CHAPTER 9

Train tracks and the Mirzakhani volume recursion

1. Measured geodesic laminations and train tracks

Properties of $\mathcal{MGL}$ provide information about Teichmüller space, the mapping class group and the Teichmüller and WP metrics. As described in Chapter 6, the intersection product $i(\ ,\ )$, defined initially for the set $\mathcal{S}$ of free homotopy classes of simple closed curves, defines an embedding i of $\mathcal{MGL}$ into $\mathbb{R}^{\mathcal{S}}$. The image is homeomorphic to $\mathbb{R}^{6g-6+n}$. An immediate consequence of the embedding is that the topology of intersection masses with transverses is equivalent to the topology of intersection numbers with $\mathcal{S}$ elements. A homeomorphism f between marked hyperbolic surfaces R and S induces a mapping between sets of simple closed geodesics by choosing the geodesic representatives of free homotopy classes. The mapping preserves the intersection product and consequently the mapping extends to a homeomorphism between $\mathcal{MGL}(R)$ and $\mathcal{MGL}(S)$. By construction, homotopic maps provide the same homeomorphism. The mapping of spaces of measured geodesic laminations for marked hyperbolic surfaces is equivariant for the action of the mapping class group. Furthermore given a marked hyperbolic surface, the mapping class group acts on the associated space $\mathcal{MGL}$. *Basic point:* there is a natural identification between spaces of measured geodesic laminations for marked hyperbolic surfaces and MCG acts on $\mathcal{MGL}$.

Thurston introduced a finite system to provide a piecewise linear symplectic structure for $\mathcal{MGL}$.

Definition 9.1. *A train track for a hyperbolic surface R with cusps is an embedded smooth 1-complex τ with vertices called switches and edges called branches. At each switch there is a unique tangent line and a neighborhood of each switch is a union of disjoint smoothly embedded arcs. Furthermore no component of the track complement is: a disc with zero switches on its boundary and zero or one interior cusps; a disc with one or two switches on the boundary and no interior cusps, or an annulus with no switches or interior cusps. A geodesic lamination μ is carried by a track τ provided there is a differentiable map f of R, homotopic to the identity, such that $f(\mu) \subset \tau$ and the differential df vanishes nowhere on a leaf.*

The complementary region condition ensures that a closed curve carried by a train track is essential (neither null homotopic or homotopic to a cusp). The essential reference for train tracks is the monograph of Penner and Harer [**PH92**] and a summary is provided in the exposition [**Ham07**]. Each geodesic lamination is carried by a train track. For a measured geodesic lamination μ, the carrying map to a track τ provides for a transverse measure on the track. Define the measure $u(e)$ of a track branch e to be the measure of leaves carried by the branch. At a switch the weights satisfy the switch condition: $u(e_1) + \cdots + u(e_i) = u(e_{i+1}) + \cdots + u(e_{j+1})$ for $e_1, \ldots, e_i$ the incoming edges and $e_{i+1}, \ldots, e_{j+1}$ the outgoing edges. Train tracks can be considered as spines for ϵ-neighborhoods of geodesic laminations. We see in Lemma 9.2 that for typical measured geodesic laminations, modulo homeomorphisms of the surface, the spine of an ϵ-neighborhood is locally constant in $\mathcal{MGL}$. This property enables local parameterization of measured geodesic laminations by train tracks. An equivalence relation for train tracks is generated by considering isotopy of tracks and a refining step, called splitting. The equivalence relation is the formalization of the affect of decreasing the ϵ-neighborhood.

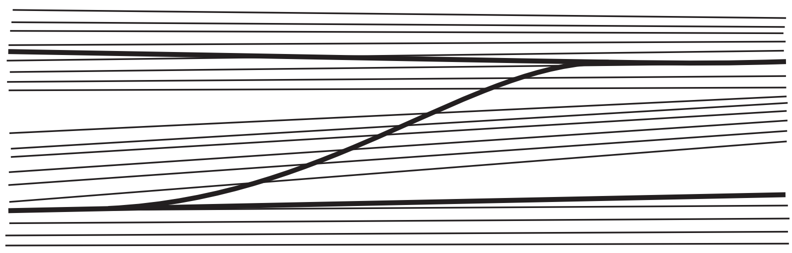

FIGURE 1. A closeup of a train track carrying a lamination.

Splitting and isotopy allow for relating parameterizations of measured geodesic laminations given by different train tracks. A fundamental result is that carrying a common measured geodesic lamination is equivalent to equivalence of train tracks [**PH92**, Thrm. 2.8.5].

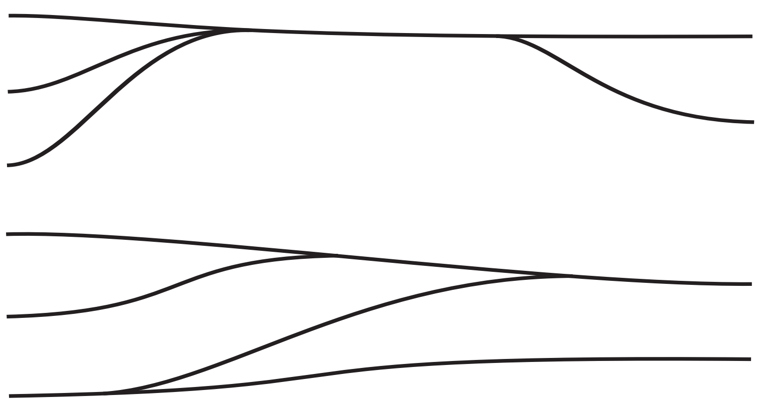

FIGURE 2. Splitting a train track.

A *rational* element of $\mathcal{MGL}$ is a union of disjoint simple closed geodesics with rational weights. An *integral* element of $\mathcal{MGL}$ is a finite union of disjoint simple closed geodesics with integral weights or equivalently, allowing parallel curves, a finite union of disjoint simple closed curves, called a multi curve (a simplex of the curve complex with integral weights). Rational (resp. integral) elements are represented on a carrying train track by rational (resp. integral) valued measures. A track is *recurrent* if each branch is in the carrying image of an essential simple closed curve. A track is *transversely recurrent* if each branch is essentially intersected by a closed curve. Train tracks provide local linearizations of $\mathcal{MGL}$ and the intersection product $i(\ ,\ )$ as follows.

Let $W(\tau)$ be the vector space of transverse measures, allowing positive and negative weights, for the track τ and $W_+(\tau)$ the cone of positive measures. If train tracks τ, τ' intersect efficiently, then the intersection product $i(\ ,\ )$ is given by a bilinear form on the product $W(\tau) \times W(\tau')$ [**PH92**, Sec. 3.4]. A piecewise symplectic structure is defined on $\mathcal{MGL}$ as follows. By splitting a track, a neighborhood of a given transverse measure in $W(\tau)$ is represented by transverse measures for a refined track τ' having only trivalent switches. For a trivalent track τ' with incoming edges $e_1(s), e_2(s)$ at the switch s, define for transverse measures $u, v \in W(\tau')$ the pairing

$$\{u, v\} = \frac{1}{2} \sum_{switches\ s} \det \begin{pmatrix} v(e_1(s)) & v(e_2(s)) \\ u(e_1(s)) & u(e_2(s)) \end{pmatrix}.$$

Basic properties of the train track parameterization are as follows.

Lemma 9.2. Train track coordinates and Thurston volume. *For a recurrent, transversely recurrent train track τ, then* $\dim W_+(\tau) = 6g - 6 + 2n$ *and the cone $W_+(\tau)$ parameterizes an open cone in $\mathcal{MGL}$. The bilinear form $\{\ ,\ \}$ for $W(\tau)$ is symplectic; the form is alternating, closed and non degenerate on $W(\tau)$. The form is unchanged under equivalence of train tracks and defines a volume element $\mu_{Thurston}$ on $\mathcal{MGL}$. On a cone $W_+(\tau)$ the Thurston volume element is a multiple of the Euclidean coordinate volume element.*

The justification for non degeneracy is as follows. For a connected, orientable train track τ, elements of $W(\tau)$ determine real-valued singular homology classes and $\{\ ,\ \}$ is the homology intersection pairing. For a general track τ, introduce a double cover of the surface with one branch point in each complementary track region. Transverse measures then lift to transverse measures for the oriented double cover of the track. The lifted measures correspond to the singular homology classes on the cover odd with respect to the covering transformation. The homology intersection pairing is natural with respect to the covering transformation; the pairing is non degenerate on the odd subspace. The original track pairing is non degenerate.

Understanding the action of MCG on $\mathcal{MGL}$ is important for understanding measured geodesic laminations. Masur established the fundamental result [**Mas85**].

THEOREM 9.3. The action of MCG on $\mathcal{MGL}$ is ergodic. *Modulo normalization, the Thurston volume element is the unique* MCG-*invariant measure in its measure class.*

The total length of measured geodesic laminations encodes the uniformization. A comparison is given between the topology and geometry of a hyperbolic surface R by considering the unit geodesic-length ball $\mathbf{B}_R = \{\mu \in \mathcal{MGL} \mid \ell_\mu(R) \leq 1\}$ and its Thurston volume $\mathbf{B}(R) = \mu_{Thurston}(\mathbf{B}_R)$. Bounds for lengths of geodesics in terms of the topology of curves lead to bounds for $\mathbf{B}(R)$. The Thurston volume is an ingredient in Mirzakhani's prime simple geodesic Theorem 10.2.

2. McShane-Mirzakhani length identity

Mirzakhani generalized McShane's identity showing that particular MCG-sums of geodesic lengths give functions constant over Teichmüller space. In the next section we explain how the identity provides a MCG-invariant partition of unity on $\mathcal{T}$ and an approach for evaluating the WP volume $\int_{\mathcal{T}/\,\mathrm{MCG}} dV$.

The basic summand for the length identity is a rational exponential function. Define the function H on $\mathbb{R}^2$ by

$$H(x,y) = \frac{1}{1+e^{\frac{x+y}{2}}} + \frac{1}{1+e^{\frac{x-y}{2}}} \tag{20}$$

and corresponding functions $\mathcal{D}, \mathcal{R}$ on $\mathbb{R}^3$ by

$$\begin{aligned} \frac{\partial}{\partial x}\mathcal{D}(x,y,z) &= H(y+z,x), \quad \mathcal{D}(0,0,0) = 0 \quad \text{and} \\ 2\,\frac{\partial}{\partial x}\mathcal{R}(x,y,z) &= H(z,x+y) + H(z,x-y), \quad \mathcal{R}(0,0,0) = 0. \end{aligned} \tag{21}$$

Let now $\mathcal{T}(L_1,\dots,L_n)$ be the Teichmüller space of genus g marked hyperbolic structures R, possibly with cusps, with n geodesic boundaries $\beta_1,\dots,\beta_n$, respectively having lengths $L_1,\dots,L_n$. We present the identity in the case of positive length boundaries. The identity has counterparts for hyperbolic surfaces with cusps and cone points [**AMS06, Bow96, McS98, McS04, Mir07a, TWZ06a**]; an overview of the sizable literature on length identities is provided in [**TWZ06b**].

THEOREM 9.4. [**Mir07a**, Thrm. 4.2] The McShane-Mirzakhani length identity. *For a hyperbolic surface R*

$$\sum_{\alpha_1,\alpha_2} \mathcal{D}(L_1, \ell_{\alpha_1}(R), \ell_{\alpha_2}(R)) \;+\; \sum_{j=2}^{n} \sum_{\alpha} \mathcal{R}(L_1, L_j, \ell_\alpha(R)) \;=\; L_1,$$

where the first sum is over all unordered pairs of simple closed geodesics with $\beta_1, \alpha_1, \alpha_2$ *bounding an embedded pair of pants, and the double sum is over simple closed geodesics with* β_1, β_j, α *bounding an embedded pair of pants.*

The length identities are based on McShane's observations on the behavior and classification of complete geodesics normal to the boundary.

Proof sketch. We follow the approach of Tan-Wong-Zhang [**TWZ06a**] and describe their considerations for the case of a single boundary β. Let $\mathcal{G}_{ns}, \mathcal{G}_{sn}, \mathcal{G}_{snn}$ be the sets of complete geodesics, as follows, with at least one endpoint normal to the boundary β. The set $\mathcal{G}_{ns}$ consists of all non simple geodesics. The set $\mathcal{G}_{sn}$ consists of all simple geodesics normal at all intersections with the boundary. The set $\mathcal{G}_{snn}$ consists of all simple geodesics not in $\mathcal{G}_{sn}$ (so somewhere also intersecting the boundary obliquely). The set $\mathcal{G}_{ns} \cup \mathcal{G}_{snn}$ is defined with the relatively open condition of a transverse or non normal intersection; the complement set $\mathcal{G}_{sn}$ is closed. The intersection of $\mathcal{G}_{ns} \cup \mathcal{G}_{snn}$ with β has full measure in β as follows. As noted in Chapter 6, the union of all simple geodesics has Hausdorff dimension 1 [**BS85**]; consequently the set of simple geodesics $\mathcal{G}_{snn} \cup \mathcal{G}_{sn}$ has zero measure intersection with β. It follows that the open intersection of $\mathcal{G}_{ns} \cup \mathcal{G}_{snn}$ with β is a union of maximal open intervals whose hyperbolic lengths sum to L_1. In fact designated pairs of maximal open intervals are in one-to-one correspondence with the simple geodesics that have both endpoints normal to β as follows. The endpoints of the elements of $\mathcal{G}_{sn}$ are topological midpoints of maximal open intervals of $\mathcal{G}_{ns} \cup \mathcal{G}_{snn}$. The endpoints of a geodesic $\gamma \in \mathcal{G}_{sn}$ separate β into segments β_+ and β_-. The closed curves $\gamma \cup \beta_+$ and $\gamma \cup \beta_-$ describe free homotopy classes respectively having geodesics α_1 and α_2. (In the special genus one case $\alpha_1 = \alpha_2$.) An ϵ-neighborhood of $\gamma \cup \beta$ is a topological pair of pants; $\beta, \alpha_1, \alpha_2$ bound an embedded geometric pair of pants. In summary designated pairs of maximal intervals are in one-to-one correspondence with pairs of pants including β as a boundary. The authors show that a pair of maximal intervals I_+ and I_- has: endpoints with normal simple geodesics spiraling left and right to the geodesics α_1 and α_2. Hyperbolic trigonometry provides that the hyperbolic lengths I_+ and I_- sum to $\mathcal{D}(L_1, \ell_{\alpha_1}, \ell_{\alpha_2})$. The considerations for the case of a single boundary are complete. The considerations for multiple boundaries includes analyzing geodesics connecting distinct boundaries. A simple geodesic γ connecting β and β_j determines a simple geodesic α, freely homotopic to the boundary of an ϵ-neighborhood of $\beta_1 \cup \gamma \cup \beta_j$. The geodesics β_1, β_j, α bound an embedded geometric pair of pants and the geodesic γ corresponds to an open interval on β_1 of length $\mathcal{R}(L_1, L_j, \ell_\alpha)$. The terms of the sums in the identity describe the maximal open subintervals of the boundary β_1. $\square$

3. Mirzakhani volume recursion

Mirzakhani discovered that the Fenchel-Nielsen construction of hyperbolic surfaces and the length identity underly a recursion for WP volume integrals. Volumes of moduli spaces can be computed using coverings by intermediate moduli spaces by considering decompositions of a surface by configurations of subsurfaces. We first describe the finite symmetry considerations for cutting open a surface and then present the recursion.

For a labeled oriented multi curve $\{\gamma_1, \ldots, \gamma_k\}$ (repetitions allowed) on the surface F, write γ for the unlabeled collection curves without orientations. The quantity γ is considered as a set of elements with multiplicities. Write $\mathrm{Stab}(\gamma)$ for the subgroup of MCG elements stabilizing the unlabeled collection of free homotopy classes of curves with multiplicities and without orientations. Similarly write $\mathrm{Stab}_0(\gamma_j)$ for the subgroup of MCG elements fixing the free homotopy class of the oriented curve γ_j. The finite quotient group $\mathrm{Sym}(\gamma) = \mathrm{Stab}(\gamma)/\cap_j \mathrm{Stab}_0(\gamma_j)$ is a subgroup of the permutation group of the set of oriented free homotopy classes of γ. The volume recursion involves the infinite covering of the moduli space by the moduli space of pairs: a hyperbolic structure and a labeled oriented multi curve. The covering will be considered in two steps according to the containments $\cap_j \mathrm{Stab}_0(\gamma_j) \subset \mathrm{Stab}(\gamma) \subset \mathrm{MCG}$. Multi curves will be used to decompose surfaces into configurations of subsurfaces. For the volume recursion the multi curve is either a single curve or a pair of curves. We will see that $\mathrm{Stab}(\gamma)$ is either the trivial group or the group $\mathbb{Z}_2$.

A second symmetry consideration involves components of $F \setminus \gamma$ that are tori with a single boundary. In particular if a curve γ_j separates off a torus with one boundary then $\mathrm{Stab}_0(\gamma_j)$ contains a half Dehn twist - an element that acts as the elliptic involution on the torus (as the conformal involution $z \mapsto -z$ when the torus is represented as a lattice $\mathbb{C}/\Lambda$; the involution reverses the orientation of lattice generators). Accordingly, in the Fenchel-Nielsen construction a fundamental interval for the twist parameter τ_j is $0 < \tau_j < \ell_j/2$, since a half twist results in attaching the same structure. Mirzakhani discusses the half twist in [**Mir07a**, pgs. 214-215] and addresses the matter in the general integration [**Mir07a**, Thrm. 7.1] and the volume recursion by inserting factors of 1/2. Mulase and Safnuk [**MS08**] discuss the matter in their Section 1 (along with the difference between the stack and orbifold definition of the moduli space) and address the matter in the volume recursion in Section 2.4 by using the value $V_1(L) = (4\pi^2 + L^2)/48$, 1/2 the value used by Mirzakhani [**Mir07a**, Sec. 5]. We use the Mulase-Safnuk normalization.

Finally volumes are computed for the symplectic form 2ω; the normalization of the 2-form is discussed in [**Wlp07**, Sec. 5] and in [**Mir07a**, Remark on pg. 180].

Statement of the volume recursion [**Mir07a**, Sec. 5]**.** Denote by $\mathcal{T}_g(L_1, \ldots, L_n)$ the Teichmüller space of genus g marked hyperbolic surfaces with geodesic boundaries $\beta_1, \ldots, \beta_n$ respectively of lengths $L_1, \ldots, L_n$ and MCG the pure mapping class group (classes of homeomorphisms preserving labeled boundaries). The WP volume $V_g(L_1, \ldots, L_n)$ of the moduli space

$\mathcal{T}_g(L_1,\dots,L_n)/\operatorname{MCG}$ is a symmetric function of boundary lengths as follows.

- For $L_1, L_2, L_3 \geq 0$, formally set
$$V_{0,3}(L_1,L_2,L_3) = 1$$
and
$$V_{1,1}(L_1) = \frac{\pi^2}{12} + \frac{L_1^2}{48}.$$
- For $L = (L_1,\dots,L_n)$, let $\widehat{L} = (L_2,\dots,L_n)$ and for $(g,n) \neq (1,1)$ or $(0,3)$, the volume satisfies
$$\frac{\partial}{\partial L_1} V_g(L) = \mathcal{A}_g^{con}(L) + \mathcal{A}_g^{dcon}(L) + \mathcal{B}_g(L)$$
where
$$\mathcal{A}_g^*(L) = \frac{1}{2}\int_0^\infty \int_0^\infty \widehat{\mathcal{A}}_g^*(x,y,L)\, xy\, dxdy$$
and
$$\mathcal{B}_g(L) = \int_0^\infty \widehat{\mathcal{B}}_g(x,L)\, x\, dx.$$
The quantities $\widehat{\mathcal{A}}_g^{con}$, $\widehat{\mathcal{A}}_g^{dcon}$ are defined in terms of the function H (see (20)) and moduli volumes for subsurfaces
$$\widehat{\mathcal{A}}_g^{con}(x,y,L) = H(x+y,L_1)V_{g-1}(x,y,\widehat{L})$$
and
$$\widehat{\mathcal{A}}_g^{dcon}(x,y,L) = \sum_{\substack{g_1+g_2=g \\ I_1 \amalg I_2 = \{2,\dots,n\}}} H(x+y,L_1)V_{g_1}(x,L_{I_1})V_{g_2}(y,L_{I_2}),$$
where in the second sum only decompositions for pairs of hyperbolic structures are considered and the unordered sets I_1, I_2 provide a partition. The third quantity $\widehat{\mathcal{B}}_g$ is defined by the sum
$$\frac{1}{2}\sum_{j=1}^{n}\big(H(x,L_1+L_j)+H(x,L_1-L_j)\big)\, V_g(x,L_2,\dots,\widehat{L_j},\dots,L_n),$$
where L_j is omitted from the argument list of V_g.

Basic point: the volume $V_g(L_1,\dots,L_n)$ is an appropriate integral of volumes for surfaces formed with *one* fewer pairs of pants.

A simplified direct consideration provides the $V_1(L)$ formula. McShane's original length identity is
$$\sum_{\alpha\ simple} \mathcal{D}(L,\ell_\alpha(R),\ell_\alpha(R)) = L,$$
where $\alpha_1 = \alpha_2$ and for a surface of topological type $(1,1)$ the free homotopy classes of simple closed curves are the MCG-orbit of any single element.

Combining the length identity; the Chapter 3 formulas $\ell_\alpha \circ h^{-1} = \ell_{h(\alpha)}$, $2\omega = d\ell \wedge d\tau$; the description of the MCG-action and that the $\mathrm{Stab}(\alpha)$ action on $\mathcal{T}_1(L)$ is generated by the Dehn twist $(\tau, \ell) \mapsto (\tau + \ell, \ell)$ (the half Dehn twist acts trivially on $\mathcal{T}_1(L)$) gives the unfolding

$$LV_1(L) = \int_{\mathcal{T}_1(L)/\,\mathrm{MCG}} \sum_{h \in \mathrm{MCG}\,/\,\mathrm{Stab}(\alpha)} \mathcal{D}(L, \ell_\alpha \circ h^{-1}, \ell_\alpha \circ h^{-1})\, 2\omega$$

$$= \int_{\mathcal{T}_1(L)/\,\mathrm{Stab}(\alpha)} \mathcal{D}(L, \ell, \ell)\, d\tau d\ell = \int_0^\infty \int_0^\ell \mathcal{D}(L, \ell, \ell)\, d\tau d\ell.$$

The definition (21) of $\mathcal{D}$ provides

$$\frac{\partial}{\partial L} LV_1(L) = \int_0^\infty \Big(\frac{1}{1 + e^{x + \frac{L}{2}}} + \frac{1}{1 + e^{x - \frac{L}{2}}} \Big)\, x dx$$

and evaluation of the integral gives

$$\frac{\partial}{\partial L} LV_1(L) = \frac{\pi^2}{6} + \frac{L^2}{8} \quad \text{or} \quad V_1(L) = \frac{\pi^2}{6} + \frac{L^2}{24},$$

consistent with the initial values for the recursion after the adjustment for the half Dehn twist.

Unfolding is the basis for the general integration and recursion. Begin with a multi curve $\gamma = \sum_{j=1}^m a_j \gamma_j$ and a function f small at infinity and introduce the MCG-invariant sum

$$f_\gamma(R) = \sum_{\mathrm{MCG}\,/\,\mathrm{Stab}(\gamma)} f\Big(\sum_{j=1}^m a_j \ell_{h(\gamma_j)}(R) \Big). \tag{22}$$

The stabilizer $\mathrm{Stab}(\gamma)$ is now the subgroup of MCG elements stabilizing the unlabeled weighted curves. Now for the hyperbolic surface R, write $R(\gamma)$ for the surface cut open on γ - each geodesic corresponds to two new labeled boundaries. The surface $R(\gamma)$ may have multiple components and additional boundaries. Let $\mathcal{T}(R)$ be the space of marked hyperbolic surfaces R and $\mathcal{T}(R(\gamma); \mathbf{x})$ the space of marked hyperbolic surfaces $R(\gamma)$, where the pair of boundaries for γ_j have length x_j, $\mathbf{x} = (x_1, \ldots, x_m)$. Denote by $\mathrm{MCG}(R(\gamma))$ the product of mapping class groups of the components of $R(\gamma)$ and $\mathcal{T}(R(\gamma); \mathbf{x})/\,\mathrm{MCG}(R(\gamma))$ the corresponding moduli space. For the product of symplectic forms corresponding to the components of $R(\gamma)$, the volume $V(R(\gamma); \mathbf{x})$ is the product of the volumes for the components, where the pair of boundaries for γ_j have common length x_j (and using the Mulase-Safnuk value for $V_1(L)$). The following is the basis for the integration recursion.

THEOREM 9.5. [**Mir07a**, Thrm. 7.1] *For a weighted multi curve* $\gamma = \sum_{j=1}^m a_j \gamma_j$ *and the* MCG*-sum of a function* f*, small at infinity, then*

$$\int_{\mathcal{T}(R)/\,\mathrm{MCG}} f_\gamma\, dV = (|Sym(\gamma)|)^{-1} \int_{\mathbb{R}^m_{>0}} f(|\mathbf{x}|) V(R(\gamma); \mathbf{x})\, \mathbf{x} \cdot d\mathbf{x}$$

for $|\mathbf{x}| = \sum_j a_j x_j$ *and* $\mathbf{x} \cdot d\mathbf{x} = x_1 \cdots x_m dx_1 \cdots dx_m$.

Proof sketch. There is a split short exact sequence for mapping class groups, corresponding to the components of $R(\gamma)$,

$$1 \longrightarrow \prod_j \operatorname{Dehn}(\gamma_j) \longrightarrow \bigcap_j \operatorname{Stab}_0(\gamma_j) \longrightarrow \prod_{R(\gamma)\ \text{components}} \operatorname{MCG}(R') \longrightarrow 1$$

and from Fenchel-Nielsen coordinates a factorization for Teichmüller space

$$\mathcal{T}(R) = \prod_{R(\gamma)\ \text{components}} \mathcal{T}(R') \times \prod_{\gamma_j} \mathbb{R}_{>0} \times \mathbb{R}.$$

(The short exact sequence places half twists in the mapping class groups of the tori with single boundaries.) The $d\ell \wedge d\tau$ formula, Theorem 3.14, provides that the second factorization describes a product of symplectic manifolds.

To establish the integral formula, first write coset sums as

$$\sum_{\operatorname{MCG}/\cap_j \operatorname{Stab}_0(\gamma_j)} f \;=\; \sum_{\operatorname{MCG}/\operatorname{Stab}(\gamma)} \;\sum_{\operatorname{Stab}(\gamma)/\cap_j \operatorname{Stab}_0(\gamma_j)} f \;=\; |\operatorname{Sym}(\gamma)|\, f_\gamma$$

using that f is $\operatorname{Sym}(\gamma)$ invariant for the second equality. Substitute the resulting formula for f_γ, and unfold the sum to obtain the equality

$$\int_{\mathcal{T}(R)/\operatorname{MCG}} f_\gamma\, dV = (|\operatorname{Sym}(\gamma)|)^{-1} \int_{\mathcal{T}(R)/\cap_j \operatorname{Stab}_0(\gamma_j)} f\, dV.$$

Substitute the factorization of spaces

$$\mathcal{T}(R)/\bigcap_j \operatorname{Stab}_0(\gamma_j) = \prod_{R(\gamma)\ \text{components}} \mathcal{T}(R')/\operatorname{MCG}(R') \times \prod_{\gamma_j} (\mathbb{R}_{>0} \times \mathbb{R})/\operatorname{Dehn}_*(\gamma_j)$$

where $\operatorname{Dehn}_*(\gamma_j)$ is generated by a half twist if the curve bounds a torus with a single boundary and otherwise is generated by a simple twist. Then substitute the factorization of the volume element

$$dV = \prod_{R(\gamma)\ \text{components}} dV(R') \times \prod_{\gamma_j} d\ell_j \wedge d\tau_j.$$

The function f depends only on the values $\mathbf{x}$. For the values $\mathbf{x}$ fixed, perform the $\prod \mathcal{T}(R')/\operatorname{MCG}(R')$ integration to obtain the product volume $V(R(\gamma); \mathbf{x})$. Finally $\operatorname{Dehn}_*(\gamma_j)$ acts only on the variable τ_j with fundamental domain $0 < \tau_j < \ell_j/2$ if γ_j bounds a torus with a single boundary or otherwise with fundamental domain $0 < \tau_j < \ell_j$. For a torus with a single boundary, the action is accounted for by using a value $V_1(L)$ that is 1/2 the integral. The right hand side of the formula is established. □

The first application is establishing the recursion formula for volumes of moduli spaces. The length identity, Theorem 9.4, has the form of MCG-sums. By a counterpart of Theorem 5.11, the length identity sums are a finite number of MCG-orbit sums with the $\mathcal{D}, \mathcal{R}$ summands depending on the length of a multi curve. We sketch the approach, beginning with the Teichmüller space $\mathcal{T}(L_1, \dots, L_n)$ of surfaces R with boundaries $\beta_1, \dots, \beta_n$

and integrating the length identity over $\mathcal{T}(L_1,\dots,L_n)/\operatorname{MCG}$. The *right hand side* is $L_1V_g(L_1,\dots,L_n)$. The *left hand side* $\mathcal{D}$-sum is a finite sum of MCG-sums as follows. Each MCG-sum corresponds to a topological configuration (a MCG-orbit) for the pair of pants bounded by β_1,α_1 and α_2. The first configuration is the *connected case* that $R\setminus P$ is connected, genus $g-1$, with boundaries $\alpha_1,\alpha_2,\beta_2,\dots,\beta_n$. (Elements of MCG interchange the labeled curves α_1 and α_2 and $\operatorname{Sym}(\beta_1,\alpha_1,\alpha_2)=2$; in effect the unlabeled pair is counted a single time.) Theorem 9.5 is applied and from (21) the $\frac{\partial}{\partial L_1}$ derivative inside the integral replaces $\mathcal{D}(L_1,\ell_{\alpha_1},\ell_{\alpha_2})$ with $H(\ell_{\alpha_1}+\ell_{\alpha_2},L_1)$. The resulting product of H and the moduli volume is $\widehat{\mathcal{A}}_g^{con}(\ell_{\alpha_1},\ell_{\alpha_2},L)$ and the integral is $\mathcal{A}_g^{con}(L)$. The *disconnected case* is that $R\setminus P$ has two components with combined boundaries $\alpha_1,\alpha_2,\beta_2,\dots,\beta_n$. The finite number of configurations modulo the action of $\operatorname{MCG}(R)$ are indexed exactly by the partitions of the genera $g_1+g_2=g$ of the $R\setminus P$ components and the partitions of the boundaries $\beta_2,\dots,\beta_n$ onto the first and second components. (The group $\operatorname{Sym}(\beta_1,\alpha_1,\alpha_2)$ is trivial since the labeled boundaries already distinguish the components.) Theorem 9.5 is applied and from (21) the $\frac{\partial}{\partial L_1}$ derivative inside the integral replaces $\mathcal{D}(L_1,\ell_{\alpha_1},\ell_{\alpha_2})$ with $H(\ell_{\alpha_1}+\ell_{\alpha_2},L_1)$. The surface $R\setminus P$ is disconnected and the moduli volume is a product of volumes. The sum over configurations of the resulting product of H and volumes is $\widehat{\mathcal{A}}_g^{dcon}(\ell_{\alpha_1},\ell_{\alpha_2},L)$ and the integral is $\mathcal{A}_g^{dcon}(L)$. The contribution of the $\mathcal{R}$-sums is similar. The contribution of the pair of boundaries β_1,β_j is a single MCG-orbit. The surface $R\setminus P$ is connected, genus g, with boundaries $\alpha,\beta_2,\dots,\widehat{\beta_j},\dots,\beta_n$. (The group $\operatorname{Sym}(\beta_1,\beta_j,\alpha)$ is trivial.) Again Theorem 9.5 and (21) are applied for the $\frac{\partial}{\partial L_1}$ derivative, to replace $\mathcal{R}(L_1,L_j,\ell_\alpha)$ with $(H(\ell_\alpha,L_1+L_j)+H(\ell_\alpha,L_1-L_j))/2$. The resulting sum of products of H and volumes is $\widehat{\mathcal{B}}_g(\ell_\alpha,L)$ and the integral is $\mathcal{B}_g(L)$. The volume recursion is established. The example $V_1(L)$ calculation follows the general calculation scheme provided we begin with the normalization $V_{0,3}(L_1,L_2,L_3)=1$ (the half twist acts trivially on $\mathcal{T}_1(L)$ and there is no attaching; there is no factor of $1/2$).

4. Moduli volumes, symplectic reduction and tautological classes

4.1. Volumes of moduli spaces. The first matter is to understand the integrals occurring in the recursion. The length identity is given in terms of the function $H(x,y)$. The recursion involves integrating one and two boundary lengths. The resulting integrals

$$(23)\qquad \int_0^\infty x^{2j+1}H(x,t)dx \quad\text{and}\quad \int_0^\infty\int_0^\infty x^{2j+1}y^{2k+1}H(x+y,t)dxdy$$

are by calculation each polynomials in t^2, the first of degree $j+1$ and the second of degree $j+k+2$, with all coefficients positive rational multiples of powers of π. The result is summarized in the following.

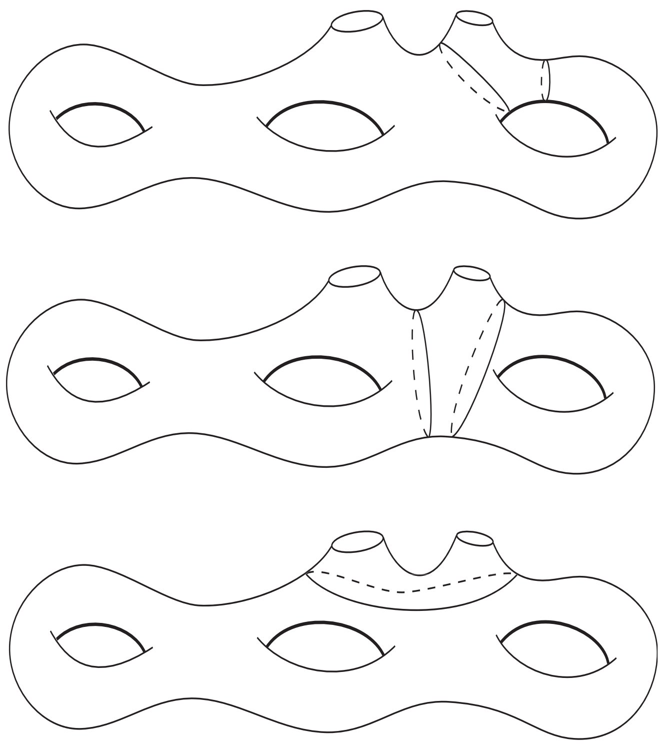

FIGURE 3. Connected, disconnected and boundary pair configurations.

THEOREM 9.6. [**Mir07a**, Sec. 6] *The volume function* $V_g(L)$ *is a polynomial of total degree* $6g-6+2n$ *in the boundary lengths as follows*

$$V_g(L) = \sum_{|j|\leq 3g-3+n} v_j L^{2j}$$

where L *is the tuple of lengths,* j *is a* n*-multi index of non negative integers,* $|j|$ *is the sum of values, and the coefficient* $v_j > 0$ *lies in* $\pi^{6g-6+2n-|j|}\mathbb{Q}$*.*

The result fore shadows the symplectic reduction and positive intersection numbers of characteristic classes on the moduli space, discussed in the paragraphs below. In particular the polynomial coefficients are intersection numbers for tautological classes on the compactified moduli space $\overline{\mathcal{M}}_{g,n}$. The considerations give volume formulas for $\mathcal{M}_{g,n} \setminus \mathcal{M}_{g,n}(\epsilon)$, the set of surfaces with systole at most ϵ. In particular for ϵ small, the geodesics of length at most ϵ on a surface R prescribe a multi curve and a cutting open $R(\gamma)$. The volume of the neighborhood of surfaces with the given multi curve of length at most ϵ is computed using the product structure description, as in the proof of Theorem 9.5. The volumes $V(R(\gamma); L)$ of moduli level sets for the lengths of the multi curve are integrated $\prod_j d\tau_j d\ell_j$ to obtain the volume of a neighborhood. Symmetries must be accounted for and the inclusion-exclusion principle applied to account for neighborhoods overlapping, corresponding to inclusions of multi curves. *Basic point:* the volume of $\mathcal{M}_{g,n} \setminus \mathcal{M}_{g,n}(\epsilon)$ is a polynomial in ϵ of degree $6g - 6 + 2n$.

Investigation of the volume recursion is connected to the larger investigation of Witten-Kontsevich theory for intersection numbers and Korteweg-de Vries (KdV) hierarchies [**Kon92, Wit91, Wit92**]. There is extensive research and literature on the subject; a beginning reference is Looijenga's Bourbaki Seminar [**Loo93**]. A brief update is included in the survey [**LV10**]. We now mention immediate considerations for the volume recursion. Recursions for calculation of WP volumes and intersection numbers are simplified and generalized by Liu-Xu [**LX09a, LX09b**] and Mulase-Safnuk [**MS08**]. The Tan-Wong-Zhang analysis for the length identity also covers the case of hyperbolic structures with cone points of arbitrary angle [**TWZ06a**]. The identity for cone points is given by substituting $i\phi$ in place of a length ℓ, where ϕ is a cone angle. Manin and Zograf investigate intersection numbers and volumes in [**MZ00**]. Do and Norbury find that the volume recursion generalizes to the case of cone points, including the important application of cone angle 2π - describing eliminating a cone point [**DN09**]. Zograf experimentally considers the large genus asymptotics of volumes [**Zog08**].

4.2. Symplectic reduction and tautological classes. The construction begins by describing principal S^1-bundles [**MS74, Mir07b**]. Considerations are for orbifolds, spaces that locally are the quotient of a manifold by a finite group action. Introduce $\widehat{\mathcal{T}}_g(L_1, \dots, L_n)$ the space of genus g marked hyperbolic structures with n pointed geodesic boundaries $(\beta_1, p_1), \dots, (\beta_n, p_n)$; p_j is a point on the boundary β_j of length L_j. For simplicity of discussion we assume no cusps. The projection of $\widehat{\mathcal{T}}_g(L_1, \dots, L_n)$ to $\mathcal{T}_g(L_1, \dots, L_n)$ gives the former the structure of a principal $(S^1)^n$-bundle over $\mathcal{T}_g(L_1, \dots, L_n)$. The family $\widehat{\mathcal{T}}_g(L_1, \dots, L_n)$ in L is a manifold of $\mathbb{R}$-dimension $6g - 6 + 2n$; the location of the points on the boundaries provides n additional degrees of freedom. A formal Fenchel-Nielsen twist parameter τ_j is introduced to describe the location of p_j on β_j.

A model for $\widehat{\mathcal{T}}_g(L_1, \dots, L_n)$ is given by the Teichmüller space $\mathcal{T}_g^{\beta}(0, \dots, 0)$ for genus g marked hyperbolic structures with $2n$ cusps (boundaries of length zero) and a distinguished multi curve β has length tuple L. In particular at the boundary β_j of a hyperbolic structure, a pair of pants P_j with boundary length L_j and two cusps is attached using the twist parameter value τ_j. The multi curve for attaching pants is $\beta = (\beta_1, \dots, \beta_n)$. By Theorem 3.14, the partial augmentation $\overline{\mathcal{T}}_g^{\beta}(0, \dots, 0)$, where only strata simplices disjoint from β are adjoined, is symplectically isomorphic to the augmentation of the pointed boundary Teichmüller space $\overline{\widehat{\mathcal{T}}}_g(L_1, \dots, L_n)$.

The space $\mathcal{T}_g^{\beta}(0, \dots, 0)$ is a principal $(S^1)^n$-bundle over $\mathcal{T}_g(L_1, \dots, L_n)$. The fibers of the projection are the formal products $\beta_1 \times \cdots \times \beta_n$ of boundaries. The description extends to the partial augmentation $\overline{\mathcal{T}}_g^{\beta}(0, \dots, 0)$ over the augmentation $\overline{\mathcal{T}}_g(L_1,\dots,L_n)$, where for the former only strata simplices disjoint from β are adjoined. The geometry is as follows. Introduce MCG-invariant connections for the principal S^1-structures to define horizontal and vertical subspaces of $T\overline{\mathcal{T}}^{\beta}_g(0,\dots,0)$. The Fenchel-Nielsen gauges of Definition 4.8 are examples of suitable MCG-invariant connections. The connection 1-forms describe a $(S^1)^n$-invariant horizontal subspace of $T\overline{\mathcal{T}}^{\beta}_g(0,\dots,0)$ of infinitesimal deformations not varying any β_j twist parameter. The curvature forms $\mathbf{c}_1(\beta_j)$ of the connections represent the cohomology classes of the S^1-bundle factors of $\overline{\mathcal{T}}^{\beta}_g(0,\dots,0)/\operatorname{MCG}$. Each space $\overline{\mathcal{T}}^{\beta}_g(0,\dots,0)$ and $\mathcal{T}_g(L_1,\dots,L_n)$ has a symplectic form. By definition the Fenchel-Nielsen vector field $L_jt_{\beta_j}$ on $\overline{\mathcal{T}}^{\beta}_g(0,\dots,0)$ is the infinitesimal generator $2\pi\frac{d}{d\theta}$ of S^1 acting on β_j. By Theorem 3.3, each vector field has a Hamiltonian potential $L_j^2/2$ and the tuple $\mu:\overline{\mathcal{T}}^{\beta}_g(0,\dots,0)\mapsto(L_1^2/2,\dots,L_n^2/2)$ is the *moment map* for the $(S^1)^n$-action [**Gui94**].

It is important to understand how the symplectic geometry of the level sets of μ varies. For this purpose the $(S^1)^n$-quotients of the level sets are considered. By construction the maps and $(S^1)^n$-action are MCG equivariant. A form of the Duistermaat-Heckman theorem on the pushforward of the symplectic form by the moment map for a torus action applies [**Gui94**]. The Normal Form Theorem provides for ω_L the symplectic form on $\overline{\mathcal{T}}^{\beta}_g(L_1,\dots,L_n)$ and the forms $\omega_{\mathbf{0}}$, $\mathbf{c}_1(\beta_j)$ on $\overline{\mathcal{T}}_g(0,\dots,0)$, an equivalence in cohomology (symplectic reduction) for the MCG orbifold quotients

$$2\omega_L \;\equiv\; 2\omega_{\mathbf{0}} + \sum_{j=1}^{n}\frac{L_j^2}{2}\,\mathbf{c}_1(\beta_j)$$

[**Mir07b**, Sec. 3, Thrm. 3.2]. *Basic point:* the $\overline{\mathcal{T}}_g(L_1,\dots,L_n)/\operatorname{MCG}$ volume is given by integrating the top exterior power of the right hand side over $\overline{\mathcal{T}}_g(0,\dots,0)/\operatorname{MCG}$. An application is again that the volume of $\overline{\mathcal{T}}_g(L_1,\dots,L_n)/\operatorname{MCG}$ is a polynomial in the variables L_j^2. We next discuss that the polynomial coefficients are intersection numbers for the quotient $\overline{\mathcal{T}}^{\beta}_g(0,\dots,0)/\operatorname{MCG}=\overline{\mathcal{M}}_{g,n}$.

The cohomology classes ω and $\mathbf{c}_1(\beta_j)$ have complex geometry descriptions on the Deligne-Mumford compactification [**AC96**]. As described in Chapter 7, Section 4 the symplectic form is the π^2 multiple of κ_1, the pushdown of the square of the relative dualizing sheaf. The extension of the relation to the Deligne-Mumford compactification is presented in [**Wlp90b**]. A complex geometry description of the above principal S^1-bundles begins by attaching a unit Euclidean disc to the boundary β_j, using proportional arc length. The point p_j on β_j canonically describes a non zero tangent vector at the origin of the disc. The origin is interpreted as a puncture. The construction of the Deligne-Mumford compactification ensures that the punctures are disjoint from cusp pairs. The Thurston right twist orientation of Chapter 3, Section 2, describes counter clockwise rotation of the tangent vector. The S^1-bundle and $\mathbb{C}$-tangent lines along punctures are oppositely oriented associated oriented bundles with homotopic structure groups S^1 and $\mathbb{C}^*$. The principal S^1-bundle and $\mathbb{C}$-cotangent lines along punctures have equal cohomology classes on $\overline{\mathcal{M}}_{g,n}$. In particular $\mathbf{c}_1(\beta_j)$ is the tautological class ψ_j, the class of the cotangent lines along the j^{th} puncture [**AC96, Mir07b**].

Considerations now combine to give an expansion for WP volume in terms of tautological intersection numbers.

Theorem 9.7. [**Mir07b**, Thrm. 4.4],[**MS08**, Sec. 2.6] Volume and intersection numbers. *The volume of $\mathcal{T}_g(L)/\operatorname{MCG}$ relative to the symplectic form $2\omega_L$ is a polynomial in $L_1^2, \ldots, L_n^2$ of total degree $d = 3g-3+n$ as follows*

$$\frac{V_g(2\pi L)}{(2\pi^2)^d} = \frac{1}{d!}\int_{\overline{\mathcal{M}}_{g,n}} \Big(\kappa_1 + \sum_{j=1}^{n} L_j^2\,\psi_j\Big)^d$$
$$= \sum_{d_0+\cdots+d_n=d} \prod_{j=0}^{n} \frac{1}{d_j!}\,\langle \kappa_1^{d_0} \prod_{j=1}^{n} \tau_{d_j}\rangle_{g,n} \prod_{j=1}^{n} L_j^{2d_j}$$

for the formal variable $\tau_{d_j} = \psi_j^{d_j}$; for $\langle\ \ \rangle_{g,n}$ denoting the $\overline{\mathcal{M}}_{g,n}$ integration of a $2d$-form and otherwise defined as zero.

Theorem 9.6 summarizes the recursion of integrals for computing coefficients of the polynomial in L. Theorem 9.7 gives the coefficients as intersection numbers of the tautological classes ψ_j and κ_1 on $\overline{\mathcal{M}}_{g,n}$. An immediate consequence is that the intersection numbers are positive and satisfy the stated recursion.

5. Virasoro constraint equations and Witten-Kontsevich theory

Witten conjectured two systems of generating relations for the intersection pairings

$$\langle \tau_{d_1}\cdots\tau_{d_n}\rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\cdots\psi_n^{d_n} \quad \text{for } d_1+\cdots+d_n = d$$

in his theory of two-dimensional quantum gravity (for $d_1+\cdots+d_n \neq d$, the product is defined as zero). He conjectured that the partition function $\mathbf{F}$ for the gravity theory satisfies both the Virasoro constraint equations and a Korteweg-de Vries hierarchy [**Wit91, Wit92**]. The partition function, a generating function for the $\langle \tau_{d_1}\cdots\tau_{d_n}\rangle_g$ intersection numbers, is defined as follows

$$\mathbf{F} = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g$$

for

$$F_g(t_0,t_1,\dots) = \sum_{\{d_j\}} \langle\prod_{j=1}^{n}\tau_{d_j}\rangle_g \prod_{r=1}^{\infty}\frac{t_r^{n_r}}{n_r!}$$

where the sum is over all sequences of non negative integers $\{d_j\}$ with finitely many non zero terms and $n_r = \#\{j \mid d_j = r\}$. The generating function encodes the intersection numbers of ψ classes alone. Witten recognized that the intersection numbers of κ_1 and the ψ classes could be expressed in terms of intersection numbers of ψ classes alone [**AC96, AC98, GP98**]. Two basic intersection pairing relations give the effect of removing the 0^{th} distinguished point with the factor ψ_0^0 or ψ_0^1

- String equation $\langle\tau_0\tau_{d_1}\cdots\tau_{d_n}\rangle_g = \sum_{d_j>0}\langle\tau_{d_1}\cdots\tau_{d_j-1}\cdots\tau_{d_n}\rangle_g$,
- Dilaton equation $\langle\tau_1\tau_{d_1}\cdots\tau_{d_n}\rangle_g = (2g+n-2)\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_g$.

From the left hand to the right hand side: the number of punctures and sum of exponents decrease by one. The geometry of the equations is discussed in the section on Witten-Kontsevich theory [**HM98**, pgs. 71-75].

Genus 0 provides an example. For $n = 3$, given L there is a unique hyperbolic structure and the moduli dimension is $d = 0$. The $n = 3$ normalizations are $\langle\tau_0\tau_0\tau_0\rangle_0 = 1$ corresponding to $V_{0,3}(2\pi L)/(2\pi^2)^0 = 1$. For $d = n-3 > 0$, from Theorem 9.7 the leading coefficients formula is

$$\Big(\frac{V_0(2\pi L)}{(2\pi^2)^d}\Big)_{leading} = \frac{1}{d!}\sum_{\{d_j\}}\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_0.$$

The general genus 0 intersection formula $\langle\prod_{j=1}^n\tau_{d_j}\rangle_0 = (n-3)!/\prod_{j=1}^n d_j!$ is established as follows. Since $d < n$, it is possible to write the intersection product for n punctures in the form $\langle\tau_0\prod_{j=2}^n\tau_{d_j}\rangle_0$ with $d_2+\cdots+d_n = n-3$. Apply the String equation to obtain the sum

$$\sum_{d_j>0}\langle\tau_{d_2}\cdots\tau_{d_j-1}\cdots\tau_{d_n}\rangle_0$$

and using an induction on d, apply the intersection formula to the summands to obtain $\sum_{d_j>0}(n-4)!d_j/\prod d_j!$, and apply $d_2+\cdots+d_n = n-3$ for the conclusion.

The Witten conjecture was established by Kontsevich with his matrix Airy integral model [**Kon92, Loo93**]. Different approaches to the Witten conjectures have been developed [**CLL08, KL07, KL09, Mir07b, OP09**]. Mondello establishes pointwise convergence, as boundary lengths tend to infinity, of the WP Poisson structure to Kontsevich's piecewise linear Poisson structure on the arc complex (a variant of the curve complex) [**Mon09**]. Mirzakhani's approach for the conjecture combines the volume recursion, calculation of the integrals (23) and manipulation of sums to show that $exp(\mathbf{F})$ is annihilated by a Virasoro Lie algebra of operators [**Mir07b**, Thrm. 6.1]. In particular the ψ intersection number relations encoded in the volume recursion imply the relations encoded in the Virasoro constraint equations.

From Theorem 9.7 the ψ intersection numbers are the coefficients of the highest degree terms of the volume polynomials $V_g(L)$.

Mulase and Safnuk consider a generating function for the intersections of combinations of the κ_1 and ψ classes [**MS08**]

$$\mathbf{G}(s,t_0,t_1,\dots) = \sum_g \langle e^{s\kappa_1+\sum t_j\tau_j}\rangle_g = \sum_g \sum_{m,\{d_j\}} \langle \kappa_1^m \tau_0^{d_0}\tau_1^{d_1}\cdots\rangle_g \frac{s^m}{m!}\prod_{j=1}^{\infty}\frac{t_j^{n_j}}{n_j!}$$

where again $\langle\ \ \rangle_g$ is defined as zero for products other than $2d$-forms. Following Mirzakhani's approach, Mulase and Safnuk prove the following by combining the volume recursion and manipulation of sums.

THEOREM 9.8. [**MS08**, Thrm. 1.1] Virasoro constraints. *For each $k \geq -1$, define*

$$\mathcal{V}_k = -\frac{1}{2}\sum_{i=0}^{\infty}(2(i+k)+3)!!\frac{(-2s)^i}{(2i+1)!}\frac{\partial}{\partial t_{i+k+1}}+\frac{1}{2}\sum_{j=0}^{\infty}\frac{(2(j+k)+1)!!}{(2j-1)!!}t_j\frac{\partial}{\partial t_{j+k}}$$
$$+\frac{1}{4}\sum_{\substack{d_1+d_2=k-1\\ d_1,d_2\geq 0}}(2d_1+1)!!(2d_2+1)!!\frac{\partial^2}{\partial t_{d_1}t_{d_2}}+\frac{\delta_{k,-1}t_0^2}{4}+\frac{\delta_{k,0}}{48},$$

for the double factorial and Kronecker delta function $\delta_{*,*}$. Then

- *the operators $\mathcal{V}_k$ satisfy the Virasoro commutator relations $[\mathcal{V}_n,\mathcal{V}_m] = (n-m)\mathcal{V}_{n+m}$;*
- *the generating function $\mathbf{G}$ satisfies $\mathcal{V}_k\, exp(\mathbf{G}) = 0$ for $k \geq -1$.*

The second system of equations uniquely determines the generating function and also provides for evaluation of all intersection pairings.

In a direct display that the information of intersection numbers for κ_1 and ψ classes is equivalent to the information of intersection numbers for ψ classes, Mulase and Safnuk show that

$$\mathbf{G}(s,t_0,t_1,t_2,t_3\dots) = \mathbf{F}(t_0,t_1,t_2+\gamma_2,t_3+\gamma_3,\dots),$$

where $\gamma_j = -(-s)^{j-1}/(2j+1)j!$ [**MS08**, Thrm. 1.2]. Kaufmann, Manin and Zograf also consider the equivalence of intersection numbers in [**KMZ96**]. Further intersection number relations are developed in the work of Dijkgraaf-Verlinde-Verlinde [**DV91, DVV91**], the work of Manin-Zograf [**MZ00**] and the work of Liu-Xu [**LX07, LX09a**].