CHAPTER 7

Application of Malliavin calculus to diffusion processes

In this chapter we discuss the existence and smoothness of the density for solutions to stochastic differential equations. Suppose that $(\Omega, \mathcal{F}, P)$ is the canonical probability space associated with a d-dimensional Brownian motion $\{W^i(t), t \in [0,T],\ 1 \le i \le d\}$ on a finite interval $[0,T]$. This means $\Omega = C_0([0,T];\mathbb{R}^d)$, P is the law of the d-dimensional Brownian motion, which is called the *Wiener measure*, and $\mathcal{F}$ is the completion of the Borel σ-field of Ω with respect to P.

The underlying Hilbert space here is $H = L^2([0,T];\mathbb{R}^d)$, and for any $h \in H$, $W(h)$ is the Wiener integral

$$W(h) = \sum_{i=1}^d \int_0^T h_s^i dW_s^i.$$

Notice that we can write $H = L^2(T, \mathcal{B}, \mu)$, where $T = \mathbb{R} \times \{1, \dots, d\}$ and μ is the product of the Lebesgue measure times the counting measure on $\{1, \dots, d\}$. Then the derivative DF of a random variable $F \in \mathbb{D}^{1,2}$ will be a d-dimensional process that we denote by $\{D_t^i F, t \in [0,T], i = 1, \dots, d\}$. For example

$$D_s^i W_t^j = \delta_{i,j} \mathbf{1}_{[0,t]}(s).$$

The Hilbert space $H = L^2([0,T];\mathbb{R}^d)$ corresponds to the white-noise interpretation of the Brownian motion. However, it is possible to choose another Hilbert space, the *Cameron-Martin* space H^1, isometric to H, defined as the set of functions $h : [0,T] \to \mathbb{R}^d$ such that

$$h(t) = \int_0^t \dot{h}_s ds,$$

and $\dot{h} \in L^2([0,T];\mathbb{R}^d)$, equipped with the scalar product

$$\langle h, g \rangle_{H^1} = \langle \dot{h}, \dot{g} \rangle_H.$$

Notice that H^1 is densely embedded into Ω. In this way (Ω, H^1, P), is an *Abstract Wiener space* in the sense of Gross [**15**]. In this case, for any $h \in H^1$ we write

$$W(h) = \sum_{i=1}^d \int_0^T \dot{h}_s^i dW_s^i.$$

Consider the m-dimensional stochastic differential equation:

$$X_t = x_0 + \sum_{j=1}^d \int_0^t A_j(X_s) dW_s^j + \int_0^t B(X_s) ds, \tag{7.1}$$

where $A_j, B : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$, $1 \le j \le d$ are measurable functions.

We know that under the Lipschitz condition

$$\max\left(|A_j(x)-A_j(y)|,|B(x)-B(y)|\right)\leq K|x-y|, \tag{7.2}$$

for all $x,y\in\mathbb{R}^m$ there exists a unique solution $X=\{X_t,\ t\in[0,T]\}$ of Equation (7.1).

7.1. Differentiability of the solution

The next proposition provides the differentiability of the solution.

PROPOSITION 7.1. *Suppose that the coefficients A_j, B are continuously differentiable and satisfy (7.2). Then, for all $t\in[0,T]$ and for all $i=1,\dots,m$ the random variable X_t^i belongs to the space $\mathbb{D}^{1,\infty}:=\cap_{p\geq 2}\mathbb{D}^{1,p}$ and the derivative $D_r^jX_t^i$ satisfies the following linear stochastic differential equation for $r\leq t$*

$$D_r^jX_t=A_j(X_r)+\sum_{k=1}^m\sum_{l=1}^d\int_r^t\partial_kA_l(X_s)D_r^jX_s^kdW_s^l+\sum_{k=1}^m\int_r^t\partial_kB(X_s)D_r^jX_s^kds. \tag{7.3}$$

Furthermore, if the coefficients are infinitely differentiable in the space variable and their partial derivatives of all orders are uniformly bounded, then $X^i(t)$ belongs to $\mathbb{D}^\infty$.

PROOF. Consider the Picard approximations given by

$$\begin{aligned} X_t^{(0)} &= x_0, \\ X_t^{(n+1)} &= x_0+\sum_{j=1}^d\int_0^tA_j(X_s^{(n)})dW_s^j+\int_0^tB(X_s^{(n)})ds \end{aligned} \tag{7.4}$$

if $n\geq 0$. We will prove the following property by induction on n:

(P) $X_t^{(n),i}\in\mathbb{D}^{1,\infty}$ for all $i=1,\dots,m$, $n\geq 0$, and $t\in[0,T]$; furthermore, for all $p>1$ we have

$$\psi_n(t):=\sup_{0\leq r\leq t}E\left(\sup_{s\in[r,t]}|D_rX_s^{(n)}|^p\right)<\infty \tag{7.5}$$

and

$$\psi_{n+1}(t)\leq c_1+c_2\int_0^t\psi_n(s)ds, \tag{7.6}$$

for some constants c_1,c_2.

Clearly, (P) holds for $n=0$. Suppose it is true for n. Applying Proposition 1.10 we get

$$D_r[A_j(X_s^{(n)})]=\sum_{k=1}^m\partial_kA_j(X_s^{(n)})D_rX_s^{(n),k}\mathbf{1}_{\{r\leq s\}}$$

and

$$D_r[B(X_s^{(n)})]=\sum_{k=1}^m\partial_kB(X_s^{(n)})D_rX_s^{(n),k}\mathbf{1}_{\{r\leq s\}}.$$

Thus the processes $\{D_r[A_j(X_s^{(n)})],s\geq r\}$ and $\{D_r[B(X_s^{(n)})],s\geq r\}$ are adapted and satisfy

$$|D_r[A_j(X_s^{(n)})]|\leq K|D_rX_s^{(n)}|,\qquad |D_r[B(X_s^{(n)})]|\leq K|D_rX_s^{(n)}|.$$

Using Proposition 2.4 we deduce that for each $i = 1, \ldots, m$, the Itô integral $\int_0^t A_j^i(X_s^{(n)})dW_s^j$ belongs to the space $\mathbb{D}^{1,2}$, and for $r \leq t$ and $l = 1, \ldots, d$ we have

$$(7.7) \qquad D_r^l[\int_0^t A_j^i(X_s^{(n)})dW_s^j] = \delta_{l,j} A_l^i(X_r^{(n)}) + \int_r^t D_r^l[A_j^i(X_s^{(n)})]dW_s^j.$$

On the other hand, $\int_0^t B^i(X_s^{(n)})ds \in \mathbb{D}^{1,2}$, and for $r \leq t$ we have

$$(7.8) \qquad D_r^l[\int_0^t B^i(X_s^{(n)})ds] = \int_r^t D_r^l[B^i(X_s^{(n)})]ds.$$

From these equalities and Equation (7.5) we see that $X_t^{(n+1),i} \in \mathbb{D}^{1,\infty}$ for all $t \in [0,T]$, and we obtain

$$(7.9) \qquad E\left(\sup_{r\leq s\leq t} |D_r^j X_s^{(n+1)}|^p\right) \leq c_p \left[\gamma_p + T^{p-1}K^p \int_r^t E\left(|D_r^j X_s^{(n)}|^p\right) ds\right],$$

where

$$\gamma_p = \sup_{n,j} E(\sup_{0\leq t\leq T} |A_j(X_t^{(n)})|^p) < \infty.$$

So (7.5) and (7.6) hold for $n+1$. We know that

$$E\left(\sup_{s\leq T} |X_s^{(n)} - X_s|^p\right) \longrightarrow 0$$

as n tends to infinity. By Gronwall's lemma applied to (7.6) we deduce that the derivatives of the sequence $X_t^{(n),i}$ are bounded in $L^p(\Omega; H)$ uniformly in n for all $p \geq 2$. Therefore, from Lemma 4.6 we deduce that the random variables X_t^i belong to $\mathbb{D}^{1,\infty}$. Finally, applying the operator D to Equation (7.1) and using propositions 1.10 and 2.4, we deduce the linear stochastic differential equation (7.3) for the derivative of X_t^i.

The proof that X_t^i belongs to $\mathbb{D}^\infty$ if the coefficients are smooth follows by an induction argument. We omit the proof (see [**35**, Theorem 2.2.2]). □

Remark 7.2. If the coefficients are just Lipschitz (satisfy (7.2)), then we still have $X_t^i \in \mathbb{D}^{1,\infty}$, and Equation (7.3) holds with $\partial_k A_\alpha(X_s)$, and $\partial_k B(X_s)$ replaced by some bounded and adapted processes. The proof of this extension follows in the same way as before, but applying Proposition 1.14.

We are going to deduce a simpler expression for the derivative DX_t^i. Consider the $m \times m$ matrix-valued process defined by

$$Y_t = I + \sum_{l=1}^d \int_0^t \partial A_l(X_s) Y_s dW_s^l + \int_0^t \partial B(X_s) Y_s ds,$$

where ∂A_l denotes the $m \times m$ Jacobian matrix of the function A_l, that is,

$$(\partial A_l)_j^i = \partial_j A_l^i.$$

In the same way, ∂B denotes the $m \times m$ Jacobian matrix of B. If the coefficients of Equation (7.1) are of class $C^{1+\alpha}$, $\alpha > 0$ (see Kunita [**21**]), then there is a version of the solution $X_t(x_0)$ to this equation that is continuously differentiable in x_0, and Y_t is the Jacobian matrix $\frac{\partial X_t}{\partial x_0}$.

We claim that for any $t \in [0,T]$ the matrix Y_t is invertible. This can be proved by exhibiting a stochastic differential equation satisfied by the inverse matrix. Consider the $m \times m$ matrix-valued process Z_t solution to the equation

$$Z_t = I - \sum_{l=1}^{d} \int_0^t Z_s \partial A_l(X_s) dW_s^l - \int_0^t Z_s \left[\partial B(X_s) - \sum_{l=1}^{d} \partial A_l(X_s) \partial A_l(X_s) \right] ds.$$

Then, by means of Itô's formula, one can check that $Z_t Y_t = Y_t Z_t = I$, which implies that $Z_t = Y_t^{-1}$. In fact,

$$\begin{aligned} Z_t Y_t &= I + \sum_{l=1}^{d} \int_0^t Z_s \; \partial A_l(X_s) Y_s dW_s^l + \int_0^t Z_s \partial B(X_s) Y_s ds - \\ & - \sum_{l=1}^{d} \int_0^t Z_s \partial A_l(X_s) Y_s dW_s^l \\ & - \int_0^t Z_s \left[\partial B(X_s) - \sum_{l=1}^{d} \partial A_l(X_s) \partial A_l(X_s) \right] Y_s ds \\ & - \int_0^t Z_s \left(\sum_{l=1}^{d} \partial A_l(X_s) \partial A_l(X_s) \right) Y_s ds = I, \end{aligned}$$

and similarly we show that $Y_t Z_t = I$.

Then, the $m \times m$ matrix $(D_r X_t)_j^i = D_r^j X_t^i$ can be expressed as

$$D_r X_t = Y_t Y_r^{-1} A(X_r). \tag{7.10}$$

Indeed, it is enough to verify that the process $\{Y_t Y_r^{-1} A(X_r), t \geq r\}$ satisfies Equation (7.3):

$$\begin{aligned} & A(X_r) + \int_r^t \partial A_l(X_s) \left\{ Y_s Y_r^{-1} A(Xs) \right\} dW_s^l + \int_r^t \partial B(X_s) \left\{ Y_s Y_r^{-1} A(Xs) \right\} ds \\ = \; & A(X_r) + [Y_t - Y_r] \, Y_r^{-1} A(X_r) = Y_t Y_r^{-1} A(X_r). \end{aligned}$$

Equation (7.10) leads to the following expression for the Malliavin matrix of X_t

$$Q_t = Y_t C_t Y_t^T, \tag{7.11}$$

where

$$C_t = \sum_{l=1}^{d} \int_0^t Y_s^{-1} A_l(X_s) A_l^T(X_s) \left(Y_s^{-1} \right)^T ds = \int_0^t Y_s^{-1} \sigma(X_s) \left(Y_s^{-1} \right)^T ds,$$

and σ is the $m \times m$ diffusion matrix $\sigma = \sum_{k=1}^{d} A_k A_k^T$. Taking into account that Y_t is invertible, the nondegeneracy of the matrix Q_t will depend only on the nondegeneracy of the matrix C_t, which is usually called the *reduced Malliavin matrix*.

7.2. Existence of densities under ellipticity conditions

Consider the stopping time defined by

$$S = \inf\{t > 0 : \det \sigma(X_t) \neq 0\} \wedge T.$$

Notice that if $\det \sigma(x_0) \neq 0$, then $S = 0$. The following absolute continuity result has been established by Bouleau and Hirsch in [**8**].

THEOREM 7.3. *Let $\{X(t), t \in [0,T]\}$ be the solution of the stochastic differential equation (7.1), where the coefficients satisfy (7.2). Then for any $0 < t \leq T$ the law of $X(t)$ conditioned by $\{t > S\}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^m$.*

PROOF. Taking into account Theorem 5.1, it suffices to show that $\det Q_t > 0$ a.s. on the set $\{t > S\}$, where Q_t is the Malliavin matrix of X_t. In view of expression (7.11) it is sufficient to prove that $\det C_t > 0$ a.s. on this set. Suppose $t > S$. For any $u \in \mathbb{R}^m$ we can write

$$\begin{aligned} u^T C_t u &= \int_0^t u^T Y_s^{-1} \sigma(X_s)(Y_s^{-1})^T u ds \\ &\geq \int_0^t \inf_{v\in\mathbb{R}^m} \left(\frac{v^T \sigma(X_s) v}{\|v\|^2} \right) \|(Y_s^{-1})^T u\|^2 ds. \end{aligned}$$

Notice that $\inf_{v\in\mathbb{R}^m} \left(\frac{v^T\sigma(X_s)v}{\|v\|^2} \right)$ is the smallest eigenvalue of $\sigma(X_s)$ which is strictly positive in an open interval contained in $[0,t]$ by the definition of the stopping time S and because $t > S$. On the other hand, $\|(Y_s^{-1})^T u\| \geq \|u\| \|Y_s\|$. Therefore we obtain

$$u^T C_t u \geq k \|u\|^2,$$

for some positive constant $k > 0$, which implies that the matrix C_t is invertible. This completes the proof. $\square$

7.3. Regularity of the density under Hörmander's conditions

In this section we assume that the coefficients of Equation (7.1) are infinitely differentiable with bounded derivatives of all orders. Let us denote by $X = \{X_t, t \geq 0\}$ the solution of this equation on $[0, \infty)$. By Proposition 7.1 the random variables X_t^i belong to the space $\mathbb{D}^\infty$. We are going to impose nondegeneracy conditions on the coefficients in such a way that the solution has a smooth density. To introduce these conditions, consider the following vector fields on $\mathbb{R}^m$ associated with the coefficients of Equation (7.1):

$$\begin{aligned} A_j &= \sum_{i=1}^m A_j^i(x) \frac{\partial}{\partial x_i}, \quad j = 1, \ldots, d, \\ B &= \sum_{i=1}^m B^i(x) \frac{\partial}{\partial x_i}. \end{aligned}$$

The covariant derivative of A_k in the direction of A_j is defined as the vector field $A_j^\nabla A_k = \sum_{i,l=1}^m A_j^l \partial_l A_k^i \frac{\partial}{\partial x_i}$, and the Lie bracket between the vector fields A_j and A_k is defined by

$$[A_j, A_k] = A_j^\nabla A_k - A_k^\nabla A_j.$$

Notice that the Lie bracket $[A_j, A_k]$ coincides with the difference of compositions $A_j A_k - A_k A_j$. Set

$$A_0 = B - \frac{1}{2} \sum_{l=1}^d A_l^\nabla A_l.$$

The vector field A_0 appears when we write the stochastic differential equation (7.1) in terms of the Stratonovich integral instead of the Itô integral:

$$X_t = x_0 + \sum_{j=1}^{d} \int_0^t A_j(X_s) \circ dW_s^j + \int_0^t A_0(X_s) ds.$$

Hörmander's condition can be stated as follows:

(H) The vector space spanned by the vector fields

$$A_1, \dots, A_d, \quad [A_i, A_j], 0 \le i \le d, 1 \le j \le d, \quad [A_i, [A_j, A_k]], 0 \le i, j, k \le d, \dots$$

at point x_0 is $\mathbb{R}^m$.

For instance, if $m = d = 1$, $A_1^1(x) = a(x)$, and $A_0^1(x) = b(x)$, then Hörmander's condition means that $a(x_0) \neq 0$ or $a^n(x_0)b(x_0) \neq 0$ for some $n \ge 1$. In this situation we have the following result.

THEOREM 7.4. *Assume that Hörmander's condition (H) holds. Then for any $t > 0$ the random vector X_t has an infinitely differentiable density.*

This result can be considered as a probabilistic version of Hörmander's theorem on the hypoellipticity of second-order differential operators.

For the proof of this theorem we need the following two lemmas.

LEMMA 7.5. *Let $\{Z_t, t \ge 0\}$ be a real-valued, adapted continuous process such that $Z_0 = z_0 \neq 0$. Suppose that there exists $\alpha > 0$ such that for all $p \ge 1$ and $t \in [0, T]$,*

$$E\left(\sup_{0 \le s \le t} |Z_s - z_0|^p \right) \le C_{p,T} t^{p\alpha}.$$

Then, for all $p \ge 1$ and $t \in (0, T]$,

$$E\left[\left(\int_0^t |Z_s| ds \right)^{-p} \right] < \infty.$$

PROOF. For any $0 < \epsilon < \frac{t|z_0|}{2}$ we have

$$\begin{aligned} P\left(\int_0^t |Z_s| ds < \epsilon \right) &\le P\left(\int_0^{2\epsilon/|z_0|} |Z_s| ds < \epsilon \right) \\ &\le P\left(\sup_{0 \le s \le 2\epsilon/|z_0|} |Z_s - z_0| > 2|z_0| \right) \\ &\le \frac{C_{p,T}}{2^p |z_0|^p} \left(\frac{2\epsilon}{|z_0|} \right)^{p\alpha}, \end{aligned}$$

which implies the desired result. □

LEMMA 7.6 (Norris, [**31**]). *Consider a continuous semimartingale of the form*

$$Y_t = y + \int_0^t a(s) ds + \sum_{i=1}^{d} \int_0^t u_i(s) dW_s^i,$$

where

$$a(t) = \alpha + \int_0^t \beta(s) ds + \sum_{i=1}^{d} \int_0^t \gamma_i(s) dW_s^i.$$

Suppose that the processes a, u_i, β, γ_i are adapted and satisfy

$$c = E\left(\sup_{0\le t\le T}(|\beta(t)|+|\gamma(t)|+|a(t)|+|u(t)|)^p\right)<\infty.$$

Fix $q>8$. Then, for all $r,\nu>0$ such that $18r+9\nu<q-8$ there exists ϵ_0 such that for all $\epsilon\le\epsilon_0$ we have

$$P\left(\int_0^T Y_t^2dt<\epsilon^q, \int_0^T(|a(t)|^2+|u(t)|^2)dt\ge\epsilon\right)\le c_1\epsilon^{rp}+e^{-\epsilon^{-\nu}}.$$

Sketch of the proof of Theorem 7.4. The proof will be done in several steps.

Step 1 We want to show that for all $t>0$ and all $p\ge 2$, $E[(\det Q_t)^{-p}]<\infty$, where Q_t is the Malliavin matrix of X_t. Taking into account Equation (7.11) and the fact that $E\left(\left|\det Y_t^{-1}\right|^p+|\det Y_t|^p\right)<\infty$, it suffices to show that $E[(\det C_t)^{-p}]<\infty$ for all $p\ge 2$.

Step 2 Fix $t>0$. Then the problem is reduced to show that for all $p\ge 2$ we have

$$\sup_{|v|=1}P\{v^TC_tv\le\epsilon\}\le\epsilon^p$$

for any $\epsilon\le\epsilon_0(p)$, where the quadratic form associated to the matrix C_t is given by

$$v^TC_tv=\sum_{j=1}^d\int_0^t\langle v, Y_s^{-1}A_j(X_s)\rangle^2ds. \tag{7.12}$$

Step 3 Fix a smooth function V and use Itô's formula to compute the differential of $Y_t^{-1}V(X_t)$

$$\begin{aligned} d\left(Y_t^{-1}V(X_t)\right) &= Y_t^{-1}\sum_{k=1}^d[A_k,V](X_t)dW_t^k \\ &\quad +Y_t^{-1}\left\{[A_0,V]+\frac{1}{2}\sum_{k=1}^d[A_k,[A_k,V]]\right\}(X_t)dt. \end{aligned}\tag{7.13}$$

Step 4 We introduce the following sets of vector fields:

$$\begin{aligned}\Sigma_0 &= \{A_1,\dots,A_d\},\\ \Sigma_n &= \{[A_k,V], k=0,\dots,d, V\in\Sigma_{n-1}\}\quad\text{if } n\ge 1,\\ \Sigma &= \cup_{n=0}^\infty\Sigma_n,\end{aligned}$$

and

$$\begin{aligned}\Sigma_0' &= \Sigma_0,\\ \Sigma_n' &= \{[A_k,V], k=0,\dots,d, V\in\Sigma_{n-1}';\\ &\qquad [A_0,V]+\frac{1}{2}\sum_{j=1}^d[A_j,[A_j,V]], V\in\Sigma_{n-1}'\}\quad\text{if } n\ge 1,\\ \Sigma' &= \cup_{n=0}^\infty\Sigma_n'.\end{aligned}$$

We denote by $\Sigma_n(x)$ (resp. $\Sigma_n'(x)$) the subset of $\mathbb{R}^m$ obtained by freezing the variable x in the vector fields of Σ_n (resp. Σ_n'). Clearly, the vector spaces spanned by $\Sigma(x_0)$ or by $\Sigma'(x_0)$ coincide, and under Hörmander's condition this vector space is $\mathbb{R}^m$. Therefore, there exists an integer $j_0 \geq 0$ such that the linear span of the set of vector fields $\bigcup_{j=0}^{j_0} \Sigma'_j(x)$ at point x_0 has dimension m. As a consequence there exist constants $R > 0$ and $c > 0$ such that

$$\sum_{j=0}^{j_0} \sum_{V \in \Sigma'_j} \langle v, V(y) \rangle^2 \geq c, \tag{7.14}$$

for all v and y with $|v| = 1$ and $|y - x_0| < R$.

Step 5 For any $j = 0, 1, \ldots, j_0$ we put $m(j) = 2^{-4j}$ and we define the set

$$E_j = \left\{ \sum_{V \in \Sigma'_j} \int_0^t \langle v, Y_s^{-1} V(X_s) \rangle^2 ds \leq \epsilon^{m(j)} \right\}.$$

Notice that $\{v^T C_t v \leq \epsilon\} = E_0$ because $m(0) = 1$. Consider the decomposition

$$E_0 \subset (E_0 \cap E_1^c) \cup (E_1 \cap E_2^c) \cup \cdots \cup (E_{j_0 - 1} \cap E_{j_0}^c) \cup F,$$

where $F = E_0 \cap E_1 \cap \cdots \cap E_{j_0}$. Then for any unit vector v we have

$$P\{v^T C_t v \leq \epsilon\} = P(E_0) \leq P(F) + \sum_{j=0}^{j_0} P(E_j \cap E_{j+1}^c).$$

We are going to estimate each term of this sum.

Step 6 Let us first estimate $P(F)$. By the definition of F we obtain

$$P(F) \leq P\left(\sum_{j=0}^{j_0} \sum_{V \in \Sigma'_j} \int_0^t \langle v, Y_s^{-1} V(X_s) \rangle^2 ds \leq (j_0 + 1)\epsilon^{m(j_0)} \right).$$

Then, taking into account (7.14) we can apply Lemma 7.5 to the process

$$Z_s = \sum_{j=0}^{j_0} \sum_{V \in \Sigma'_j} \langle v, Y_s^{-1} V(X_s) \rangle^2,$$

we obtain

$$E\left(\left| \sum_{j=0}^{j_0} \sum_{V \in \Sigma'_j} \int_0^t \langle v, Y_s^{-1} V(X_s) \rangle^2 ds \right|^{-p} \right) < \infty.$$

Therefore, for any $p \geq 1$ the exists ϵ_0 such that

$$P(F) \leq \epsilon^p$$

for any $\epsilon < \epsilon_0$.

Step 7 For any $j = 0, \ldots, j_0$ the probability of the event $E_j \cap E_{j+1}^c$ is bounded by the sum with respect to $V \in \Sigma'_j$ of the probability that the two following event happens

$$\int_0^t \langle v, Y_s^{-1} V(X_s) \rangle^2 ds \leq \epsilon^{m(j)}$$

and

$$\begin{aligned} &\sum_{k=1}^{d}\int_0^t \langle v, Y_s^{-1}[A_k,V](X_s)\rangle^2 ds \\ &+\int_0^t \left\langle v, Y_s^{-1}\left([A_0,V]+\frac{1}{2}\sum_{j=1}^{d}[A_j,[A_j,V]]\right)(X_s)\right\rangle^2 ds \\ > \quad &\frac{\epsilon^{m(j+1)}}{n(j)}, \end{aligned}$$

where $n(j)$ denotes the cardinality of the set Σ_j'. Consider the continuous semimartingale $\{\langle v, Y_s^{-1}V(X_s)\rangle, s\geq 0\}$. From (7.13) we see that the quadratic variation of this semimartingale is equal to

$$\sum_{k=1}^{d}\int_0^s \langle v, Y_\sigma^{-1}[A_k,V](X_\sigma)\rangle^2 d\sigma,$$

and the bounded variation component is

$$\int_0^s \left\langle v, Y_\sigma^{-1}\left\{[A_0,V]+\frac{1}{2}\sum_{j=1}^{d}[A_j,[A_j,V]]\right\}(X_\sigma)\right\rangle d\sigma.$$

Taking into account that $8m(j+1)<m(j)$, from Lemma 7.6 applied to the semimartingale $Y_s = v^T Y_s^{-1}V(X_s)$ we get that for any $p\geq 1$ there exists $\epsilon_0>0$ such that

$$P(E_j\cap E_{j+1}^c)\leq \epsilon^p,$$

for all $\epsilon\leq\epsilon_0$. The proof of the theorem is now complete. □

REMARK 7.7. Formula (7.13) explains the role played by the Lie brackets in Hörmander's condition, and it leads immediately to the existence of the density under Hörmander's condition. In fact, from (7.12) and (7.13) we deduce that on the set $\{v^T C_t v = 0\}$, where $|v|=1$, we have $v^T Y_s^{-1}A_j(X_s)=0$, for all $j=1,\ldots,d$ and for all $s\in[0,t]$. Taking $s=0$ we get $v^T A_j(x_0)=0$. Then, if a continuous semimartingale vanishes in some set of positive probability, the quadratic variation and the bounded variation part of the semimartingale must vanish on this set. As a consequence, we obtain

$$\begin{aligned} v^T Y_s^{-1}[A_l,A_j](X_s) &= 0, \\ v^T Y_s^{-1}\left([A_0,A_j]+\frac{1}{2}\sum_{k=1}^{d}[A_k,[A_k,A_j]]\right)(X_s) &= 0, \end{aligned}$$

for all $s\in[0,t]$ and $l,j=1,\ldots,d$. Taking $s=0$, and using a recursive argument we obtain that $v^T V(x_0)=0$, for any vector field on the set $\Sigma'(x_0)$. So, this is also true for any $V\in\Sigma(x_0)$, and from this we can show that $\det C_t>0$, almost surely, under Hormander's condition.

EXAMPLE 7.8. Consider the following example.

$$\begin{aligned} dX_t^1 &= dW_t^1 + \sin X_t^2 dW_t^2, \\ dX_t^2 &= 2X_t^1 dW_t^1 + X_t^1 dW_t^2 \end{aligned}$$

with initial condition $x_0 = 0$. In this case the diffusion matrix

$$\sigma(x) = \begin{bmatrix} 1 + \sin^2 x_2 & x_1(2 + \sin x_2) \\ x_1(2 + \sin x_2) & 5x_1^2 \end{bmatrix}$$

degenerates along the line $x_1 = 0$. The Lie bracket $[A_1, A_2]$ is equal to $\begin{bmatrix} 2x_1 \cos x_2 \\ 1 - 2\sin x_2 \end{bmatrix}$. Therefore, the vector fields A_1 and $[A_1, A_2]$ at $x = 0$ span $\mathbb{R}^2$ and Hörmander's condition holds. So from Theorem 7.4 X_t has a C^∞ density for any $t > 0$.

CHAPTER 9

Central limit theorems and Malliavin Calculus

Suppose that $W = \{W(h), h \in H\}$ is a Gaussian family associated with a Hilbert space H. In this chapter we present a central limit theorem for random variables belonging to a fixed Wiener chaos. To simplify the proofs we will assume that $H = L^2(T, \mathcal{B}, \mu)$, $(T, \mathcal{B})$ is a measurable space, and μ is a σ-finite and non-atomic measure.

9.1. Central limit theorems via chaos expansions

Consider two symmetric elements $f \in L^2(T^n)$, and $g \in L^2(T^m)$. For any $r = 0, ..., n \wedge m$, we define the *contraction* of f and g of order r to be the element of $L^2(T^{n+m-2r})$ defined by

$$\begin{aligned} &(f \otimes_r g)(t_1, \dots, t_{n-r}, s_1, \dots, s_{m-r}) \\ = &\int_{T^r} f(t_1, \dots, t_{n-r}, x_1, \dots, x_r) g(s_1, \dots, s_{m-r}, x_1, \dots, x_r) dx_1 \cdots dx_r. \end{aligned}$$

We denote by $f \widetilde{\otimes}_r g$ the symmetrization of the function $f \otimes_r g$.

We recall the following formula for the product of two multiple stochastic integrals (see Itô, [**17**]):

$$I_n(f) I_m(g) = \sum_{r=0}^{n \wedge m} r! \binom{n}{r} \binom{m}{r} I_{n+m-2r}(f \otimes_r g). \tag{9.1}$$

The next theorem is the main result of this chapter, and it has been proved in [**39**] and [**37**].

THEOREM 9.1. *Fix $n \geq 2$. Consider a sequence $\{F_k = I_n(f_k), k \geq 1\}$ of square integrable random variables belonging to the nth Wiener chaos such that*

$$E\left[F_k^2\right] = n! \left\| f_k \right\|_{H^{\otimes n}}^2 \underset{k \to \infty}{\to} \sigma^2. \tag{9.2}$$

The following statements are equivalent.

1. *As k goes to infinity, the sequence $\{F_k, k \geq 1\}$ converges in distribution to the normal law $N(0, \sigma^2)$.*
2. $\lim_{k \to \infty} E[F_k^4] = 3\sigma^4$.
3. *For all $1 \leq r \leq n-1$, $\lim_{k \to \infty} f_k \otimes_r f_k = 0$, in $H^{\otimes 2(n-r)}$.*
4. $\|DF_k\|_H^2 \to n\sigma^2$ *in* $L^2(\Omega)$.

We need the following technical lemma.

LEMMA 9.2. *Consider two random variables $F = I_n(f)$, $G = I_m(g)$, where $n, m \geq 1$. Then*

$$E\left[\langle DF, DG \rangle_H^2\right] = \sum_{r=1}^{n \wedge m} \frac{(n!m!)^2 (n+m-2r)!}{((n-r)!(m-r)!(r-1)!)^2} \left\| f \widetilde{\otimes}_r g \right\|_{H^{\otimes(n+m-2r)}}^2. \tag{9.3}$$

PROOF. Equation (1.20) implies that

$$D_tF = nI_{n-1}(f(\cdot,t)), D_tG = mI_{m-1}(g(\cdot,t))$$

and, as a consequence,

$$\langle DF, DG\rangle_H^2 = nm\int_T I_{n-1}(f(\cdot,t))\, I_{m-1}(g(\cdot,t))\,\mu(dt).$$

Thanks to the product formula for multiple stochastic integrals (see (9.1)), one obtains

$$\langle DF, DG\rangle_H^2 = nm\int_T \sum_{r=0}^{n\wedge m-1} r!\binom{n-1}{r}\binom{m-1}{r} I_{n+m-2-2r}(f(\cdot,t)\otimes_r g(\cdot,t))\,\mu(dt).$$

Taking into account the orthogonality between multiple stochastic integrals of different order, we have

$$\begin{aligned} & E[\langle DF, DG\rangle_H^2] \\ = \; & n^2m^2\sum_{r=0}^{n\wedge m-1}(r!)^2\binom{n-1}{r}^2\binom{m-1}{r}^2(n+m-2r)! \\ & \times\int_{T^2}\langle f(\cdot,t)\,\widetilde{\otimes}_r g(\cdot,t), f(\cdot,s)\,\widetilde{\otimes}_r g(\cdot,s)\rangle_{H^{\otimes(n+m-2-2r)}}\,\mu(dt)\,\mu(ds). \end{aligned}$$

Notice that

$$\int_T f(\cdot,t)\otimes_r g(\cdot,t)\,\mu(dt) = f\otimes_{r+1} g,$$

and, as a consequence,

$$\int_T f(\cdot,t)\,\widetilde{\otimes}_r g(\cdot,t)\,\mu(dt) = f\widetilde{\otimes}_{r+1} g.$$

Therefore,

$$\begin{aligned} E[\langle DF, DG\rangle_H^2] \;=\; & n^2m^2\sum_{r=0}^{n\wedge m-1}(r!)^2\binom{n-1}{r}^2\binom{m-1}{r}^2 \\ & \times(n+m-2r)!\left\|f\widetilde{\otimes}_{r+1}g\right\|^2_{H^{\otimes(n+m-2-2r)}}, \end{aligned}$$

which implies the desired result. □

PROOF OF THEOREM 9.1. To simplify we assume $\sigma^2 = 1$. We will prove the following implications

$$(4) \Rightarrow (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4).$$

$(4) \Rightarrow (1)$ The sequence of random variables F_k is tight because it is bounded in $L^2(\Omega)$ by condition (9.2). Then, by Prohorov's theorem we have that F_k is relatively compact, and it suffices to show that the limit of any subsequence converging in distribution is $N(0,1)$. Therefore, we can assume that F_k converges in law to G, and it suffices to show that G has law $N(0,1)$. By condition (9.2) $G \in L^2(\Omega)$. Therefore, the characteristic function $\varphi(t) = E[e^{itG}]$ is differentiable and $\varphi'(t) = iE[Ge^{itG}]$. For every $k \geq 1$, define $\varphi_k(t) = E[e^{itF_k}]$. We have $\varphi_k'(t) = iE[F_ke^{itF_k}]$.

Clearly, $F_ke^{itF_k}$ converges in law to Ge^{itG}, and the boundedness in $L^2(\Omega)$ implies convergence of the expectations. Hence, we obtain

$$\varphi_k'(t) \to \varphi'(t), \tag{9.4}$$

as k tends to infinity. On the other hand, using the definition of the operator L, Proposition 3.9, and the duality relationship (2.2) we have

$$\begin{aligned} E[F_k e^{itF_k}] &= -\frac{1}{n} E[LF_k e^{itF_k}] = -\frac{1}{n} E[-\delta D(F_k) e^{itF_k}] \\ &= \frac{1}{n} E[\langle DF_k, D\left(e^{itF_k}\right)\rangle_H] = \frac{it}{n} E[e^{itF_k} \left\|DF_k\right\|_H^2]. \end{aligned}$$

Therefore,

$$\varphi_k'(t) = -\frac{t}{n} E[e^{itF_k} \left\|DF_k\right\|_H^2],$$

and by hypothesis (4) we obtain

$$\varphi_k'(t) \to -t\varphi(t), \tag{9.5}$$

as k tends to infinity. From (9.4) and (9.5) we deduce that $\varphi(t)$ satisfies the following differential equation

$$\begin{aligned} \varphi'(t) &= -t\varphi(t) \\ \varphi(0) &= 1, \end{aligned}$$

which is the differential equation satisfied by the characteristic function of the $N(0,1)$.

(1) $\Rightarrow$ (2) This implication is immediate because for any $p > 2$, the hypercontractivity property of the Ornstein-Uhlembeck operator (see Equation (3.5)), implies

$$\sup_k \left\|F_k\right\|_p \le (p-1)^{\frac{n}{2}} \sup_k \left\|F_k\right\|_2 < \infty.$$

Therefore, the moments of order 3 converge to the moment of order 3 of the law $N(0,1)$, which is equal to 3.

(2) $\Rightarrow$ (3) Using the product formula for multiple stochastic integrals (see (9.1)) yields

$$\begin{aligned} I_n(f_k)^2 &= \sum_{r=0}^{n} r! \binom{n}{r}^2 I_{2(n-r)}\left(f_k \otimes_r f_k\right) \\ &= n! \left\|f_k\right\|^2 + I_{2n}(f_k \otimes f_k) + \sum_{r=1}^{n-1} r! \binom{n}{r}^2 I_{2(n-r)}\left(f_k \otimes_r f_k\right). \end{aligned}$$

As a consequence,

$$\begin{aligned} E\left[I_n(f_k)^4\right] &= (n!)^2 \left\|f_k\right\|_{H^{\otimes n}}^4 + (2n)! \left\|f_k \widetilde{\otimes} f_k\right\|_{H^{\otimes(2n)}}^2 \\ &\quad + \sum_{r=1}^{n-1} \left[r! \binom{n}{r}^2\right]^2 (2(n-r))! \left\|f_k \widetilde{\otimes}_r f_k\right\|_{H^{\otimes 2(n-r)}}^2. \end{aligned}$$

Notice that

$$\left\|f_k \widetilde{\otimes} f_k\right\|_{H^{\otimes(2n)}}^2 = \left(f_k \otimes f_k\right) \otimes_{2n} \left(f_k \widetilde{\otimes} f_k\right)$$

will be the sum of $\frac{(2n)!}{(n!)^2}$ terms, each of them of the form $\frac{(n!)^2}{(2n)!} \left\|f_k \otimes_a f_k\right\|_{H^{2(n-a)}}^2$, where $a = 0, \ldots, n$. For $a = 0, n$, we have

$$\left\|f_k \otimes_a f_k\right\|_{H^{\otimes 2(n-a)}}^2 = \left\|f_k\right\|_{H^{\otimes n}}^2.$$

Therefore, we have

$$E\left[I_n(f_k)^4\right] = 3\,(n!)^2 \left\|f_k\right\|_{H^{\otimes n}}^4 + R_k$$

where $R_k \geq 0$. From (9.2) and Hypothesis ii) we get $\lim_k R_k = 0$, which implies (3).

(3) $\Rightarrow$ (4) Taking into account the computations

$$\begin{aligned} E[(\|DF_k\|_H^2 - n)^2] &= E[\|DF_k\|_H^4] - 2nE[\|DF_k\|_H^2] + n^2 \\ &= E[\|DF_k\|_H^4] - 2n^2 n! \, \|f_k\|_{H^{\otimes n}}^2 + n^2, \end{aligned}$$

it is enough to prove that (3) implies $\lim_{k\to\infty} E[\|DF_k\|_H^4] = n^2$. Using Lemma 9.2 we obtain

$$E[\|DF_k\|_H^4] = \sum_{r=1}^{n-1} \frac{(n!)^4 (2(n-r))!}{((n-r)!)^4((r-1)!)^2} \|f_k \widetilde{\otimes}_r f_k\|_{H^{\otimes 2(n-r)}}^2 + n^2 (n!)^2 \|f_k\|_{H^{\otimes n}}^4.$$

It follows that $E[\|DF_k\|_H^4]$ converges to n^2 if and only if

$$\lim_{k\to\infty} \left\|f_k \widetilde{\otimes}_r f_k\right\|_{H^{\otimes 2(n-l)}}^2 = 0,$$

for $r = 1, \ldots, n-1$. As

$$\left\|f_k \widetilde{\otimes}_r f_k\right\|_{H^{\otimes 2(n-l)}}^2 \leq \|f_k \otimes_r f_k\|_{H^{\otimes 2(n-l)}}^2,$$

we conclude the proof. □

The equivalence of the properties (1)–(3) in Theorem 9.1 was proved by Nualart and Peccati in [**39**], using the Dambis-Dubins-Schwartz characterization of continuous martingales as a Brownian motion with a time change. In a subsequent paper, Peccati and Tudor [**43**] provide a multidimensional version of this characterization. In particular, they proved the following result: Given a sequence of random vectors

$$F_k = (F_k^1, \ldots, F_k^d) = (I_{n_1}(f_k^1), \ldots, I_{n_d}(f_k^d)),$$

where $1 \leq n_1 < \cdots < n_d$, whose components satisfy (9.2), then if one of the above equivalent conditions (1)–(4) holds true for each component, the sequence F_k converges in law to a random vector with law $N(0, I_d)$.

The following general central limit theorem has been proved in [**16**] Hu and Nualart.

THEOREM 9.3. *Let $F_k = \sum_{n=1}^{\infty} I_n(f_{n,k})$, $k \geq 1$, be a sequence of square integrable random variables. Suppose that*

(1) *For all $n \geq 1$, $\lim_{k\to\infty} n! \, \|f_{n,k}\|_{H^{\otimes n}}^2 = \sigma_n^2$.*
(2) *For all $n \geq 1$, and $1 \leq r \leq n-1$,$\lim_{k\to\infty} f_{n,k} \otimes_r f_{n,k} = 0$ in $H^{\otimes(n-2r)}$.*
(3) *$n! \, \|f_{n,k}\|_{H^{\otimes n}}^2 \leq \delta_n$, where $\sum_{n=1}^{\infty} \delta_n < \infty$.*

Then, as k tends to infinity, the sequence F_n converges in distribution to the law $N(0, \sigma^2)$, where $\sigma^2 = \sum_{n=1}^{\infty} \sigma_n^2$.

REMARK 9.4. Assuming condition (1), condition (2) is equivalent to

$$\lim_{k\to\infty} E(I_n(f_{n,k})^3) = 3\sigma_n^4,$$

or to $\lim_{k\to\infty} \|DI_n(f_{n,k})\|_H^2 = n\sigma_n^2$, for each $n \geq 2$.

The above theorem implies the convergence of the whole sequence $(I_n(f_{n,k}), n \geq 1)$ to an infinite dimensional Gaussian vector with independent components. We call this phenomenon *chaotic central limit theorem*,. Examples of this behavior are weak convergence of the renormalized self-intersection local time of the fractional

Brownian motion (cf. [**16**]), and the fluctuations of the centered p-variations of some Gaussian processes (see, for instance [**11**] and the references therein). As an example of the application of these theorems we will show the following simple result.

PROPOSITION 9.5. *Let B be a fractional Brownian motion with Hurst parameter $H < \frac{1}{2}$ and fix an odd integer $k \geq 1$. Then, the sequence*

$$Z_n = n^{kH-\frac{1}{2}} \sum_{j=1}^{k} \left(B_{j/n} - B_{(j-1)/n} \right)^k$$

converges in distribution, as k tends to infinity, to the law $N(0, \sigma^2)$, where

$$\sigma^2 = E(X_1^{2k}) + 2 \sum_{j=1}^{\infty} E\left[(X_1 X_{1+j})^k \right]$$

and $X_j = B_j - B_{j-1}$.

PROOF. By the scaling property of the fBm, the sequence Z_n has the same distribution as

$$Y_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (B_j - B_{j-1})^k = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_j^k. \tag{9.6}$$

The sequence $\{X_j, j \geq 1\}$ is Gaussian, centered, stationary with covariance

$$\rho_H(n) = E(X_1 X_{n+1}) = \frac{1}{2} \left[(n+1)^{2H} - 2n^{2H} + (n-1)^{2H} \right].$$

Notice that ρ_n^H behaves as $H(2H-1)n^{2H-2}$ as n tends to infinity. Consider the expansion of the power x^k into a finite sum of Hermite polynomials.

$$x^k = \sum_{m=1}^{k} c_m H_m(x). \tag{9.7}$$

From (9.6) and (9.7) we obtain the expansion of Y_n in the Wiener chaos:

$$Y_n = \sum_{m=1}^{k} \left(\frac{1}{\sqrt{n}} \sum_{j=1}^{n} c_m H_m(X_j) \right).$$

Then, in view of Theorem 9.3, the convergence in distribution of Y_k to the law $N(0, \sigma^2)$ will be a consequence of the following results for any $1 \leq m \leq k$ and for some constants σ_m^2

(1) $\lim_{n\to\infty} E((J_m Y_n)^2) = \sigma_m^2$,
(2) $\lim_{n\to\infty} \|DJ_m Y_n\|_{\mathcal{H}}^2 = m\sigma_m^2$ in L^2.

To show (1) we write

$$\begin{aligned} E((J_m Y_n)^2) &= \frac{c_m^2}{n} \sum_{j,l=1}^{n} E\left(H_m(X_j) H_m(X_l) \right) = \frac{c_m^2}{nm!} \sum_{j,l=1}^{n} \rho_H^m(j-l) \\ &\to \frac{c_m^2}{m!} \left(1 + 2 \sum_{j=1}^{\infty} \rho_H^m(j) \right) := \sigma_m^2. \end{aligned}$$

Notice that the series $\sum_{j=1}^{\infty} \rho_H^m(j)$ is convergent for $m = 1, \ldots, k$ because $H < \frac{1}{2}$.

To show (2) let us compute

$$\begin{aligned}\|DJ_mY_n\|_H^2 &= \frac{c_m^2}{n}\left\|\sum_{j=1}^n H_{m-1}(X_j)\mathbf{1}_{(j-1,j]}\right\|_{\mathcal{H}}^2 \\ &= \frac{c_m^2}{n}\sum_{j,l=1}^n H_{m-1}(X_j)H_{m-1}(X_l)\rho_H(j-l) \\ &= \frac{c_m^2}{n}\sum_{j=1}^n H_{m-1}(X_j)^2 + 2\frac{c_m^2}{n}\sum_{j=1}^n\sum_{i=1}^{n-j} H_{m-1}(X_j)H_{m-1}(X_{j+i})\rho_H(i).\end{aligned}$$

The series $\xi_j = \sum_{i=1}^\infty H_{m-1}(X_j)H_{m-1}(X_{j+i})\rho_H(i)$ converges in L^2 and defines a stationary ergodic sequence $\{\xi_j, j\ge 1\}$. As a consequence, by the ergodic theorem

$$\begin{aligned}\|DJ_mY_n\|_H^2 &\to c_m^2\left(E\left(H_{m-1}(X_1)^2\right) + 2\sum_{j=1}^\infty E\left(H_{m-1}(X_1)H_{m-1}(X_{1+i})\right)\rho_H(i)\right) \\ &= \frac{c_m^2}{(m-1)!}\left(1+2\sum_{j=1}^\infty \rho_H^m(j)\right) = m\sigma_m^2,\end{aligned}$$

where the convergence holds in L^2. This completes the proof. □

9.2. Stein's method and Malliavin calculus

Suppose that $\{F_n, n\ge 1\}$ is a sequence of zero-mean random variables which converges in distribution to a random variable N with law $N(0,1)$. In the recent papers [**32**] and [**33**] the authors combine Malliavin calculus with Stein's method to provide explicit bounds for the distance $d(F_n,N)$ between the laws of F_n and N, where d is a suitable distance between probabilities on the real line. For example, consider the Kolmogorov distance defined by

$$d_{\mathrm{Kol}}(X,Y) = \sup_{z\in\mathbb{R}} |P(X\le z) - P(Y\le z)|. \tag{9.8}$$

Fix $z\in\mathbb{R}$, and consider the *Stein equation*

$$\mathbf{1}_{(-\infty,z]}(x) - \Phi(z) = f'(x) - xf(x), \tag{9.9}$$

$x\in\mathbb{R}$, where $\Phi(z) = P(N\le z)$. It is well-known that for every fixed z, Equation (9.9) admits a solution f_z bounded by $\frac{\sqrt{2\pi}}{4}$ and such that $\|f_z'\|_\infty \le 1$. In fact, it suffices to take

$$\begin{aligned}f_z(x) &= e^{x^2/2}\int_{-\infty}^x \left[\mathbf{1}_{(-\infty,z]}(a) - \Phi(z)\right]e^{-a^2/2}da \\ &= \begin{cases}\sqrt{2\pi}e^{x^2/2}\Phi(x)(1-\Phi(z)) & \text{if } x\le z \\ \sqrt{2\pi}e^{x^2/2}\Phi(z)(1-\Phi(x)) & \text{if } x> z.\end{cases}\end{aligned}$$

Then, we have the following result.

THEOREM 9.6. *Let F be a random variable with $E(F)=0$, $F\in\mathbb{D}^{1,2}$, and F has an absolutely continuous law. Then,*

$$d_{\mathrm{Kol}}(F,N) \le \sqrt{E\left[\left(1-\langle DF, -DL^{-1}F\rangle_H\right)^2\right]}. \tag{9.10}$$

In the particular case $F = I_q(f)$, for some $q \geq 2$, then $\left\langle DF, -DL^{-1}F \right\rangle_H = q^{-1} \|DF\|_H^2$, and therefore

$$d_{\mathrm{Kol}}(F,N) \leq \sqrt{E\left[\left(1 - q^{-1}\|DF\|_H^2\right)^2\right]}. \tag{9.11}$$

Proof. From Equations (9.8) and (9.9) we get

$$d_{\mathrm{Kol}}(F,N) \leq \sup_{z\in\mathbb{R}} |P(F \leq z) - \Phi(z)| = \sup_{z\in\mathbb{R}} |E\left[f_z'(F) - Ff_z(F)\right]|.$$

We can write $F = LL^{-1}F = -\delta DL^{-1}F$. Using the integration by parts formula and the fact that $D(f_z(F)) = f_z'(F)DF$, we obtain

$$E(Ff_z(F)) = -E\left(\delta\left(DL^{-1}F\right)f_z(F)\right) = E\left(f_z'(F)\left\langle DF, -DL^{-1}F\right\rangle_H\right),$$

finally it suffices to apply Cauchy-Schwarz inequality. □

Remark 9.7. The bound (9.10) can be infinite, because we only know that $\left\langle DF, -DL^{-1}F\right\rangle_H \in L^1$. Also notice that Equation (9.11) applied to a sequence of multiple stochastic integrals of order q, yields

$$d_{\mathrm{Kol}}(F_k,N) \leq \sqrt{E\left[\left(1 - q^{-1}\|DF_k\|_H^2\right)^2\right]},$$

and this provides an estimate of the rate of convergence in Theorem 9.1 (compare with condition (4)).

Let $\{F_n, n \geq 1\}$ be a sequence of centered random variables in $\mathbb{D}^{1,2}$, such that $E(F_n^2) \to 1$, as n tends to infinity, with absolutely continuous law. Set

$$\varphi(n) = \sqrt{E\left[\left(1 - \langle DF_n, -DL^{-1}F_n\rangle_H\right)^2\right]}.$$

Then, if $\lim_{n\to\infty}\varphi(n) = 0$, the sequence F_n converges in law to $N(0,1)$. For more details on applications of these ideas we refer the reader to [**32**] and [**33**].