Preface

The factors of any integer can be found quickly by a quantum computer. Since P. Shor discovered this efficient quantum factoring algorithm in 1994 [**S**], people have started to work on building these new machines. As one of those people, I joined Microsoft Station Q in Santa Barbara to pursue a topological approach in 2005. My dream is to braid non-abelian anyons. So long hours are spent on picturing quasiparticles in fractional quantum Hall liquids. From my office on UCSB campus, I often see small sailboats sailing in the Pacific Ocean. Many times I am lost in thought imagining that the small sailboats are anyons and the ocean is an electron liquid. Then to carry out a topological quantum computation is as much fun as jumping into such small sailboats and steering them around each other.

Will we benefit from such man-made quantum systems besides knowing factors of large integers? A compelling reason for a yes comes from the original idea of R. Feynman: a quantum computer is an efficient universal simulator of quantum mechanics. This was suggested in his original paper [**Fe82**]. Later, an efficient simulation of topological quantum field theories was given by M. Freedman, A. Kitaev, and the author [**FKW**]. These results provide support for the idea that quantum computers can efficiently simulate quantum field theories, although rigorous results depend on mathematical formulations of quantum field theories. So quantum computing literally promises us a new world. More speculatively, while the telescope and microscope have greatly extended the reach of our eyes, quantum computers would enhance the power of our brains to perceive the quantum world. Would it then be too bold to speculate that useful quantum computers, if built, would play an essential role in the ontology of quantum reality?

Topological quantum computation is a paradigm to build a large-scale quantum computer based on topological phases of matter. In this approach, information is stored in the lowest energy states of many-anyon systems, and processed by braiding non-abelian anyons. The computational answer is accessed by bringing anyons together and observing the result. Topological quantum computation stands uniquely at the interface of quantum topology, quantum physics, and quantum computing, enriching all three subjects with new problems. The inspiration comes from two seemingly independent themes which appeared around 1997. One was Kitaev's idea of fault-tolerant quantum computation by anyons [**Ki1**]; the other was Freedman's program to understand the computational power of topological quantum field theories [**Fr1**]. It turns out the two ideas are two sides of the same coin: the algebraic theory of anyons and the algebraic data of a topological quantum field theory are both modular tensor categories. The synthesis of the two ideas ushered in topological quantum computation. The topological quantum computational model is efficiently equivalent to other models of quantum computation, such as the quantum circuit model, in the sense that all models solve the same class of problems in polynomial time [**FKW, FLW1, FKLW**].

Besides its theoretical esthetic appeal, the practical merit of the topological approach lies in its error-minimizing hypothetical hardware: topological phases of matter are fault-avoiding or deaf to most local noises, and unitary gates are implemented with exponential accuracy. There exist semi-realistic local model Hamiltonians whose ground states are proven to be error-correcting codes such as the celebrated toric code. It is an interesting question to understand whether fault-avoidance will survive in more realistic situations, such as at finite temperatures or with thermal fluctuations. Perhaps no amount of modeling can be adequate for us to completely understand Mother Nature, who has repeatedly surprised us with her magic.

We do not have any topological qubits yet. Since scalability is not really an issue in topological quantum computation—rather, the issue is controlling more anyons in the system—it follows that demonstrating a single topological qubit is very close to building a topological quantum computer. The most advanced experimental effort to build a topological quantum computer at this writing is fractional quantum Hall quantum computation. There is evidence both experimentally and numerically that non-abelian anyons exist in certain 2-dimensional electron systems that exhibit the fractional quantum Hall effect. Other experimental realizations are conceived in systems such as rotating bosons, Josephson junction arrays, and topological insulators.

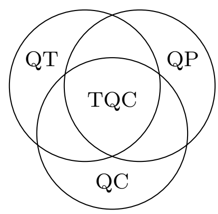
This book expands the plan of the author's 2008 NSF-CBMS lectures on knots and topological quantum computing, and is intended as a primer for mathematically inclined graduate students. With an emphasis on introduction to basic notions and current research, the book is almost entirely about the mathematics of topological quantum computation. For readers interested in the physics of topological quantum computation with an emphasis on fractional quantum Hall quantum computing, we recommend the survey article [**NSSFD**]. The online notes of J. Preskill [**P**] and A. Kitaev's two seminal papers [**Ki1**] [**Ki2**] are good references for physically inclined readers. The book of F. Wilczek [**Wi2**] is a standard reference for the physical theory of anyons, and contains a collection of reprints of classic papers on the subject.

The CBMS conference gave me an opportunity to select a few topics for a coherent account of the field. No efforts have been made to be exhaustive. The selection of topics is personal, based on my competence. I have tried to cite the original reference for each theorem along with references which naturally extend the exposition. However, the wide-ranging and expository nature of this monograph makes this task very difficult if not impossible. I apologize for any omission in the references.

The contents of the book are as follows: Chapters 1,2,4,5,6 are expositions, in some detail, of Temperley-Lieb-Jones theory, the quantum circuit model, ribbon fusion category theory, topological quantum field theory, and anyon theory, while Chapters 3,7,8 are sketches of the main results on the selected topics. Chapter 3 is on the additive approximation of the Jones polynomial, Chapter 7 is on the universality of certain anyonic quantum computing models, and Chapter 8 is on the mathematical models of topological phases of matter. Finally, Chapter 9 lists a few open problems. Chapters 1,2,3 give a self-contained treatment of the additive approximation algorithm. Moreover, universal topological quantum computation models can be built from some even-half theories of Jones algebroids such as the Fibonacci theory. Combining the results together, we obtain an equivalence of the topological quantum computational model with the quantum circuit model. Chapters 1,2,3, based on graphical calculus of ribbon fusion categories, are accessible to entry-level graduate students in mathematics, physics, or computer science. A ribbon fusion category, defined with 6j symbols, is just some point up to equivalence on an algebraic variety of polynomial equations. Therefore the algebraic theory of anyons is elementary, given basic knowledge of surfaces and their mapping class groups of invertible self-transformations up to deformation.

Some useful books on related topics are: for mathematics, Bakalov-Kirillov [**BK**], Kassel [**Kas**], Kauffman-Lins [**KL**], and Turaev [**Tu**]; for quantum computation, Kitaev-Shen-Vyalyi [**KSV**] and Nielsen-Chuang [**NC**]; and for physics, Altland-Simons [**AS**], Di Francesco–Mathieu-Senechal [**DMS**], and Wen [**Wen7**].

Topological quantum computation sits at the triple juncture of quantum topology, quantum physics, and quantum computation:



The existence of topological phases of matter (TPM) with non-abelian anyons would lead us to topological quantum computation (TQC) via unitary modular tensor categories (UMTC):

$$\text{TPM} \longrightarrow \text{UMTC} \longrightarrow \text{TQC}$$

Thus the practical aspect of topological quantum computation hinges on the existence of non-abelian topological states.

Will we succeed in building a large scale quantum computer? Only time will tell. To build a useful quantum computer requires unprecedented precise control of quantum systems, and complicated dialogues between the classical and quantum worlds. Though Nature seems to favor simplicity, she is also fond of complexity as evidenced by our own existence. Therefore, there is no reason to believe that she would not want to claim quantum computers as her own.