Introduction

The early history of mirror symmetry has been told many times; we will only summarize it briefly here. The story begins with the introduction of Calabi-Yau compactifications in string theory in 1985 [**11**]. The idea is that, since superstring theory requires a ten-dimensional space-time, one reconciles this with the observed universe by requiring (at least locally) that space-time take the form

$$\mathbb{R}^{1,3} \times X,$$

where $\mathbb{R}^{1,3}$ is usual Minkowski space-time and X is a very small six-dimensional Riemannian manifold. The desire for the theory to preserve the supersymmetry of superstring theory then leads to the requirement that X have SU(3) holonomy, i.e., be a Calabi-Yau manifold. Thus string theory entered the realm of algebraic geometry, as any non-singular projective threefold with trivial canonical bundle carries a metric with SU(3) holonomy, thanks to Yau's proof of the Calabi conjecture [**113**].

This generated an industry in the string theory community devoted to producing large lists of examples of Calabi-Yau threefolds and computing their invariants, the most basic of which are the Hodge numbers $h^{1,1}$ and $h^{1,2}$.

In 1989, a rather surprising observation came out of this work. Candelas, Lynker and Schimmrigk [**12**] provided a list of Calabi-Yau hypersurfaces in weighted projective space which exhibited an obvious symmetry: if there was a Calabi-Yau threefold with Hodge numbers given by a pair $(h^{1,1}, h^{1,2})$, then there was often also one with Hodge numbers given by the pair $(h^{1,2}, h^{1,1})$. Independently, guided by certain observations in conformal field theory, Greene and Plesser [**36**] studied the quintic threefold and its mirror partner. If we let X_ψ be the solution set in $\mathbb{P}^4$ of the equation

$$x_0^5 + \cdots + x_4^5 - \psi x_0 x_1 x_2 x_3 x_4 = 0$$

for $\psi \in \mathbb{C}$, then for most ψ, X_ψ is a non-singular quintic threefold, and as such, has Hodge numbers

$$h^{1,1}(X_\psi) = 1, \quad h^{1,2}(X_\psi) = 101.$$

On the other hand, the group

$$G = \frac{\{(a_0, \ldots, a_4) | a_i \in \mu_5, \prod_{i=0}^4 a_i = 1\}}{\{(a,a,a,a,a) | a \in \mu_5\}}$$

acts diagonally on $\mathbb{P}^4$, via

$$(x_0, \ldots, x_4) \mapsto (a_0 x_0, \ldots, a_4 x_4).$$

Here μ_5 is the group of fifth roots of unity. This action restricts to an action on X_ψ, and the quotient X_ψ / G is highly singular. However, these singularities can be resolved via a proper birational morphism $\check{X}_\psi \to X_\psi/G$ with $\check{X}_\psi$ a new Calabi-Yau threefold with Hodge numbers

$$h^{1,1}(\check{X}_\psi) = 101, \quad h^{1,2}(\check{X}_\psi) = 1.$$

These examples were already a surprise to mathematicians, since at the time very few examples of Calabi-Yau threefolds with positive Euler characteristic were known (the Euler characteristic coinciding with $2(h^{1,1} - h^{1,2})$).

Much more spectacular were the results of Candelas, de la Ossa, Green and Parkes [**10**]. Guided by string theory and path integral calculations, Candelas et al. conjectured that certain period calculations on the family $\check{X}_\psi$ parameterized by ψ would yield predictions for numbers of rational curves on the quintic threefold. They carried out these calculations, finding agreement with the known numbers of rational curves up to degree 3. We omit any details of these calculations here, as they have been exposited in many places, see e.g., [**43**]. This agreement was very surprising to the mathematical community, as these numbers become increasingly difficult to compute as the degree increases. The number of lines, 2875, was known in the 19th century, the number of conics, 609250, was computed only in 1986 by Sheldon Katz [**66**], and the number of twisted cubics, 317206375, was only computed in 1990 by Ellingsrud and Strømme [**22**].

Throughout the history of mathematics, physics has been an important source of interesting problems and mathematical phenomena. Some of the interesting mathematics that arises from physics tends to be a one-off — an interesting and unexpected formula, say, which once verified mathematically loses interest. Other contributions from physics have led to powerful new structures and theories which continue to provide interesting and exciting new results. I like to believe that mirror symmetry is one of the latter types of subjects.

The conjecture raised by Candelas et al., along with related work, led to the study of Gromov-Witten invariants (defining precisely what we mean by "the number of rational curves") and quantum cohomology, a way of deforming the usual cup product on cohomology using Gromov-Witten invariants. This remains an active field of research, and by 1996, the theory was sufficiently developed to allow proofs of the mirror symmetry formula for the quintic by Givental [**34**], Lian, Liu and Yau [**75**] and subsequently others, with the proofs getting simpler over time.

Concerning mirror symmetry, Batyrev [**6**] and Batyrev-Borisov [**7**] gave very general constructions of mirror pairs of Calabi-Yau manifolds occurring as complete intersections in toric varieties. In 1994, Maxim Kontsevich [**68**] made his fundamental Homological Mirror Symmetry conjecture, a profound effort to explain the relationship between a Calabi-Yau manifold and its mirror in terms of category theory.

In 1996, Strominger, Yau and Zaslow proposed a conjecture, [**108**], now referred to as the SYZ conjecture, suggesting a much more concrete geometric relationship between mirror pairs; namely, mirror pairs should carry dual special Lagrangian fibrations. This suggested a very explicit relationship between a Calabi-Yau manifold and its mirror, and initial work in this direction by myself [**37, 38, 39**] and Wei-Dong Ruan [**97, 98, 99**] indicates the conjecture works at a topological level. However, to date, the analytic problems involved in proving a full-strength version of the SYZ conjecture remain insurmountable. Furthermore, while a proof of the SYZ conjecture would be of great interest, a proof alone will not explain the finer aspects of mirror symmetry. Nevertheless, the SYZ conjecture has motivated several points of view which appear to be yielding new insights into mirror symmetry: notably, the rigid analytic program initiated by Kontsevich and Soibelman in [**69, 70**] and the program developed by Siebert and myself using log geometry, [**47, 48, 51, 49**].

These ideas which grew out of the SYZ conjecture focus on the base of the SYZ fibration; even though we do not know an SYZ fibration exists, we have a good guess as to what these bases look like. In particular, they should be *affine manifolds*, i.e., real manifolds with an atlas whose transition maps are affine linear transformations. In general, these manifolds have a singular locus, a subset not carrying such an affine structure. It is not difficult to write down examples of such manifolds which we expect to correspond, say, to hypersurfaces in toric varieties.

More precisely,

DEFINITION 0.1. An *affine manifold* B is a real manifold with an atlas of coordinate charts

$$\{\psi_i : U_i \to \mathbb{R}^n\}$$

with $\psi_i \circ \psi_j^{-1} \in \mathrm{Aff}(\mathbb{R}^n)$, the affine linear group of $\mathbb{R}^n$. We say B is *tropical* (respectively *integral*) if $\psi_i \circ \psi_j^{-1} \in \mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{Z}) \subseteq \mathrm{Aff}(\mathbb{R}^n)$ (respectively $\psi_i \circ \psi_j^{-1} \in \mathrm{Aff}(\mathbb{Z}^n)$, the affine linear group of $\mathbb{Z}^n$).

In the tropical case, the linear part of each coordinate transformation is integral, and in the integral case, both the translational and linear parts are integral.

Given a tropical manifold B, we have a family of lattices $\Lambda \subseteq \mathcal{T}_B$ generated locally by $\partial/\partial y_1, \ldots, \partial/\partial y_n$, where $y_1, \ldots, y_n$ are affine coordinates. The condition on transition maps guarantees that this is well-defined. Dually, we have a family of lattices $\check{\Lambda} \subseteq \mathcal{T}_B^*$ generated by $dy_1, \ldots, dy_n$, and then we get two torus bundles

$$f : X(B) \to B$$
$$\check{f} : \check{X}(B) \to B$$

with

$$X(B) = \mathcal{T}_B/\Lambda, \quad \check{X}(B) = \mathcal{T}_B^*/\check{\Lambda}.$$

Now $X(B)$ carries a natural complex structure. Sections of Λ are flat sections of a connection on $\mathcal{T}_B$, and the horizontal and vertical tangent spaces of this connection are canonically isomorphic. Thus we can write down an almost complex structure J which interchanges these two spaces, with an appropriate sign-change so that $J^2 = -\mathrm{id}$. It is easy to see that this almost complex structure on $\mathcal{T}_B$ is integrable and descends to $X(B)$.

On the other hand, $\mathcal{T}_B^*$ carries a canonical symplectic form which descends to $\check{X}(B)$, so $\check{X}(B)$ is canonically a symplectic manifold.

We can think of $X(B)$ and $\check{X}(B)$ as forming a mirror pair; this is a simple version of the SYZ conjecture. In this simple situation, however, there are few interesting compact examples, in the Kähler case being limited to the possibility that $B = \mathbb{R}^n/\Gamma$ for a lattice Γ (shown in [**15**]). Nevertheless, we can take this simple case as motivation, and ask some basic questions:

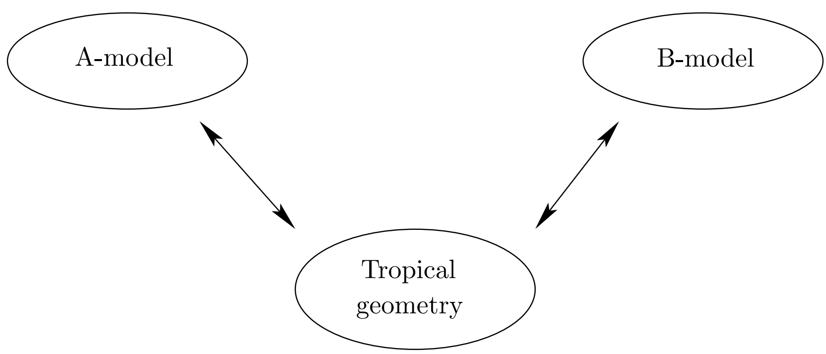
(1) What geometric structures on B correspond to geometric structures of interest on $X(B)$ and $\check{X}(B)$?
(2) If we want more interesting examples, we need to allow B to have singularities, i.e., have a tropical affine structure on an open set $B_0 \subseteq B$ with

$B \setminus B_0$ relatively small (e.g., codimension at least two). How do we deal with this?

By 2000, it was certainly clear to many of the researchers in the field that holomorphic curves in $X(B)$ should correspond to certain sorts of piecewise linear graphs in B. Kontsevich suggested the possibility that one might be able to actually carry out a curve count by counting these graphs. In 2002, Mikhalkin [**79, 80**] announced that this was indeed possible, introducing and proving curve-counting formulas for toric surfaces. This was the first evidence that one could really compute invariants using these piecewise linear graphs. For historical reasons which will be explained in Chapter 1, Mikhalkin called these piecewise linear graphs "tropical curves," introducing the word "tropical" into the field.

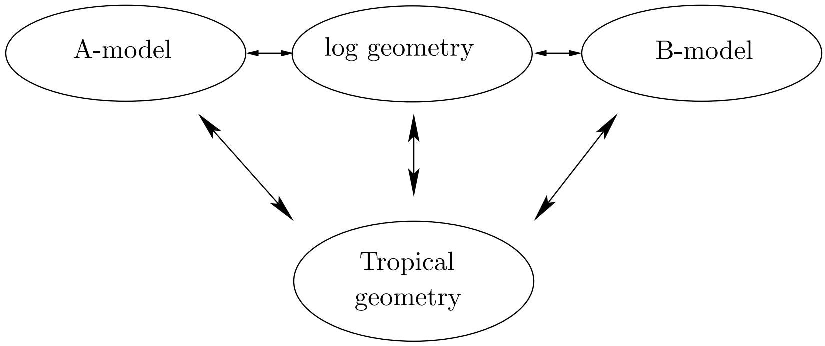
This brings us to the following picture. Mirror symmetry involves a relationship between two different types of geometry, usually called the A-model and the B-model. The A-model involves symplectic geometry, which is the natural category in which to discuss such things as Gromov-Witten invariants, while the B-model involves complex geometry, where one can discuss such things as period integrals.

This leads us to the following conceptual framework for mirror symmetry:



Here, we wish to explain mirror symmetry by identifying what we shall refer to as tropical structures in B which can be interpreted as geometric structures in the A- and B-models. However, the interpretations in the A- and B-models should be different, i.e., mirror, so that the fact that these structures are given by the same tropical structures then gives a conceptual explanation for mirror symmetry. For the most well-known aspect of mirror symmetry, namely the enumeration of rational curves, the hope should be that tropical curves on B correspond to (pseudo)-holomorphic curves in the A-model and corrections to period calculations in the B-model.

The main idea of my program with Siebert is to try to understand how to go between the tropical world and the A- and B-models by passing through another world, the world of log geometry. One can view log geometry as half-way between tropical geometry and classical geometry:



As this program with Siebert is ongoing, with much work still to be done, my lectures at the CBMS regional conference in Manhattan, Kansas were intended to give a snapshot of the current state of this program. This monograph closely follows the outline of those lectures. The basic goal is threefold.

First, I wish to explain explicitly, at least in special cases, all the worlds suggested in the above diagram: the tropical world, the "classical" world of the A- and B-model, and log geometry.

Second, I would like to explain one very concrete case where the full picture has been worked out for both the A- and B-models. This is the case of $\mathbb{P}^2$. For the A-model, curve counting is the result of Mikhalkin, and here I will give a proof of his result adapted from a more general result of Nishinou and Siebert [**86**], as that approach is more in keeping with the philosophy of the program. For the B-model, I will explain my own recent work [**42**] which shows how period integrals extract tropical information.

Third, I wish to survey some of the results obtained by Siebert and myself in the Calabi-Yau case, outlining how this approach can be expected to yield a proof of mirror symmetry. While for $\mathbb{P}^2$ I give complete details, this third part is intended to be more of a guide for reading the original papers, which unfortunately are quite long and technical. I hope to at least convey an intuition for this approach.

I will take a very ahistorical approach to all of this, starting with the basics of tropical geometry and working backwards, showing how a study of tropical geometry can lead naturally to other concepts which first arose in the study of mirror symmetry. In a way, this may be natural. To paraphrase Witten's statement about string theory, mirror symmetry often seems like a piece of twenty-first century mathematics which fell into the twentieth century. Its initial discovery in string theory represents some of the more difficult aspects of the theory. Even an explanation of the calculations carried out by Candelas et al. can occupy a significant portion of a course, and the theory built up to define and compute Gromov-Witten invariants is even more involved. On the other hand, the geometry that now seems to underpin mirror symmetry, namely tropical geometry, is very simple and requires no particular background to understand. So it makes sense to develop the discussion from the simplest starting point.

The prerequisites of this volume include a familiarity with algebraic geometry at the level of Hartshorne's text [**57**] as well as some basic differential geometry. In addition, familiarity with toric geometry will be very helpful; the text will recall many of the basic necessary facts about toric geometry, but at least some previous experience will be useful. For a more in-depth treatment of toric geometry, I recommend Fulton's lecture notes [**27**]. We shall also, in Chapter 3, make use of sheaves in the étale topology, which can be reviewed in [**83**], Chapter II. However, this use is not vital to most of the discussion here.

I would like to thank many people. Foremost, I would like to thank Ricardo Castaño-Bernard, Yan Soibelman, and Ilia Zharkov for organizing the NSF-CBMS conference at Kansas State University. Second, I would like to thank Bernd Siebert; the approach in this book grew out of our joint collaboration. I also thank the many people who answered questions and commented on the manuscript, including Sean Keel, M. Brandon Meredith, Rahul Pandharipande, D. Peter Overholser, Daniel Schultheis, and Katharine Shultis.

Parts of this book were written during a visit to Oxford; I thank Philip Candelas for his hospitality during this visit. The final parts of the book were written during the fall of 2009 at MSRI; I thank MSRI for its financial support via a Simons Professorship.

I would like to thank Lori Lejeune for providing the files for Figures 17, 18 and 19 of Chapter 1 and Figures 7 and 8 of Chapter 6. Finally, and definitely not least, I thank Arthur Greenspoon, who generously offered to proofread this volume.

Convention. Throughout this book $\Bbbk$ denotes an algebraically closed field of characteristic zero. $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, 2, \ldots\}$.