CHAPTER 6

TQFTs in Nature

This chapter introduces the algebraic theory of anyons using unitary ribbon fusion categories. It follows that quantum invariants of colored links are amplitudes of anyon trajectories.

6.1. Emergence and anyons

TQFTs are very special quantum field theories. A physical Hamiltonian of interacting electrons in real materials exhibits no topological symmetries. Then it begs the question, is TQFT relevant to our real world? The answer is a resounding yes; it is saved by the so-called emergence phenomenon. The idea is expressed well by a line in an old Chinese poem:

草色遥看近却无

Word for word it is: grass color far see close but not. It means that in early spring, one sees the color of grass in a field from far away, yet no particular green spot can be pointed to. TQFTs do exist in Nature as effective theories, though they are rare and difficult to discover.

It is extremely challenging for experimental physicists to confirm the existence of TQFTs in Nature. Physical systems whose low-energy effective theories are TQFTs are called topological states or phases of matter. Elementary excitations in topological phases of matter are particle-like, called quasiparticles to distinguish them from fundamental particles such as the electron. But the distinction has become less and less clear-cut, so very often we call them particles. In our discussion, we will have a physical system of electrons or maybe some other particles in a plane. We will also have quasiparticles in this system. To avoid confusion, we will call the particles in the underlying system constituent particles or slave particles or sometimes just electrons, though they might be bosons or atoms, or even quasiparticles. If we talk about a Hamiltonian, it is often the Hamiltonian for the constituent particles.

While in classical mechanics the exchange of two identical particles does not change the underlying state, quantum mechanics allows for more complex behavior [**LM**]. In three-dimensional quantum systems the exchange of two identical particles may result in a sign-change of the wave function which distinguishes fermions from bosons. Two-dimensional quantum systems—such as electrons in FQH liquids—can give rise to exotic particle statistics, where the exchange of two identical (quasi)particles can in general be described by either abelian or non-abelian statistics. In the former, the exchange of two particles gives rise to a complex phase $e^{i\theta}$, where $\theta = 0, \pi$ correspond to the statistics of bosons and fermions respectively, and $\theta \neq 0, \pi$ is referred to as the statistics of abelian *anyons* [**Wi1**].

More exotic are non-abelian anyons, whose statistics are described by $k \times k$ unitary matrices acting on a degenerate ground state manifold with $k > 1$ [**FM, FG**]. These unitary matrices form a non-abelian group when $k > 1$, hence the term non-abelian anyons.

Anyons appear as emergent quasiparticles in fractional quantum Hall states [**Hal, MR, Wen3**] and as excitations in microscopic models of frustrated quantum magnets that harbor topological quantum liquids [**Ki1, Ki2, Fr2, FNSWW, LW1**]. While for most quantum Hall states the exchange statistics are abelian, there are quantum Hall states at certain filling fractions, e.g., $\nu = 5/2$ and $\nu = 12/5$, for which non-abelian quasiparticle statistics have been proposed, namely those of so-called Ising and Fibonacci theories, respectively [**RR**].

If many particles live in the same configuration space X, then the configuration space of n such particles taken together depends on their distinguishability. For example, if the n particles are pairwise distinct and not allowed to coincide (called hard-core particles), then their configuration space is the n-fold Cartesian product X^n with the big diagonal $\Delta = \{(x_1, \ldots, x_n) \mid x_i = x_j \text{ for some } i \neq j\}$ removed. But if the n particles are instead identical, then the symmetric group S_n acts on $X^n \setminus \Delta$ freely, and the configuration space becomes the quotient space $(X^n \setminus \Delta)/S_n$, denoted as $C_n(X)$.

Now suppose $X = \mathbb{R}^m$, $m \geq 1$. The configuration space $C_n(\mathbb{R}^m)$ describes the possible states of n identical hard-core particles in $\mathbb{R}^m$. If the n particles are subject to a quantum description, then their states will correspond to nonzero vectors in some Hilbert space $\mathbb{L}$. Let H be the Hamiltonian, with eigenvalues λ_i ordered as $0 = \lambda_0 < \lambda_1 < \cdots$, where we normalize the lowest energy λ_0 to 0. So the state space $\mathbb{L}$ can be decomposed into energy eigenspaces $\mathbb{L} = \bigoplus_i \mathbb{L}_i$, where $\mathbb{L}_i$ is the eigenspace of the eigenvalue λ_i of H. States in $\mathbb{L}_0$ have the lowest energy, and are called the ground states. States in $\mathbb{L}_i$ for $i > 0$ have higher energy, hence are called excited. Normally we are only interested in excited states in $\mathbb{L}_1$. The minimal possible states in $\mathbb{L}_1$ which violate local constraints in the Hamiltonian are called elementary excitations. Suppose the non-local properties of the ground states can be isolated into a subspace V_n of $\mathbb{L}_0$. Then for n particles at $p_1, \ldots, p_n$, their non-local properties will be encoded in a nonzero vector $|\psi\rangle = |\psi(p_1, \ldots, p_n)\rangle \in V_n$. Furthermore, let us assume that the non-local properties encoded in V_n are protected by some physical mechanism such as an energy gap. Now start with n particles at positions $p_1, \ldots, p_n$ with the non-local properties in a state $|\psi_0\rangle \in V_n$. Suppose the n particles are transformed back to the original positions as a set after some time t, and the non-local properties are in a state $|\psi_1\rangle \in V_n$. If V_n has an orthonormal basis $\{e_i\}_1^k$, and we start with $|\psi_0\rangle = e_i$, then $|\psi_1\rangle$ will be a linear combination of $\{e_i\}$: $e_i \mapsto \sum_{j=1}^k a_{ji} e_j$. The motion of the n particles traverses a loop b in the configuration space $C_n(\mathbb{R}^m)$. If the non-local properties are topological, then the associated unitary matrix $U(b) = (a_{ij})$ depends only on the homotopy class of b. Hence we get a unitary projective representation $\pi_1(C_n(\mathbb{R}^m)) \to \mathrm{U}(V_n)$, which will be called the statistics of the particles.

Definition 6.1. *Given n identical anyons in $\mathbb{R}^2$, their statistics are representations $\rho\colon \pi_1(C_n(\mathbb{R}^2)) \to \mathrm{U}(V_n)$ for some Hilbert space V_n. Anyons with* $\dim(V_n) = 1$ *for all n are called abelian anyons; otherwise they are non-abelian. Note in our convention, abelian anyons can have bosonic or fermionic statistics.*

It is well-known:

$$\pi_1(C_n(\mathbb{R}^m)) = \begin{cases} 1, & m = 1, \\ B_n, & m = 2, \\ S_n, & m \geq 3. \end{cases}$$

Therefore braid group representations and anyon statistics are the same in dimension two [**Wu**].

6.2. FQHE and Chern-Simons theory

The only real materials that we are certain are in topological states are electron liquids, which exhibit the fractional quantum Hall effect (FQHE).

Eighteen years before the discovery of the electron, E. Hall was studying Maxwell's book *Electricity and Magnetism*. He was puzzled by a statement in the book and performed an experiment to disprove it, discovering the so-called Hall effect. In 1980, K. von Klitzing discovered the integer quantum Hall effect (IQHE), which won him the 1985 Nobel Prize. Two years later, H. Stormer, D. Tsui, and A. Gossard discovered the FQHE, which led to the 1998 Nobel Prize for Stormer, Tsui, and R. Laughlin. They were all studying electrons in a plane immersed in a perpendicular magnetic field. Laughlin's prediction of the fractional charge $e/3$ of quasiparticles in $\nu = 1/3$ FQH liquids was experimentally confirmed. Braid statistics of quasiparticles were deduced for $\nu = 1/3$, and experiments to confirm braid statistics are making progress.

FQH liquids are new phases of matter that cannot be described with Landau's theory. A new concept—topological order—was proposed, and modular transformations S, T of the torus were used to characterize this new exotic quantum order [**Wen1, Wen2**].

6.2.1. Electrons in flatland. The classical Hall effect (Fig. 6.1) is characterized by a Hall current with resistance $R_{xy} = \alpha B$ for some non-universal constant α, where B is a perpendicular magnetic field. One explanation is as follows. Electrons

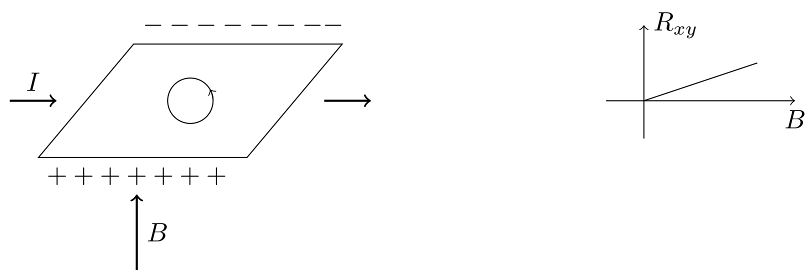


FIGURE 6.1. Classical Hall effect.

in the square $\{(x, y) \mid 0 \leq x, y \leq 1\}$ immersed in a magnetic field in the z-direction feel the Lorentz force $F = q(v \times B + E)$, where q is the charge of one electron, v the velocity, and E the electric field. When a current flows in the x-direction, they consequently move in circles. Electrons on the front edge $y = 0$ will drift to the back edge $y = 1$ due to collisions. Eventually electrons accumulate at the back edge and a current, called Hall current, starts in the y-direction. The Hall resistance depends linearly on B. But when temperature lowers and B strengthens, a surprise is discovered. The Hall resistance is no longer linear with respect to B. Instead it develops so-called plateaux and quantization (Fig. 6.2). What is more astonishing

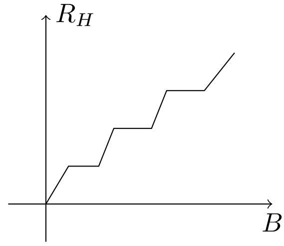


FIGURE 6.2. FQH plateaux.

is the quantized value: it is always $R_{xy} = \nu^{-1}h/e^2$, where $\nu =$ integer up to an error of 10^{-10}, where e is the charge of an electron and h is Planck's constant. When B becomes even bigger, around 30 Tesla, another surprise occurs: ν can be a fraction with odd denominator, such as $\nu = 1/3, 2/5, \ldots$. In 1987 an even denominator FQH liquid was discovered at $\nu = 5/2$.

The quantum mechanical problem of an electron in a perpendicular magnetic field was solved by L. Landau in the 1930s. But the fact that there are about 10^{11} electrons per cm^2 in FQH liquids makes the solution of the realistic Hamiltonian for such electron systems impossible, even numerically. The approach in condensed matter physics is to write down an effective theory at low energy and long wavelength which describes the universal properties of the electron systems. The electrons are strongly interacting with each other to form an incompressible electron liquid when the FQHE could be observed. Landau's solution for a single electron in a magnetic field shows that quantum mechanically an electron behaves like a harmonic oscillator. Therefore its energy is quantized to so-called Landau levels. For a finite size sample of a 2-dimensional electron system in a magnetic field, the number of electrons in the sample divided by the number of flux quanta in the perpendicular magnetic field is called the Landau filling fraction ν. The state of an electron system depends strongly on the Landau filling fraction. For $\nu < 1/5$, the electron system is a Wigner crystal: the electrons are pinned at the vertices of a triangular lattice. When ν is an integer, the electron system is an IQH liquid, where the interaction among electrons can be neglected. When ν is certain fractions such as $1/3, 2/5, \ldots$, the electrons are in a FQH state. Both IQHE and FQHE are characterized by the quantization of the Hall resistance $R_{xy} = \nu^{-1}h/e^2$, where e is the electron charge and h the Planck constant, and the exponential vanishing of the longitudinal resistance R_{xx}. There are about 80 such fractions and the quantization of R_{xy} is reproducible up to 10^{-10}. How could an electron system with so many uncontrolled factors such as the disorders, sample shapes, and variations of the magnetic field strength quantize so precisely?

The IQHE has a satisfactory explanation both physically and mathematically. The mathematical explanation is based on noncommutative Chern classes. For the FQHE at filling fractions with odd denominators, the composite fermion theory based on U(1) Chern-Simons theory is a great success: electrons combine with vortices to form composite fermions, which form their own integer quantum Hall liquids. The exceptional case is the observed FQHE $\nu = 5/2$ and its particle-hole conjugate $\nu = 7/2$. The most promising model for $\nu = 5/2$ is the Pfaffian state;

the low-energy effective theory for $5/2$ is essentially the Ising TQFT or the closely related $SU(2)_2$. If this theory holds, the Jones polynomial at 4th roots of unity would have a direct bearing on experimental data for $\nu = 5/2$.

6.2.2. Chern-Simons theory as effective theory. The discovery of the FQHE has cast some doubts on the completeness of Landau theory for states of matter. It is believed that the electron liquid in a FQHE state is in a topological state with a Chern-Simons TQFT as an effective theory. Since topological states are described by TQFTs, we can ask what TQFT represents the $\nu = 1/3$ Laughlin state. It turns out this is not a simple question to answer because TQFTs such as Chern-Simons theories describe bosons rather than fermions. To work with fermions, the answer is a spin TQFT. To work with bosons, we use the so-called flux attachment to convert the electrons into charge-flux composites, which are bosonic objects. For the rest of the book, we will follow this bosonic approach.

How do physicists conclude that topological properties of FQH liquids can be modeled by Chern-Simons TQFTs?

From an emergent perspective, if a system is examined from longer and longer wavelengths, the behavior of the system is dominated by the lowest derivative terms: m derivatives under the Fourier transform become k^m, where k is the momentum, and the long wavelength limit is $k \to 0$. Therefore Chern-Simons terms dominate in the long wavelength limit. To make a contact with FQH liquid, we can derive the equation of motion, agreeing with the off-diagonality of Hall resistance.

More definite evidence comes from edge theories of FQH liquids and path integral manipulation. Witten [**Witt**] discovered that the boundary theory of a Chern-Simons TQFT is a Wess-Zumino-Witten (WZW) CFT. Such a TQFT-CFT connection has two applications in FQH liquids: one modeling the boundary (1+1)-system [**Wen4**], the other modeling a (2+0) fixed time slice [**MR**]. Based on this connection, Wen proposed a Luttinger liquid theory, which has experimental confirmation. For a (2+0) fixed time slice, the electron ground states can be described by a wave function $\psi(z_1, \ldots, z_N)$, where z_i is the position of the ith electron. The Laughlin theory which predicted the charge $e/3$ for quasiparticles in $\nu = 1/3$ FQH liquids is based on the famous Laughlin wave function

$$\prod_{i<j}(z_i - z_j)^3 e^{-\frac{1}{4}\sum |z_i|^2}$$

obtainable as the conformal block of a U(1) CFT. As generalized later, electron wave functions are conformal blocks of the corresponding CFTs. Physically one can also "derive" abelian Chern-Simons theory starting from electrons using path integrals. Of course, many steps are not rigorous, and based on certain physical assumptions. Considering all the evidence together, we are confident that Chern-Simons theory describes FQH liquids.

While the case for the abelian Chern-Simons modeling of FQH liquids is convincing, the description of $\nu = 5/2$ with non-abelian Chern-Simons theory has less evidence. In particular, the physical "derivation" of abelian Chern-Simons theory does not apply to $\nu = 5/2$. How is it possible to have non-abelian anyons from electrons? We still don't know. But one possibility is that electrons first organize themselves into states with abelian anyons. Then a phase transition drives them into a non-abelian phase.

6.2.3. Ground states and statistics. To describe new states of matter such as FQH electron liquids, we need new concepts and methods. Consider the following Gedanken experiment. Suppose an electron liquid is confined to a closed oriented surface Σ, e.g., a torus. The lowest energy states of the system form a Hilbert space $L(\Sigma)$, called the ground state manifold. Furthermore, suppose $L(\Sigma)$ decomposes as $V(\Sigma) \otimes V^{\mathrm{local}}(\Sigma)$, where $V^{\mathrm{local}}(\Sigma)$ encodes the local degrees of freedom, such as anyon positions. In an ordinary quantum system, the ground state will be unique, so $V(\Sigma)$ is 1-dimensional. But for topological states of matter, $V(\Sigma)$ is often degenerate (more than 1-dimensional), i.e., there are several orthonormal ground states with exponentially small energy differences. This ground state degeneracy in $V(\Sigma)$ is a new quantum number. Hence a topological quantum system assigns each closed oriented surface Σ a Hilbert space $V(\Sigma)$, which is exactly the rule for a TQFT. A FQH electron liquid always has an energy gap in the thermodynamic limit which is equivalent to the incompressibility of the electron liquid. Therefore the ground state manifold is stable if controlled below the gap. Since the ground state manifold has exponentially close energy, the Hamiltonian of the system restricted to the ground state manifold is constant, hence there will be no continuous evolution except an overall abelian phase due to ground state energy. In summary, ground state degeneracy, energy gap, and the vanishing of the Hamiltonian are all salient features of topological quantum systems.

Although the Hamiltonian for a topological system is a constant, there are still discrete dynamics induced by topological changes besides an overall abelian phase. As we mentioned before, given a realistic system, even the ground states have local degrees of freedom. Topological changes induce evolution of the whole system, so within the ground state, states in $V(\Sigma)$ evolve through $V(\Sigma) \otimes V^{\mathrm{local}}(\Sigma)$.

Elementary excitations of FQH liquids are anyons, which are strict labels for a TQFT; anyon types serve as labels. Suppose a topological quantum system confined to a surface Σ has elementary excitations of types $a_1, \ldots, a_n$ localized at well-separated points $p_1, \ldots, p_n$ on Σ. Then the ground states of the system outside some small neighborhoods of p_i form a Hilbert space. Suppose this Hilbert space splits into $V(\Sigma; a_i) \otimes V^{\mathrm{local}}(\Sigma; p_i, a_i)$ as before. Then associated to the surface with small neighborhoods of p_i removed and each resulting boundary circle labeled by the corresponding anyon is a Hilbert space $V(\Sigma; a_1, \ldots, a_n)$. There are discrete evolutions of the ground states induced by topological changes, diffeomorphisms of Σ which preserve the boundaries and their labels. An interesting case is the mapping class group of the disk with n punctures—the famous braid group on n strands, B_n. Suppose the anyons can be braided adiabatically so that the quantum system remains in the ground states. Then we have a unitary transformation from the ground states at time t_0 to the ground states at time t_1. Then $V(\Sigma; a_1, \ldots, a_n)$ is a projective representation of the mapping class group of Σ. Therefore an anyonic system provides an assignment from a closed oriented surface Σ with anyons of types $a_1, \ldots, a_n$ at $p_1, \ldots, p_n$ to a Hilbert space $V(\Sigma; a_1, \ldots, a_n)$ of topological ground states, and from braiding of anyons to a unitary transformation of $V(\Sigma; a_1, \ldots, a_n)$.

6.3. Algebraic theory of anyons

A unitary MTC (UMTC) $\mathcal{C}$ gives rise to a modular functor $V_{\mathcal{C}}$, which assigns a Hilbert space $V(Y)$ to each surface Y with extra structure and a projective representation of the mapping class group of Y. Therefore it is natural to use a

UMTC to model the topological properties of anyonic systems. We will always assume our categories are strict in this section.

How does an anyon look? Nobody knows. But it is a particle-like topological quantum field. It is important that an anyon can be transported from one location to another by local operators. Although a single anyon cannot be created or removed, its physical size can be changed by local operators. Therefore anyons are mobile, indestructible, yet shrinkable by local operators. The mathematical model under UMTCs is a framed point in the plane: a point with a small arrow. Therefore its world line in $\mathbb{R}^3$ is not really an arc; it is a ribbon. Hence we are interested in framed links instead of just links. In $\mathbb{R}^3$, the information of the ribbon can be encoded by the winding number of the two boundary curves or the linking number of two boundary circles for a closed trajectory (oriented in the same direction). In FQH liquids, an anyon is considered to be a pointlike defect in the uniform electron liquid, called a quasiparticle or quasihole. They are attracted to impurities in the sample. In the wave function model of FQH liquids, a quasihole is a coherent superposition of edge excitations.

A dictionary of terminologies is given in Table 6.1. Given a unitary TQFT,

UMTC	*Anyonic system*
simple object	anyon
label	anyon type or topological charge
tensor product	fusion
fusion rules	fusion rules
triangular space V_{ab}^{c} or V_{c}^{ab}	fusion/splitting space
dual	antiparticle
birth/death	creation/annihilation
mapping class group representations	anyon statistics
nonzero vector in $V(Y)$	ground state vector
unitary F-matrices	recoupling rules
twist $\theta_x = e^{2\pi i s_x}$	topological spin
morphism	physical process or operator
tangles	anyon trajectories
quantum invariants	topological amplitudes

Table 6.1

there exists a unique topological Hermitian product on the modular functor $V(Y)$ so that the representation of the mapping class group is unitary [**Tu**]. Therefore we can always choose a unitary realization of the F-symbols. It is shown in [**HH**] for the $\frac{1}{2}E_6$ theory that the F-matrices cannot be all real, hence the two hexagon axioms are independent. Strictly speaking, for physical application, we only need the recoupling rules to preserve probability, so anti-unitary transformations should also be allowed. We also need caution when interpreting tangles as anyon trajectories and quantum invariants as amplitudes. For example, suppose we create from the vacuum 1 a particle-antiparticle pair x, x^*, separate them, and then annihilate. Surely they will return to the vacuum. But on the other hand, its quantum dimension d_x is supposed to tell us the probability of going back to the vacuum. The point is that when we create a particle-antiparticle pair, we cannot be certain of their types. Therefore creating a particle-antiparticle pair is a probabilistic process. The probability of creating a particle-antiparticle pair of type a is given by d_a^2/D^2, where d_a is the quantum dimension of a and D is the global quantum dimension of $\mathcal{C}$. Therefore the bigger the quantum dimension, the better the chance to be created given enough energy. In general, a tangle is an operator, therefore it does not have a well-defined amplitude without specifying initial and final states.

One of the most exciting predictions is that in $\nu = 5/2$ FQH liquids, a certain electric current quantity σ_{xx} in interferometric measurement is governed by the Jones polynomial at a 4th root of unity: $\sigma_{xx} \propto |t_1|^2 + |t_2|^2 + 2\mathrm{Re}\big(t_1^* t_2 e^{i\alpha} \langle \psi | M_n | \psi \rangle\big)$, where M_n is the Jones representation of a certain braid [**FNTW**]. More generally, if a FQH state exists at $\nu = 2 + \frac{k}{k+2}$, its non-abelian statistics are conjectured to be closely related to $\mathrm{SU}(2)_k$ [**RR**]. If so, then experimental data directly manifest Jones evaluations. For further applications to FQH liquids, see [**Bo**].

6.3.1. Particle types and fusion rules. To describe a system of anyons, we list the species of the anyons in the system, called the particle types, topological charges, superselection sectors, labels, and other names; we also specify the antiparticle type of each particle type. We will list the particle types as $\{i\}_{i=0}^{n-1}$, and use $\{x_i\}_{i=0}^{n-1}$ to denote a representative set of anyons, where the type of x_i is i.

In any anyonic system, we always have a trivial particle type denoted by 0, which represents the ground states of the system or the vacuum. In the list of particle types above, we assume $x_0 = 0$. The trivial particle is its own antiparticle. The antiparticle of x_i, denoted as x_i^*, is always of the type of another x_j. If x_i and x_i^* are of the same type, we say x_i is self-dual.

To have a nontrivial anyonic system, we need at least one more particle type besides 0. The Fibonacci anyonic system is such an anyonic system with only two particle types: the trivial type 0, and the nontrivial type τ. Anyons of type τ are called the Fibonacci anyons. They are self-dual: the antiparticle type of τ is also τ. We need to distinguish between anyons and their types. For Fibonacci anyons, this distinction is unnecessary, as for any TQFT with trivial Frobenius-Schur indicators.

Anyons can be combined in a process called fusion, which is tensoring two simple objects. Repeated fusions of the same two anyons do not necessarily result in anyons of the same type: the resulting anyons may be of several different types, each with a certain probability. In this sense we can also think of fusion as a measurement. It follows that given two anyons x, y of types i, j, the particle type of the fusion, denoted as $x \otimes y$, is in general not well-defined.

If fusion of an anyon x with any other anyon y (maybe x itself) is always well-defined, then x is called abelian. If neither x nor y is abelian, then there will be anyons of more than one type as the possible fusion results. We say such fusion has *multi-fusion channels* of x and y.

Given two anyons x, y, we write the fusion result as $x \otimes y \cong \bigoplus_i n_i x_i$, where $\{x_i\}$ is a representative set of isomorphism classes of simple objects, and each n_i is a nonnegative integer, called the multiplicity of the occurrence of anyon x_i. Multi-fusion channels correspond to $\sum_i n_i > 1$. Given an anyonic system with anyon representative set $\{x_i\}_{i=0}^{n-1}$, we have $i \otimes j = \bigoplus_{k=0}^{n-1} N_{ij}^k k$. The nonnegative integers N_{ij}^k are called the fusion rules of the anyonic system; the matrix N_i with (j, k)-entry

N_{ij}^k is called the ith fusion matrix. If $N_{ij}^k \neq 0$, we say fusion of x_i and x_j to x_k is admissible.

DEFINITION 6.2.

(1) An anyon x_i is abelian, also called a simple current, if $\sum_k N_{ij}^k = 1$ for every j. Otherwise it is non-abelian.

(2) An anyon x_i such that $x_i^2 = 1$ is called a boson if $\theta_i = 1$, a fermion if $\theta_i = -1$, and a semion if $\theta_i = \pm i$.

PROPOSITION 6.3.

(1) The quantum dimension of an anyon x_i is the Perron-Frobenius eigenvalue of N_i.

(2) An anyon x_i is abelian iff $d_i = 1$.

PROOF. An anyon is a simple object in a UMTC, so $d_i \geq 1$ [**Tu**]. But d_i is the Perron-Frobenius eigenvalue of the fusion matrix N_i. □

6.3.2. Many-anyon states and fusion tree bases. A defining feature of non-abelian anyons is the existence of multi-fusion channels. Suppose we have three anyons a, b, c localized in the plane and well-separated. We would like to know, when all three anyons are brought together to fuse, what kinds of anyons will this fusion result in? When anyons a and b are combined, we may see several anyons. Taking each resulting anyon and combining with c, we would have many possible outcomes. Hence the fusion result is not necessarily unique. Moreover, even if we fix the resulting outcome, there is an alternative arrangement of fusions given by fusing b and c first. For three or more anyons to be fused, there are many such arrangements, each represented graphically by a *fusion tree.*

FIGURE 6.3. Fusion trees.

A *fusion path* is a labeling of a fusion tree whereby each edge is labeled by a particle type, and the three labels around any trivalent vertex represent a fusion admissible by the fusion rules. The top edges are labeled by the anyons to be fused, drawn along a horizontal line; the bottom edge represents the fusion result, also called the total charge of the fused anyons.

In general, given n anyons in the plane localized at certain well-separated places, we will fix a total charge at the ∞ boundary. In theory any superposition of anyons is possible for the total charge, but it is physically reasonable to assume that such a superposition will decohere into a particular anyon if left alone. Let us arrange the n anyons along the real axis of the plane. When we fuse them consecutively, we have a fusion tree as in Fig. 6.4. In our convention, fusion trees go downward. If we want to interpret a fusion tree as a physical process in time, we should also introduce the Hermitian conjugate operator of fusion: splitting of anyons from one to two. Then as time goes upward, a fusion tree can be interpreted as a splitting of one anyon into many.

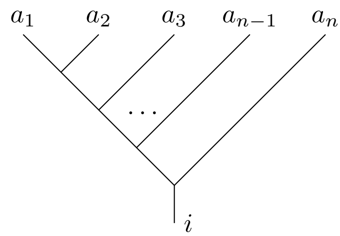


FIGURE 6.4. "Consecutive" fusion tree for anyons $a_1, \ldots, a_n$ with total charge i.

The ground state manifold of a multi-anyon system in the plane even when the positions of the anyons are fixed might be degenerate: there may be more than one ground state. (In reality the energy differences between the different ground states go to 0 exponentially as the anyon separations go to infinity; we will ignore such considerations here, and always assume that anyons are well-separated until they are brought together for fusion.) Such degeneracy is necessary for non-abelian statistics. We claim that fusion paths over a fixed fusion tree represent an orthonormal basis of the degenerate ground state manifold when appropriately normalized. (Complications from fusion coefficients $N_{ij}^k > 1$ are ignored.)

The fusion tree basis of a multi-anyon system then leads to a combinatorial way to compute the degeneracy: count the number of labelings of the fusion tree, i.e., the number of fusion paths. For example, consider n τ-anyons in the plane with total charge τ, and denote the ground state degeneracy as F_n. Simple counting shows that $F_0 = 0$ and $F_1 = 1$; easy induction then gives $F_{n+1} = F_n + F_{n-1}$. This is exactly the Fibonacci sequence, hence the name of Fibonacci anyons.

6.3.3. F-matrices and pentagons. In the discussion of the fusion tree basis above, we fuse anyons one by one from left to right, e.g., the left fusion tree in Fig. 6.3. We may as well choose any other arrangement of fusions, e.g., the right fusion tree in Fig. 6.3. Given n anyons with a certain total charge, each arrangement of fusions is represented by a fusion tree, whose admissible labelings form a basis of the multi-anyon system.

The change from the left fusion tree to the right in Fig. 6.3 is called the F-move. Since both fusion tree bases describe the same degenerate ground state manifold of 3 anyons with a certain total charge, they should be related by a unitary transformation. The associated unitary matrix is called the F-matrix, denoted as F_d^{abc}, where a, b, c are the anyons to be fused, and d is the resulting anyon or total charge.

For more than 3 anyons, there are many more fusion trees. To have a consistent theory, a priori we must specify the change of basis matrices for any number of anyons in a consistent way. For instance, the leftmost and rightmost fusion trees of 4 anyons in Fig. 4.3 are related by two different sequences of applications of F-moves, whose consistency will be referred to as the pentagon. Mac Lane's coherence theorem [**Ma**] guarantees that pentagons suffice, i.e., imply all other consistencies. Note that pentagons are just polynomial equations in F-matrix entries.

To set up the pentagons, we need to explain the consistency of fusion tree bases for any number of anyons. Consider a decomposition of a fusion tree T into two fusion subtrees T_1, T_2 by cutting an edge e into two new edges, each still referred to as e. The fusion tree basis for T has a corresponding decomposition: if i's are the particle types of the theory, for each i we have a fusion tree basis for T_1, T_2 with the edge e labeled by i. Then the fusion tree basis for T is the direct sum over all i of the tensor product: (the fusion tree basis of T_1) $\otimes$ (the fusion tree basis of T_2).

In the pentagons, an F-move is applied to part of the fusion tree in each step. The fusion tree decomposes into two pieces: the part where the F-move applies, and the remaining part. It follows that the fusion tree basis decomposes as a direct sum of several terms corresponding to admissible new labels.

Given a set of fusion rules N_{ij}^k, solving the pentagons turns out to be a difficult task (even with the help of computers). However, certain normalizations can be made to simplify the solutions. If one of the indices a, b, c of the F-matrix is the trivial type 0, we may assume $F_d^{abc} = 1$. We cannot do so in general if d is trivial.

EXAMPLE 6.4 (Fibonacci F-matrix). *Recall $\tau^2 = 1 \oplus \tau$. A priori there are only two potentially nontrivial F-matrices, which we will denote as*

$$F_1^{\tau\tau\tau} = t, \qquad\qquad F_\tau^{\tau\tau\tau} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

where $p, q, r, s, t \in \mathbb{C}$. There are many pentagons even for the Fibonacci fusion rules depending on the four anyons to be fused and their total charges: a priori $2^5 = 32$. But a pentagon is automatically trivial if one of the anyons to be fused is trivial, leaving only two pentagons to solve. Drawing fusion tree diagrams and keeping track of the various F-moves among ordered fusion tree bases, the pentagons become:

$$\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}^2 = F_\tau^{\tau\tau\tau} \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} F_\tau^{\tau\tau\tau}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & F_\tau^{\tau\tau\tau} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & F_\tau^{\tau\tau\tau} \end{pmatrix} = \begin{pmatrix} p & 0 & q \\ 0 & t & 0 \\ r & 0 & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & F_\tau^{\tau\tau\tau} \end{pmatrix} \begin{pmatrix} p & 0 & q \\ 0 & t & 0 \\ r & 0 & s \end{pmatrix}$$

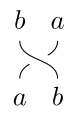
These matrix equations expand into thirteen polynomial equations over p, q, r, s, t, instances of the pentagon equation for 6j symbol systems (Defn. 4.7). Solving them and constraining the F-matrices to be unitary, we obtain

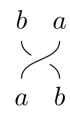
$$F_1^{\tau\tau\tau} = 1, \qquad\qquad F_\tau^{\tau\tau\tau} = \begin{pmatrix} \phi^{-1} & \bar{\xi}\phi^{-1/2} \\ \xi\phi^{-1/2} & -\phi^{-1} \end{pmatrix} \tag{6.5}$$

where $\phi = \frac{\sqrt{5}+1}{2}$ is the golden ratio and ξ is an arbitrary phase, w.l.o.g. $\xi = 1$.

6.3.4. R-matrix and hexagons. Given n anyons y_i in a surface S, well-separated at fixed locations p_i, the ground states $V(S; p_i, y_i)$ of this quantum system form a projective representation of the mapping class group of S punctured n times. If S is the disk, the mapping class group is the braid group. In a nice basis of $V(S; p_i, y_i)$, the braiding matrix R_{ij} becomes diagonal.

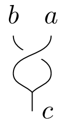
To describe braidings carefully, we introduce some conventions. When we exchange two anyons a, b in the plane, there are two different exchanges which are topologically inequivalent: their world lines are given by braids.





In our convention time goes upwards. We will refer to the left picture as the left-handed braiding R_{ab}^{-1} and the right picture as the right-handed braiding R_{ab}.

Let V_c^{ab} be the ground state manifold of two anyons of types a, b with total charge c. Let us assume each space V_c^{ab} is one-dimensional when (a, b, c) is admissible, and let e_c^{ab} be its fusion tree basis. When anyons a and b are braided by R_{ab}, the state e_c^{ab} in V_c^{ab} is changed into a state $R_{ab}e_c^{ab}$ in V_c^{ba}. Since both $R_{ab}e_c^{ab}$ and e_c^{ba} are nonzero vectors in a one-dimensional Hilbert space V_c^{ba}, they are equal up to a phase, denoted as R_c^{ba}, i.e, $R_{ab}e_c^{ab} = R_c^{ba} e_c^{ba}$.



$= R_c^{ba}$


Here R_c^{ba} is a phase, but in general it is a unitary matrix called an R-matrix. A matrix representing an arbitrary braiding of many anyons can be obtained from R- and F-matrices (and their inverses) via composition and tensoring with identity matrices. We should mention that in general $R_c^{ab} \neq (R_c^{ba})^{-1}$; their product involves twists of particles.

THEOREM 6.6.

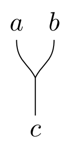
(1) *For any particle types a, b, c,*

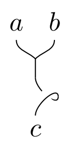
$R^{ab}R^{ba} = \theta_a^{-1}\theta_b^{-1}\theta_c \mathrm{id}\colon V_c^{ab} \to V_c^{ab}$ *if* $N_c^{ab} \geq 1$, *whence*

$R_c^{ab}R_c^{ba} = \theta_a^{-1}\theta_b^{-1}\theta_c$ *if* $N_c^{ab} = 1$.

(2) *Spin-statistics connection:* $\mathrm{tr}(c_{aa}) = \theta_a d_a$.

PROOF. For statement (1),

θ_c


$=$


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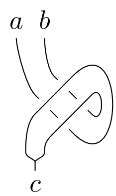
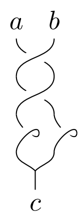
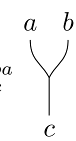
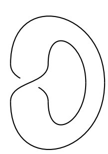
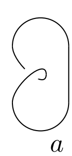


$=$


$= \theta_a\theta_b R_c^{ab}R_c^{ba}$


For statement (2),

$\mathrm{tr}(c_{aa}) =$

$=$


$= \theta_a d_a$ □

Statement (1) above is called the suspender formula. The spin-statistics connection was pointed out to me by N. Read.

As we have seen before, anyons can be fused or split, so braidings should be compatible with fusion and splitting. For example, given two anyons c, d, we may first split d to a, b, then braid c with a and then with b, or we may braid c with d first, then split d into a, b. These two processes are physically equivalent, so their resulting matrices should be the same. Applying the two operators to the fusion tree basis e_m^{cd}, we have an identity in pictures (Fig. 6.5). Both sides are threefold

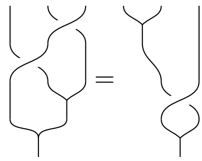


FIGURE 6.5. Right-handed hexagon.

compositions of F-moves and braidings. It follows that a certain product of six matrices equals the identity (Fig. 4.4). This equation is called a hexagon. There is another family of hexagons obtained by replacing all right-handed braidings with left-handed ones. In general, these two families of hexagons are independent of each other. The hexagons imply all other consistency equations for braidings.

EXAMPLE 6.7 (Fibonacci braiding). *A priori there are eight right-handed Fibonacci hexagons. But braiding with the vacuum is trivial, i.e.,* $R_\tau^{\tau 1} = R_\tau^{1\tau} = R_1^{11} = 1$*. It follows that a hexagon is trivial if one of the three upper labels is trivial, leaving only two right-handed hexagons to solve:*

$$(R_\tau^{\tau\tau})^2 = R_1^{\tau\tau}$$

$$\begin{pmatrix} R_1^{\tau\tau} & 0 \\ 0 & R_\tau^{\tau\tau} \end{pmatrix} F_\tau^{\tau\tau\tau} \begin{pmatrix} R_1^{\tau\tau} & 0 \\ 0 & R_\tau^{\tau\tau} \end{pmatrix} = F_\tau^{\tau\tau\tau} \begin{pmatrix} 1 & 0 \\ 0 & R_\tau^{\tau\tau} \end{pmatrix} F_\tau^{\tau\tau\tau}$$

These expand into five polynomial equations manifesting Defn. 4.13. *Left-handed braidings are the same, but with inverted R-symbols. Using eqn.* (6.5)*, our ten-polynomial system boils down to* $R_1^{\tau\tau} = e^{4\pi i/5}$ *and* $R_\tau^{\tau\tau} = e^{-3\pi i/5}$.

6.3.5. Morphisms as operators. Suppose anyons a, b, c on the x-axis undergo a process adiabatically as in Fig. 6.6, from $t = 0$ to $t = 1$. It is common to

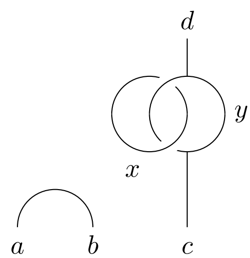
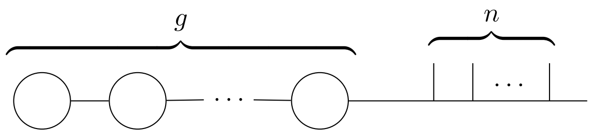


FIGURE 6.6. Particle trajectories constituting a morphism.

interpret the morphism in $\mathrm{Hom}(a \otimes b \otimes c, d)$ as particle trajectories. Then we may ask, what is the amplitude of this process? This question is not quite well-defined for non-abelian anyons because at time $t = 0$, the ground state is not unique. Hence we should instead ask for matrix elements because a morphism in $\mathrm{Hom}(a \otimes b \otimes c, d)$ is an operator.

Given two states at $t = 0$ and $t = 1$, how do we compute matrix elements? Supposing the anyonic system is given by a UMTC $\mathcal{C}$, such matrix elements are part of the operator invariant from $\mathcal{C}$. Then they can be computed by recoupling rules when statistics are given in fusion tree bases. More generally, if n anyons $x_1, \ldots, x_n$ are fixed at $p_1, \ldots, p_n$ on a genus g closed orientable surface Σ_g, the ground state manifold has a generalized fusion graph basis obtained from labeling the following graph:



6.3.6. Measurement. Measurement is performed by fusing anyons. A particular outcome is given by a fusion graph. Hence the amplitude of measuring a certain outcome is just the matrix element for the initial state and outcome state.

6.4. Intrinsic entanglement

An interesting feature of the tensor product of vector spaces is that neither tensor factor is a canonical subspace of a tensor product. In quantum theory, the Hilbert space of a composite system is the tensor product of the Hilbert spaces of the constituent subsystems.

DEFINITION 6.8. *Consider a Hilbert space $\mathbb{L} = \bigotimes_{i=1}^n \mathbb{L}_i$ with a fixed tensor decomposition and $n \geq 2$. A nonzero vector $v \in \mathbb{L}$ is a product (or separable or decomposable) state if v can be written as $v = \bigotimes_{i=1}^n v_i$, where $v_i \in \mathbb{L}_i$. Otherwise v is entangled. Classical states are products.*

EXAMPLE 6.9. *The spin-singlet state $\frac{|01\rangle - |10\rangle}{\sqrt{2}}$ is an entangled 2-qubit state.*

Any two vectors v, w in $\mathbb{C}^N$ span a parallelogram, degenerate iff $v \propto w$, whose area will be denoted as $A(v, w)$. Recall any $v \in (\mathbb{C}^2)^{\otimes n}$ is a linear combination $v = \sum v_I |I\rangle$. For each $1 \leq i \leq n$ and $x \in \{0, 1\}$, let $\partial_x^i \colon (\mathbb{C}^2)^{\otimes n} \to (\mathbb{C}^2)^{\otimes (n-1)}$ be the linear map given by $|b_1 \cdots b_n\rangle \mapsto \delta_{x,b_i} |b_1 \cdots \hat{b}_i \cdots b_n\rangle$, where $\hat{}$ denotes deletion. What ∂_x^i does is identify $(\mathbb{C}^2)^{\otimes (n-1)}$ with the subspace of $(\mathbb{C}^2)^{\otimes n}$ spanned by all n-bit strings with ith bit x.

DEFINITION 6.10. *Given $v \in (\mathbb{C}^2)^{\otimes n}$, let $E(v) = \sum_{i=0}^n A^2(\partial_0^i(v), \partial_1^i(v))$.*

THEOREM 6.11.
(1) $0 \leq E(v) \leq n/4$.
(2) $E(v)$ is invariant under local unitary transformations $\mathrm{U}(2)^{\otimes n}$.
(3) $E(v) = 0$ iff v is a product state.

This theorem is from [**MW**].

Note that it takes exponentially many steps to compute $E(V)$ with respect to n. For $n = 9$, it attains a maximum value on $(|000\rangle + |111\rangle)^{\otimes 3}$, which shows some weakness of $E(V)$ as an entanglement measure.

Topological order is an exotic quantum order with non-local entanglement. Since topological ground state manifolds have no natural tensor decomposition, it is hard to quantify entanglement. In [**LW2, KP**], it was discovered that intrinsic entanglement of a topological order can be quantified by $\ln D$, where D is the positive global quantum dimension. Consider the ground state $|\psi\rangle$ on S^2, and a disk whose size is large relative to the correlation length. If the constituent degree of freedom is split along the disk, and the outside degree of freedom is traced out, we obtain a density matrix $\rho_L^{\text{ins}}(|\psi\rangle)$. The von Neumann entropy $\rho \ln \rho$ of $\rho_L^{\text{ins}}(|\psi\rangle)$ grows as $\alpha L - \gamma + O(1/L)$ as $L \to \infty$. The linear coefficient α is not universal and dictated by local physics around the perimeter, but γ is universal.

THEOREM 6.12. *$\gamma = \ln D$ for a doubled unitary TQFT.*

The physical argument in [**KP**] applies to all unitary TQFTs. It would be interesting to find a connection between $E(|\psi\rangle)$ with respect to a lattice realization and $\ln D$. It is possible that a topological ground state has maximal entanglement in any lattice realization.

Topological entanglement entropy of other surfaces such as the torus is computed physically for Witten-Chern-Simons TQFTs [**DFLN**]. In general the answer is given in terms of entries of the modular S-matrix. Very likely, the computation could be made mathematical.