = \int_{\mathbb{R}^n} (|x_1|^q + \cdots + |x_n|^q)^{-p/q} \hat{\phi}(x) dx =$$

$$= \frac{q}{\Gamma(p/q)} \int_0^\infty y^{p-1} \left\langle e^{-y^q(|x_1|^q+\cdots+|x_n|^q)}, \hat{\phi}(x) \right\rangle dy$$

$$= \frac{q}{\Gamma(p/q)} \int_0^\infty y^{-n+p-1} \left(\int_{\mathbb{R}^n} \prod_{k=1}^n \gamma_q(\xi_k/y) \phi(\xi) d\xi \right) dy$$

$$= \left\langle \frac{q}{\Gamma(p/q)} \int_0^\infty y^{-n+p-1} \prod_{k=1}^n \gamma_q(\xi_k/y) dy,\ \phi(\xi) \right\rangle. \tag{2.1}$$

The latter is the action of the distribution

$$\xi \to \int_0^\infty t^{n-p-1} \prod_{k=1}^n \gamma_q(t\xi_k)\ dt$$

on the test function ϕ. Since ϕ is an arbitrary even test function, the result follows.

□

Let us note that the result of Lemma 2.1 remains true for a larger set of p, as follows from an analytic extension argument. First, differentiating by p under the integral one can show that

$$p \to \int_{\mathbb{R}^n} \|x\|^{-p} \hat{\phi}(x) dx = \langle(\|x\|_\infty^{-p})^\wedge, \phi\rangle$$

is an analytic function of p in the domain $\Re p < n$. On the other hand, the last expression in (2.1) is an analytic function of p in the domain $\Re p \in (-qn, n)$, $-p/q \notin \mathbb{N} \cup \{0\}$, if ϕ is supported outside of the coordinate planes in $\mathbb{R}^n$. This follows from the fact that $\gamma_q(t) \sim t^{-1-q}$ at infinity (and has exponential decay if q is an even integer); see [**K9**, Section 2.8]. Therefore by uniqueness of analytic continuation, the formula of Lemma 2.1 holds for $-nq < p < n$, $-p/q \notin \mathbb{N} \cup \{0\}$ with ξ outside of the coordinate planes in $\mathbb{R}^n$.

Putting $p = n-1$ in Lemma 2.1 and comparing the result with formula (1.2), we see that for every $q > 0$ and every $\xi \in \mathbb{R}^n$ the volume of the section of B_q^n by the central hyperplane perpendicular to ξ is equal to

$$\mathrm{Vol}_{n-1}(B_q^n \cap \xi^\perp) = \frac{1}{\pi(n-1)} (\|x\|_q^{-n+1})^\wedge(\xi).$$

We are going to show now that the latter formula is true for any star body in place of B_q^n, namely

THEOREM 2.2. *(*[**K4**]*) Let D be an origin-symmetric star body in $\mathbb{R}^n$. Then for every $\xi \in S^{n-1}$*

$$\mathrm{Vol}_{n-1}(D \cap \xi^\perp) = \frac{1}{\pi(n-1)} (\|x\|_D^{-n+1})^\wedge(\xi).$$

Before we prove this theorem (which we do in Remark 2.5), we need several definitions and simple facts. Recall that a compact set D in $\mathbb{R}^n$ is called a *star body* if every straight line through the origin intersects the boundary of D at exactly two points different from the origin and the *Minkowski functional* of D defined by

$$\|x\|_D = \min\{a \geq 0 : x \in aD\}$$

is a continuous function on $\mathbb{R}^n$.

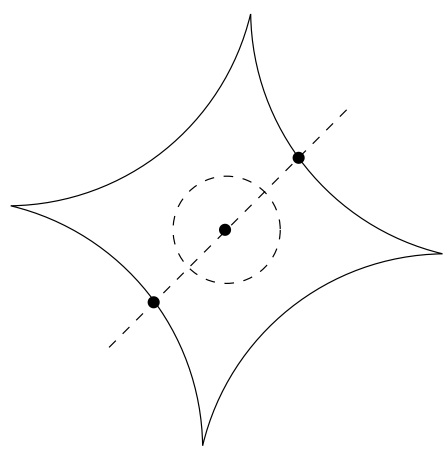

Figure 5. Star body.

It is easy to see that $\|x\|_D$ is a homogeneous function of degree 1 on $\mathbb{R}^n$, and if $x \in S^{n-1}$, then

$$\rho_D(x) = \|x\|_D^{-1}$$

is the radius of D in the direction of x.

We need the following *polar formula for the volume of hyperplane sections*:

$$\mathrm{Vol}_{n-1}(D \cap \xi^\perp) = \int_{\xi^\perp} \chi(\|x\|_D)dx =$$

$$= \int_{S^{n-1}\cap\xi^\perp} \left(\int_0^\infty r^{n-2}\chi(r\|\theta\|_D)dr \right) d\theta =$$

$$= \int_{S^{n-1}\cap\xi^\perp} \left(\int_0^{1/\|\theta\|_D} r^{n-2}dr \right) d\theta = \frac{1}{n-1}\int_{S^{n-1}\cap\xi^\perp} \|\theta\|_D^{-n+1}d\theta$$

Analogously,

$$\mathrm{Vol}_n(D) = \frac{1}{n}\int_{S^{n-1}} \|\theta\|_D^{-n}d\theta.$$

The *spherical Radon transform* is a linear operator $R : C(S^{n-1}) \to C(S^{n-1})$ defined by

$$Rf(\xi) = \int_{S^{n-1}\cap\xi^\perp} f(x)dx, \qquad \xi \in S^{n-1}.$$

Hence, in terms of the spherical Radon transform we have

$$\mathrm{Vol}_{n-1}(D\cap\xi^\perp) = \frac{1}{n-1}R(\|\theta\|_D^{-n+1})(\xi).$$

It is known that the spherical Radon transform is self-dual (see e.g. [**Gr**, Lemma 1.3.3]), meaning that for any $f,\ g \in C(S^{n-1})$,

$$\int_{S^{n-1}} Rf(\xi)g(\xi)d\xi = \int_{S^{n-1}} f(\xi)Rg(\xi)d\xi.$$

Let ϕ be a function integrable on $\mathbb{R}^n$ and all hyperplanes. The *Radon transform* of ϕ is a function of two variables $\xi \in S^{n-1}$ and $t \in \mathbb{R}$ defined by

$$\mathcal{R}\phi(\xi;t) = \int_{(x,\xi)=t} \phi(x)dx.$$

The following is the well-known connection between the Radon and Fourier transforms.

Lemma 2.3. *For a fixed* $\xi \in S^{n-1}$,

$$(\mathcal{R}(\xi;t))^\wedge_t(z) = \hat{\phi}(z\xi).$$

Proof. By definition of the Fourier transform and a simple change of variables we get

$$\hat{\phi}(z\xi) = \int_{\mathbb{R}^n} \phi(x)e^{-iz(x,\xi)}dx = \int_{\mathbb{R}} e^{-izt}(\int_{(x,\xi)=t} \phi(x)dx)dt$$

$$= \int_{\mathbb{R}} e^{-izt}\mathcal{R}(\xi;t)dt.$$

□

The next result is more general than Theorem 2.2. In fact, we get the formula of Theorem 2.2 from it by choosing $f(\theta) = \|\theta\|_D^{-n+1}$. Note that this lemma first appeared in the work of Semyanistyi [**Se**], and later was independently proved in [**K4**]. The proof below comes from [**K4**].

LEMMA 2.4. *Let f be an even homogeneous of degree $-n+1$ function, continuous on $\mathbb{R}^n \setminus \{0\}$. Then the Fourier transform of f is an even homogeneous function of degree -1, continuous on $\mathbb{R}^n \setminus \{0\}$, whose restriction to the sphere equals*

$$Rf(\xi) = \frac{1}{\pi}\hat{f}(\xi), \qquad \forall \xi \in S^{n-1}.$$

PROOF. Since f is even, it is enough to consider only even test functions. For any even test function ϕ,

$$\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle = \int_{\mathbb{R}^n} f(x)\hat{\phi}(x)dx$$

$$= \int_{S^{n-1}} f(\theta) \left(\int_0^\infty \hat{\phi}(t\theta)dt \right) d\theta.$$

By Lemma 2.3, the Fourier transform of the even function $t \to \hat{\phi}(t\theta)$ equals

$$z \to 2\pi \int_{(x,\theta)=z} \phi(x)dx.$$

Since ϕ is even, we have

$$\int_0^\infty \hat{\phi}(t\theta)dt = \frac{1}{2}\int_{-\infty}^\infty \hat{\phi}(t\theta)dt = \frac{1}{2}(\hat{\phi}(t\theta))_t^\wedge(0) = \pi \int_{(x,\theta)=0} \phi(x)dx.$$

Hence, using self-duality of the spherical Radon transform we get

$$\langle \hat{f}, \phi \rangle = \pi \int_{S^{n-1}} f(\theta) \left(\int_{S^{n-1}\cap\theta^\perp} \left(\int_0^\infty r^{n-2}\phi(r\xi)dr \right) d\xi \right) d\theta =$$

$$= \pi \int_{S^{n-1}} \left(\int_0^\infty r^{n-2}\phi(r\theta)dr \right) \left(\int_{S^{n-1}\cap\theta^\perp} f(\xi)d\xi \right) d\theta =$$

$$= \pi \int_{\mathbb{R}^n} |x|_2^{-1} \left(\int_{S^{n-1}\cap(x/|x|_2)^\perp} f(\xi)d\xi \right) \phi(x)dx.$$

Therefore,

$$\hat{f}(x) = \pi |x|_2^{-1} \int_{S^{n-1}\cap(x/|x|_2)^\perp} f(\xi)d\xi.$$

Restricting this equality to the sphere we get the result.

□

Remark 2.5. Let us reiterate the fact that the result of Theorem 2.2 immediately follows from Lemma 2.4. To see this, consider an origin-symmetric star body D in $\mathbb{R}^n$, put $f(x) = \|x\|_D^{-n+1}$ and note that, by the polar formula for the volume of hyperplane sections,

$$Rf(x) = (n-1)\mathrm{Vol}_{n-1}(D \cap x^{\perp}).$$

Lemma 2.4 also gives a simple proof of the classical Minkowski's uniqueness theorem for sections.

Corollary 2.6. *(Minkowski's uniqueness theorem) Let K and L be origin-symmetric star bodies in $\mathbb{R}^n$ such that*

$$\mathrm{Vol}_{n-1}(K \cap \xi^{\perp}) = \mathrm{Vol}_{n-1}(L \cap \xi^{\perp}), \qquad \forall \xi \in S^{n-1}.$$

Then $K = L$.

Proof. By Theorem 2.2 we have

$$(\|x\|_K^{-n+1})^\wedge(\xi) = (\|x\|_L^{-n+1})^\wedge(\xi)$$

for all $\xi \in \mathbb{R}^n \setminus \{0\}$. Therefore,

$$\|x\|_K = \|x\|_L, \qquad \forall x \in \mathbb{R}^n,$$

which is equivalent to $K = L$.

□

We now prove a more general formula relating volume of sections to the Fourier transform, which will have numerous applications later. Let K be a star body. For a fixed $\xi \in S^{n-1}$ the *parallel section function* of K in the direction of ξ is defined by

$$A_{K,\xi}(t) = \mathrm{Vol}_{n-1}(K \cap \{\xi^{\perp} + t\xi\}), \quad t \in \mathbb{R}.$$

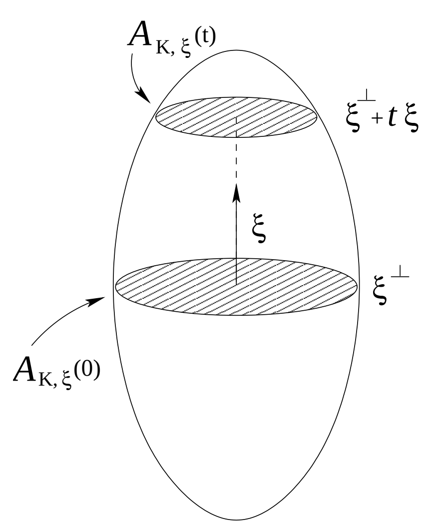

Figure 6. Parallel section function.

Suppose that the function $A_{K,\xi}(t)$ is infinitely differentiable in a neighborhood of zero, which happens in particular when K is an infinitely smooth convex body, see [**K9**, Lemma 2.4]. The *fractional derivative* of the function $A_{K,\xi}$ of order q at zero is defined as follows

$$A^{(q)}_{K,\xi}(0) = \langle \frac{t_+^{-1-q}}{\Gamma(-q)}, A_{K,\xi}(t)\rangle,$$

where $t_+ = \max\{0,t\}$. Let us explain the latter equality in more detail.

We start with $\Re q < 0$, then the function t^{-1-q} is locally integrable and the fractional derivative is equal to

$$A^{(q)}_{K,\xi}(0) = \frac{1}{\Gamma(-q)} \int_0^\infty t^{-1-q} A_{K,\xi}(t)dt.$$

This integral is equal to

$$= \frac{1}{\Gamma(-q)} \int_0^1 t^{-1-q}(A_{K,\xi}(t) - A_{K,\xi}(0))dt+$$

$$+\frac{1}{\Gamma(-q)} \int_1^\infty t^{-1-q} A_{K,\xi}(t)dt + \frac{1}{\Gamma(-q)} \frac{A_{K,\xi}(0)}{(-q)}.$$

Note that the latter expression makes sense for q with $\Re q \in (0,1)$, and this is how we define $A^{(q)}_{K,\xi}(0)$ for these values of q. Continuing this procedure, for $-1 < \Re q < m$, $q \neq 0,1,...,m-1$ we define

$$A^{(q)}_{K,\xi}(0) = \frac{1}{\Gamma(-q)} \int_0^1 t^{-1-q}\Big(A_{K,\xi}(t) - A_{K,\xi}(0) - \cdots$$

$$-A^{(m-1)}_{K,\xi}(0)\frac{t^{m-1}}{(m-1)!}\Big)dt$$

$$+\frac{1}{\Gamma(-q)} \int_1^\infty t^{-1-q} A_{K,\xi}(t)dt + \frac{1}{\Gamma(-q)} \sum_{k=0}^{m-1} \frac{A^{(k)}_{K,\xi}(0)}{k!(k-q)}.$$

If $k \geq 0$ is an integer, we define the fractional derivative of the order k as the limit of the latter expression as $q \to k$, then we get

$$A^{(k)}_{K,\xi}(0) = (-1)^k \frac{d^k}{dt^k} A_{K,\xi}(t)|_{t=0},$$

i.e. fractional derivatives of integral orders coincide up to a sign with usual derivatives.

With this definition, $q \to A^{(q)}_{K,\xi}(0)$ is an entire function of the variable $q \in \mathbb{C}$. If K is origin-symmetric, then $A_{K,\xi}(t)$ is an even function and its derivatives of odd orders at zero are equal to zero. Therefore, for $m-2 < \Re q < m$ we have (this is an elementary calculation)

$$A^{(q)}_{K,\xi}(0) = \frac{1}{\Gamma(-q)} \int_0^\infty t^{-1-q} \left(A_{K,\xi}(t) - \sum_{j=0}^{(m-2)/2} \frac{t^{2j}}{(2j)!} A^{(2j)}_{K,\xi}(0) \right) dt.$$

In particular, for $q \in (0, 2)$:

$$A^{(q)}_{K,\xi}(0) = \frac{1}{\Gamma(-q)} \int_0^\infty t^{-1-q} \left(A_{K,\xi}(t) - A_{K,\xi}(0) \right) dt.$$

We are now ready to present a formula relating the derivatives of the parallel section function at zero to the Fourier transform of powers of the Minkowski functional of the body. This formula will later play the crucial role in the solution to the Busemann-Petty problem.

THEOREM 2.7. *([**GKS**]) Let D be an infinitely smooth origin-symmetric convex body in $\mathbb{R}^n$, $\xi \in S^{n-1}$. Then for every $q \in (-1, \infty)$, $q \neq n - 1$,*

$$A^{(q)}_{D,\xi}(0) = \frac{\cos(\pi q/2)}{\pi(n-q-1)} \left(\|x\|_D^{-n+q+1} \right)^\wedge (\xi). \tag{2.2}$$

In particular, if $k \geq 0$, $k \neq n-1$ is an even integer, then

$$\left(\|x\|_D^{-n+k+1} \right)^\wedge (\xi) = (-1)^{k/2} \pi (n-k-1) A^{(k)}_{D,\xi}(0).$$

If $k \geq 0$, $k \neq n-1$ is an odd integer, then

$$\left(\|x\|_D^{-n+k+1} \right)^\wedge (\xi) = (-1)^{(k+1)/2} 2(n-k-1) k! \times$$

$$\times \int_0^\infty z^{-1-k} \left(A(z) - A(0) - \cdots - A^{(k-1)}(0) \frac{z^{k-1}}{(k-1)!} \right) dz.$$

PROOF. Let $\Re q \in (-1, 0)$. Using Fubini's theorem and passing to the polar coordinates we get

$$\begin{aligned}
A^{(q)}_{D,\xi}(0) &= \frac{1}{2\Gamma(-q)} \int_{-\infty}^\infty |z|^{-1-q} A(z) dz \\
&= \frac{1}{2\Gamma(-q)} \int_{-\infty}^\infty |z|^{-1-q} \int_{(x,\xi)=z} \chi(\|x\|_D) dx\, dz \\
&= \frac{1}{2\Gamma(-q)} \int_{\mathbb{R}^n} |(x,\xi)|^{-1-q} \chi(\|x\|_D) dx \\
&= \frac{1}{2\Gamma(-q)} \int_{S^{n-1}} |(\theta,\xi)|^{-1-q} \left(\int_0^\infty r^{n-q-2} \chi(\|r\theta\|_D) dr \right) d\theta \\
&= \frac{1}{2(n-q-1)\Gamma(-q)} \int_{S^{n-1}} |(\theta,\xi)|^{-1-q} \|\theta\|_D^{-n+q+1} d\theta.
\end{aligned}$$

Consider $A^{(q)}_{D,\xi}(0)$ as a function of $\xi \in \mathbb{R}^n \setminus \{0\}$, homogeneous of degree $-1-q$. For every even test function ϕ we have

$$\begin{aligned}
&\langle A^{(q)}_{D,\xi}(0), \phi(\xi) \rangle = \\
&= \frac{1}{2(n-q-1)\Gamma(-q)} \int_{S^{n-1}} \|\theta\|_D^{-n+q+1} \left(\int_{\mathbb{R}^n} |(\theta,\xi)|^{-1-q} \phi(\xi) d\xi \right) d\theta \\
&= \frac{1}{2(n-q-1)\Gamma(-q)} \int_{S^{n-1}} \|\theta\|_D^{-n+q+1} \left(\int_{\mathbb{R}} |z|^{-1-q} \mathcal{R}\phi(\theta; z) dz \right) d\theta.
\end{aligned}$$

In the latter expression the integral in parentheses equals

$$\langle |z|^{-1-q}, \mathcal{R}\phi(\theta;z)\rangle = \frac{1}{2\pi}\langle (|z|^{-1-q})^\wedge(t), (\mathcal{R}\phi(\theta;z))^\wedge(t)\rangle$$
$$= \frac{1}{\pi}\Gamma(-q)\cos\frac{\pi q}{2}\left\langle |t|^q, \hat{\phi}(t\theta)\right\rangle,$$

where we used the formula for the Fourier transform of $|z|^{-1-q}$; see [**K9**, p. 38]. Therefore,

$$\langle A_{D,\xi}^{(q)}(0), \phi(\xi)\rangle = \frac{\cos(\pi q/2)}{2\pi(n-q-1)}\int_{S^{n-1}} \|\theta\|_D^{-n+q+1}\left(\int_{\mathbb{R}} |t|^q\hat{\phi}(t\theta)dt\right)d\theta.$$

On the other hand

$$\langle \left(\|x\|_D^{-n+q+1}\right)^\wedge, \phi\rangle = \int_{\mathbb{R}^n} \|x\|_D^{-n+q+1}\hat{\phi}(x)dx$$
$$= \int_{S^{n-1}} \|\theta\|_D^{-n+q+1}\left(\int_0^\infty |t|^q\hat{\phi}(t\theta)dt\right)d\theta.$$

Hence, assuming that $\Re q \in (-1, 0)$, we obtain

$$\langle A_{D,\xi}^{(q)}(0), \phi(\xi)\rangle = \frac{\cos(\pi q/2)}{\pi(n-q-1)}\langle \left(\|x\|_D^{-n+q+1}\right)^\wedge(\xi), \phi(\xi)\rangle,$$

for all even ϕ. For other values of q the result follows by analytic continuation.

□

Let D be an origin-symmetric convex infinitely smooth body in $\mathbb{R}^n$. For the following particular values of k the formulas of Theorem 2.7 turn into

- $k = 0$
$$\left(\|x\|_D^{-n+1}\right)^\wedge(\xi) = \pi(n-1)A_{D,\xi}(0),$$
- $k = 1$
$$\left(\|x\|_D^{-n+2}\right)^\wedge(\xi) = -2(n-2)\int_0^\infty \frac{A_{D,\xi}(z) - A_{D,\xi}(0)}{z^2}dz,$$
- $k = 2$
$$\left(\|x\|_D^{-n+3}\right)^\wedge(\xi) = -\pi(n-3)A''_{D,\xi}(0).$$

Brunn's theorem (see, for example, [**K9**, p. 18]) implies that the central section has maximal volume among all hyperplane sections perpendicular to a given direction, i.e. for every $\xi \in S^{n-1}$,

$$A_{D,\xi}(0) \geq A_{D,\xi}(z)$$

for all z. In particular, $A''_{D,\xi}(0) \leq 0$ for every $\xi \in S^{n-1}$.

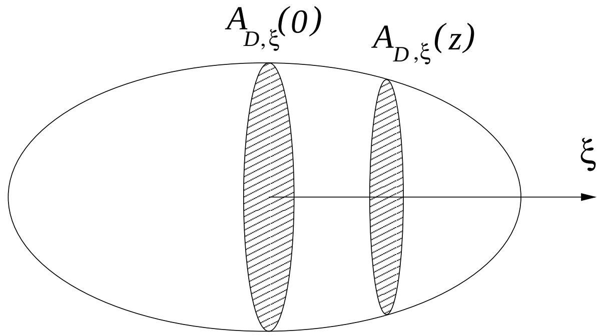

Figure 7. Central section.

In view of the formulas above $\|x\|_D^{-n+1}$, $\|x\|_D^{-n+2}$, $\|x\|_D^{-n+3}$ are positive definite distributions. Recall that a distribution f is called *positive definite* if its Fourier transform is a positive distribution, in the sense that $\langle \hat{f}, \phi\rangle \geq 0$ for any non-negative test function ϕ. Approximating any origin-symmetric convex body by infinitely smooth convex bodies, we get

Corollary 2.8. *For any origin-symmetric convex body D in $\mathbb{R}^n$, the functions $\|x\|_D^{-n+1}$, $\|x\|_D^{-n+2}$, $\|x\|_D^{-n+3}$ represent positive definite distributions.*

Using the same ideas one can generalize the result of Theorem 2.7 to sections of arbitrary dimension. Let H be an $(n-k)$-dimensional subspace of $\mathbb{R}^n$, $1 \leq k < n$. Fix an orthonormal basis ξ_1, ξ_2, ..., ξ_k in $H^\perp$ and let $u \in \mathbb{R}^k$. The $(n-k)$-dimensional parallel section function of D is defined by

$$A_{D,H}(u) = \mathrm{Vol}_{n-k}(D \cap \{H + u_1\xi_1 + \cdots u_k\xi_k\}).$$

Theorem 2.9. *([**K8**]) Let D be an infinitely smooth origin-symmetric convex body in $\mathbb{R}^n$, $1 \leq k < n$. For every $(n-k)$-dimensional subspace H of $\mathbb{R}^n$ and every $m \in \mathbb{N} \cup \{0\}$, $m \neq (n-k)/2$,*

$$\Delta^m A_{D,H}(0) = \frac{(-1)^m}{2^k \pi^k (n-2m-k)} \int_{S^{n-1}\cap H^\perp} (\|x\|_D^{-n+2m+k})^\wedge(\theta) d\theta,$$

where Δ is the Laplacian in $\mathbb{R}^k$.

In particular, for $m = 0$ we have

$$A_{D,H}(0) = \mathrm{Vol}_{n-k}(D \cap H) = \frac{1}{n-k} \int_{S^{n-1}\cap H} \|x\|_D^{-n+k} dx =$$

$$= \frac{1}{2^k \pi^k (n-k)} \int_{S^{n-1}\cap H^\perp} (\|x\|_D^{-n+k})^\wedge(\theta) d\theta. \tag{2.3}$$

One can view the last formula as the spherical analog of the following well-known fact: for any even test function ϕ

$$(2\pi)^n \int_H \phi(x) dx = \int_{H^\perp} \hat{\phi}(x) dx. \tag{2.4}$$

There is also a spherical version of another well-known fact, Parseval's formula, which says that for any $f, g \in L_2(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \widehat{f}(x)\widehat{g}(x)dx = (2\pi)^n \int_{\mathbb{R}^n} f(x)g(x)dx.$$

This formula can be projected to the sphere as follows. Let K and L be origin-symmetric infinitely smooth star bodies in $\mathbb{R}^n$ and $0 < p < n$. Then

$$\int_{S^{n-1}} (\|x\|_K^{-p})^\wedge(\xi)(\|x\|_L^{-n+p})^\wedge(\xi)d\xi = (2\pi)^n \int_{S^{n-1}} \|x\|_K^{-p}\|x\|_L^{-n+p}dx. \tag{2.5}$$

For the proof see [**K9**, Section 3.4]. Note that the projection to the sphere is possible, because the functions under the integral are homogeneous with degrees of homogeneity adding up to $-n$.