CHAPTER 1

The tropics

We start with the simplest of the three worlds, the tropical world. Tropical geometry is a kind of piecewise linear combinatorial geometry which arises when one starts to think about algebraic geometry over the so-called tropical semi-ring.

This chapter will give a rather shallow introduction to the subject. We will start with the definition of the tropical semi-ring and some elementary algebraic geometry over the tropical semi-ring. We move on to the notion of parameterized tropical curve, which features in Mikhalkin's curve counting results. Next, we introduce the type of tropical objects which arise in the Gross-Siebert program: affine manifolds with singularities. These arise naturally if one wants to think about curve counting in Calabi-Yau manifolds. We end with a duality between such objects given by the Legendre transform.

1.1. Tropical hypersurfaces

We begin with the *tropical semi-ring*,

$$\mathbb{R}^{\mathrm{trop}} = (\mathbb{R}, \oplus, \odot).$$

Here $\mathbb{R}$ is the set of real numbers, but with addition and multiplication defined by

$$\begin{aligned} a \oplus b &:= \min(a, b) \\ a \odot b &:= a + b. \end{aligned}$$

Of course there is no additive inverse. This semi-ring became known as the tropical semi-ring in honour of the Brazilian mathematician Imre Simon. The word *tropical* has now spread rapidly.

We would like to do algebraic geometry over the tropical semi-ring instead of over a field. Of course, since there is no additive identity in this semi-ring, it is not immediately obvious what the zero-locus of a polynomial should be. The correct, or rather, useful, intepretation is as follows. Let

$$\mathbb{R}^{\mathrm{trop}}[x_1, \ldots, x_n]$$

denote the space of functions $f : \mathbb{R}^n \to \mathbb{R}$ given by tropical polynomials

$$f(x_1, \ldots, x_n) = \sum_{(i_1,\ldots,i_n)\in S} a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n}$$

where $S \subseteq \mathbb{Z}^n$ is a finite index set. Here all operations are in $\mathbb{R}^{\mathrm{trop}}$, so this is really the function

$$f(x_1, \ldots, x_n) = \min\{a_{i_1,\ldots,i_n} + \sum_{k=1}^{n} i_k x_k \,|\, (i_1, \ldots, i_n) \in S\}$$

This is a piecewise linear function, and the *tropical hypersurface* defined by f, $V(f) \subseteq \mathbb{R}^n$, as a set, is the locus where f is not linear.

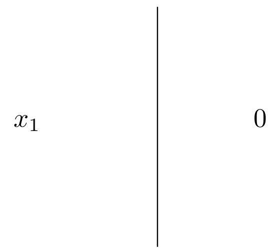

FIGURE 1. $0 \oplus (0 \odot x_1)$

In order to write these formulas in a more invariant way, in what follows we shall often make use of the notation

$$M = \mathbb{Z}^n, \quad M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}, \quad N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}), \quad N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}.$$

We denote evaluation of $n \in N$ on $m \in M$ by $\langle n, m\rangle$. We shall often use the notion of *index* of an element $m \in M \setminus \{0\}$; this is the largest positive integer r such that there exists $m' \in M$ with $rm' = m$. If the index of m is 1, we say m is *primitive*.

With this notation, we can view a tropical function as a map $f : M_{\mathbb{R}} \to \mathbb{R}$ written, for $S \subseteq N$ a finite set, as

$$f(z) = \sum_{n\in S} a_n z^n := \min\{a_n + \langle n, z\rangle \,|\, n \in S\}.$$

Now $V(f)$ will be a union of codimension one polyhedra in $\mathbb{R}^n$. Here, by a *polyhedron*, we mean:

DEFINITION 1.1. A *polyhedron* σ in $M_{\mathbb{R}}$ is a finite intersection of closed half-spaces. A *face* of a polyhedron is a subset given by the intersection of σ with a hyperplane H such that σ is contained in a half-space with boundary H.

The *boundary* $\partial\sigma$ of σ is the union of all proper faces of σ, and the *interior* $\mathrm{Int}(\sigma)$ of σ is $\sigma \setminus \partial\sigma$.

The polyhedron σ is a lattice polyhedron if it is an intersection of half-spaces defined over $\mathbb{Q}$ and all vertices of σ lie in M.

A *polytope* is a compact polyhedron.

Returning to $V(f)$, each codimension one polyhedron making up $V(f)$ separates two domains of linearity of f, in one of which f is given by a monomial with exponent $n \in N$ and in the other by a monomial with exponent $n' \in N$. Then the *weight* of this polyhedron in $V(f)$ is the index of $n' - n$. We then view $V(f)$ as a weighted polyhedral complex.

EXAMPLES 1.2. Figures 1 through 5 give examples of two-variable tropical polynomials and their corresponding “zero loci.” All edges have weight 1 unless otherwise indicated. We also indicate the monomial determining the function on each domain of linearity and the precise position of the vertices.

We now explain a simple way to see what $V(f)$ looks like. Given

$$f = \sum_{n\in S} a_n z^n,$$

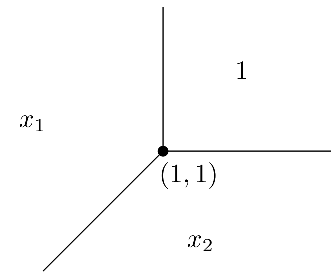

FIGURE 2. $1 \oplus (0 \odot x_1) \oplus (0 \odot x_2)$

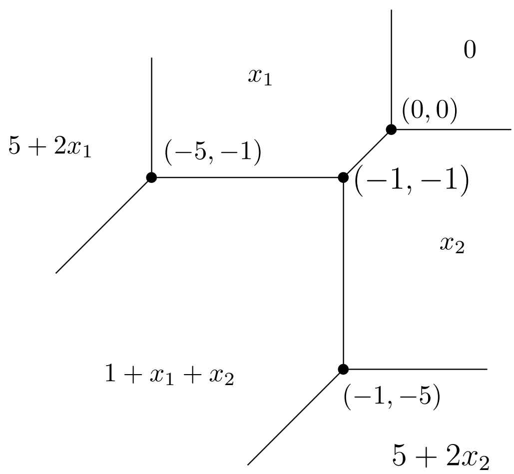

FIGURE 3. $0 \oplus (0 \odot x_1) \oplus (0 \odot x_2) \oplus (1 \odot x_1 \odot x_2) \oplus (5 \odot x_1 \odot x_1) \oplus (5 \odot x_2 \odot x_2)$

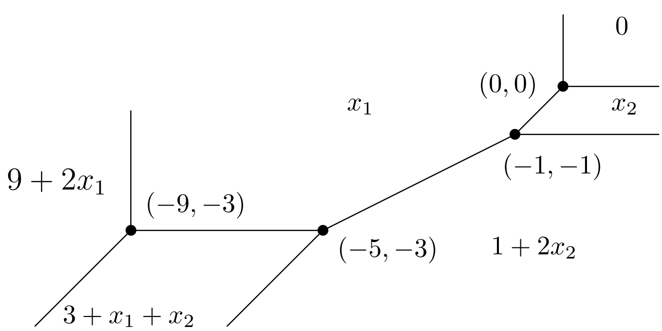

FIGURE 4. $0 \oplus (0 \odot x_1) \oplus (0 \odot x_2) \oplus (3 \odot x_1 \odot x_2) \oplus (9 \odot x_1 \odot x_1) \oplus (1 \odot x_2 \odot x_2)$

we consider the *Newton polytope* of S,

$$\Delta_S := \operatorname{Conv}(S) \subseteq N_{\mathbb{R}},$$

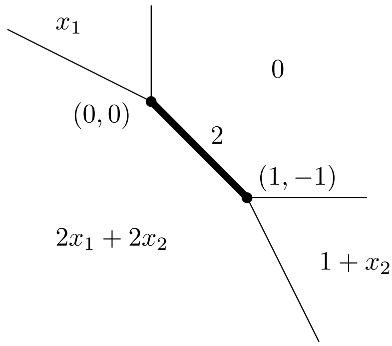

FIGURE 5. $0 \oplus (0 \odot x_1) \oplus (1 \odot x_2) \oplus (0 \odot x_1 \odot x_1 \odot x_2 \odot x_2)$

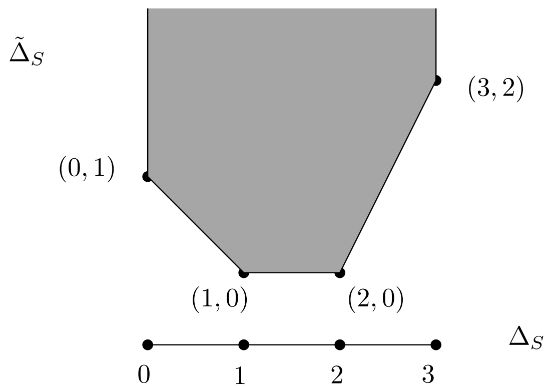

FIGURE 6

the convex hull of S in $N_{\mathbb{R}}$. The coefficients a_n then define a function

$$\varphi : \Delta_S \to \mathbb{R}$$

as follows. We consider *the upper convex hull* $\tilde{\Delta}_S$ of the set

$$\tilde{S} = \{(n, a_n) \,|\, n \in S\} \subseteq N_{\mathbb{R}} \times \mathbb{R},$$

namely

$$\tilde{\Delta}_S = \{(n, a) \in N_{\mathbb{R}} \times \mathbb{R} \,|\, \text{there exists } (n, a') \in \operatorname{Conv}(\tilde{S}) \text{ with } a \geq a'\}.$$

We then define

$$\varphi(n) = \min\{a \in \mathbb{R} \,|\, (n, a) \in \tilde{\Delta}_S\}.$$

For example, considering the univariate tropical polynomial

$$f = 1 \oplus (0 \odot x) \oplus (0 \odot x^2) \oplus (2 \odot x^3),$$

we get Δ_S and $\tilde{\Delta}_S$ as depicted in Figure 6, with the lower boundary of $\tilde{\Delta}_S$ being the graph of φ.

This picture yields a polyhedral decomposition of Δ_S:

DEFINITION 1.3. A *(lattice) polyhedral decomposition* of a (lattice) polyhedron $\Delta \subseteq N_{\mathbb{R}}$ is a set $\mathscr{P}$ of (lattice) polyhedra in $N_{\mathbb{R}}$ called *cells* such that

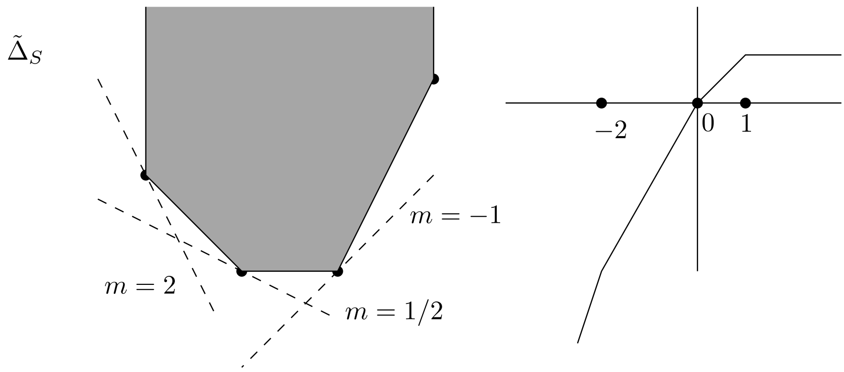

FIGURE 7. The left-hand figure is $\tilde{\Delta}_S$. The right-hand picture shows $\check{\mathscr{P}}$ on the x-axis and the graph of $\check{\varphi}$.

(1) $\Delta = \bigcup_{\sigma \in \mathscr{P}} \sigma$.
(2) If $\sigma \in \mathscr{P}$ and $\tau \subseteq \sigma$ is a face, then $\tau \in \mathscr{P}$.
(3) If $\sigma_1, \sigma_2 \in \mathscr{P}$, then $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

For a polyhedral decomposition $\mathscr{P}$, denote by $\mathscr{P}_{\max}$ the subset of maximal cells of $\mathscr{P}$. We denote by $\mathscr{P}^{[k]}$ the set of k-dimensional cells of $\mathscr{P}$.

Indeed, to get a polyhedral decomposition $\mathscr{P}$ of Δ_S, we just take $\mathscr{P}$ to be the set of images under the projection $N_{\mathbb{R}} \times \mathbb{R} \to N_{\mathbb{R}}$ of proper faces of $\tilde{\Delta}_S$. A polyhedral decomposition of Δ_S obtained in this way from the graph of a convex piecewise linear function is called a *regular decomposition* and these decompositions play an important role in the combinatorics of convex polyhedra, see e.g., [**32**].

We can now define the *discrete Legendre transform* of the triple $(\Delta_S, \mathscr{P}, \varphi)$:

DEFINITION 1.4. The *discrete Legendre transform* of $(\Delta_S, \mathscr{P}, \varphi)$ is the triple $(M_{\mathbb{R}}, \check{\mathscr{P}}, \check{\varphi})$ where:

(1)
$$\check{\mathscr{P}} = \{\check{\tau} \mid \tau \in \mathscr{P}\}$$
with
$$\check{\tau} = \left\{ m \in M_{\mathbb{R}} \;\middle|\; \begin{array}{c} \exists a \in \mathbb{R} \text{ such that } \langle -m, n \rangle + a \leq \varphi(n) \\ \text{for all } n \in \Delta_S, \text{ with equality for } n \in \tau \end{array} \right\}.$$
(2) $\check{\varphi}(m) = \max\{a \mid \langle -m, n\rangle + a \leq \varphi(n) \text{ for all } n \in \Delta_S\}$.

Let us explain this in a bit more detail. First, if $\sigma \in \mathscr{P}_{\max}$, let $m_\sigma \in M$ be the slope of $\varphi|_\sigma$. Then in fact
$$\check{\sigma} = \{-m_\sigma\},$$
as follows from the convexity of φ. Second, the formula in (2) is a fairly standard way of describing the Legendre transformed function $\check{\varphi}$. We think of $\check{\varphi}(m)$ as obtained by taking the graph in $N_{\mathbb{R}} \times \mathbb{R}$ of a linear function on $N_{\mathbb{R}}$ with slope $-m$ and moving it up or down until it becomes a supporting hyperplane for $\tilde{\Delta}_S$. The value of this affine linear function at 0 is then $\check{\varphi}(m)$; see Figure 7. Note that if $m \in \mathrm{Int}(\check{\tau})$, then the graph of $\langle -m, \cdot \rangle + \check{\varphi}(m)$ is then a supporting hyperplane for the face of $\tilde{\Delta}_S$ projecting isomorphically to τ.

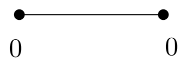

Figure 8. The Newton polytope and subdivision for Figure 1.

In fact, $\check{\varphi}$ can be described in a more familiar way. Note that

$$\check{\varphi}(m) = \min\{\varphi(n) + \langle m, n\rangle \,|\, n \in \Delta_S\}.$$

From this, it is clear that $\check{\mathscr{P}}_{\max}$ consists of the maximal domains of linearity of $\check{\varphi}$, with $\check{\varphi}|_{\check{v}}$ having slope v for v a vertex (element of $\mathscr{P}^{[0]}$) of $\mathscr{P}$. Indeed, the minimum is always achieved at some vertex, and if this vertex is v, then $\check{\varphi}(m) = \varphi(v) + \langle m, v\rangle$. Thus $\check{\varphi}$ is linear on $\check{v}$ with slope v. Furthermore, as necessarily $\varphi(v) + \langle m, v\rangle \leq \varphi(v') + \langle m, v'\rangle$ whenever $m \in \check{v}$, one sees that $\check{\varphi}$ is in fact given by the tropical polynomial

$$\sum_{n \in \mathscr{P}^{[0]}} \varphi(n) z^n.$$

This is not necessarily the original polynomial defining the function f. However, clearly the vertices of $\tilde{\Delta}_S$ are of the form (n, a_n) for $n \in \mathscr{P}^{[0]} \subseteq S$, so $\varphi(n) = a_n$ for $n \in \mathscr{P}^{[0]}$, and the tropical polynomial defining $\check{\varphi}$ is simply missing some of the terms of the original defining polynomial f. These missing terms are precisely ones of the form $a_n z^n$ with (n, a_n) not a vertex of $\tilde{\Delta}_S$. We can see that such terms are irrelevant for calculating f. Indeed, if (n, a_n) is not a vertex of $\tilde{\Delta}_S$ for some $n \in N \cap \Delta_S$, and $f(m) = \langle m, n\rangle + a_n$ for some $m \in M_{\mathbb{R}}$, then

$$\langle m, n\rangle + a_n \leq \langle m, n'\rangle + a_{n'}$$

for all $n' \in S$. But then the hyperplane in $N_{\mathbb{R}} \times \mathbb{R}$ given by

$$\{(n', r) \in N_{\mathbb{R}} \times \mathbb{R} \,|\, \langle m, n'\rangle + r = \langle m, n\rangle + a_n\}$$

is a supporting hyperplane for $\tilde{\Delta}_S$ which contains (n, a_n), and hence must also contain a vertex $(n', a_{n'})$ of $\tilde{\Delta}_S$. Then $f(m)$ coincides with $\langle m, n'\rangle + a_{n'}$, and hence the term $a_n z^n$ was irrelevant for calculating f. Thus we see

$$\check{\varphi} = f.$$

Since the domains of linearity of $\check{\varphi}$ are the polyhedra of $\check{\mathscr{P}}_{\max}$, we see that

$$V(f) = \bigcup_{\tau \in \mathscr{P}^{[1]}} \check{\tau}.$$

Since the 0-cells of $\check{\mathscr{P}}$ are the cells $\check{\sigma} = \{-m_\sigma\}$ for $\sigma \in \mathscr{P}_{\max}$, it is usally easy to draw $V(f)$ using this description. Additionally, the weights are easily determined: for $\tau \in \mathscr{P}^{[1]}$, the weight of $\check{\tau}$ is just the affine length of τ, i.e., the index of the difference of the endpoints of τ.

To summarize, the function φ determines the dual decomposition $\check{\mathscr{P}}$, whose vertices are given by slopes of φ, and $V(f)$ is the codimension one skeleton of $\check{\mathscr{P}}$.

Examples 1.5. For the examples of Figures 1 through 5, the Newton polytopes along with their regular decomposition and values of a_n are given in Figures 8 throught 12.

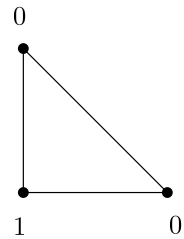

FIGURE 9. The Newton polytope and subdivision for Figure 2.

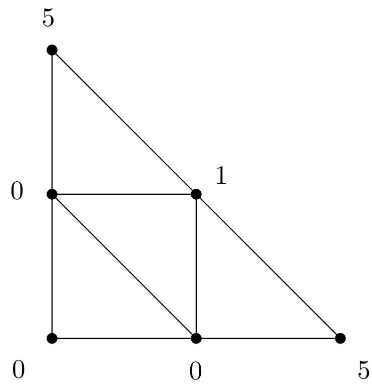

FIGURE 10. The Newton polytope and subdivision for Figure 3.

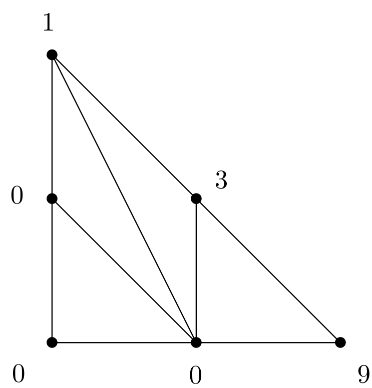

FIGURE 11. The Newton polytope and subdivision for Figure 4.

This description of $V(f)$ leads to an important condition known as the *balancing condition.* Specifically, for each $\breve{\omega} \in \breve{\mathscr{P}}^{[n-2]}$, a codimension two cell, let $\breve{\tau}_1, \ldots, \breve{\tau}_k \in \breve{\mathscr{P}}^{[n-1]}$ be the cells containing it in $V(f)$, with weights $w_1, \ldots, w_k$. Note that ω is a two-dimensional cell of $\mathscr{P}$ and $\tau_1, \ldots, \tau_k$ are the edges of ω. Let $n_1, \ldots, n_k \in N$ be primitive tangent vectors to $\tau_1, \ldots, \tau_k$, pointing in directions consistent with the orientations on $\tau_1, \ldots, \tau_k$ induced by some chosen orientation on ω. The vectors $n_1, \ldots, n_k$ are primitive normal vectors to $\breve{\tau}_1, \ldots, \breve{\tau}_k$. Indeed, the endpoints of τ_i

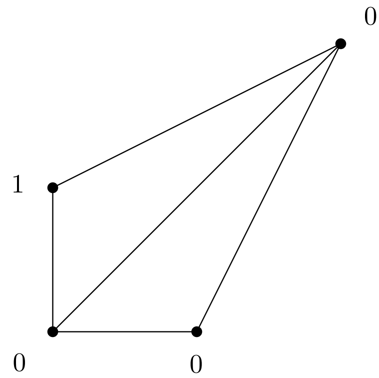

FIGURE 12. The Newton polytope and subdivision for Figure 5.

give the slopes of $\check{\varphi} = f$ on the two domains of linearity of f on either side of $\check{\tau}_i$, so n_i must be constant on $\check{\tau}_i$. Obviously, we have

$$\sum_{i=1}^{k} w_i n_i = 0. \tag{1.1}$$

We call this the *balancing condition*.

In the case when $\dim M_{\mathbb{R}} = 2$, so that $V(f)$ is a curve, it is useful to rewrite this as follows. Let $V \in \check{\mathscr{P}}^{[0]}$ be a vertex of $V(f)$, contained in edges $E_1, \ldots, E_k \in \check{\mathscr{P}}^{[1]}$ of $V(f)$, and let $m_1, \ldots, m_k \in M$ be primitive tangent vectors to $E_1, \ldots, E_k$ pointing away from V. Suppose E_i has weight w_i. Then (1.1) is equivalent to

$$\sum_{i=1}^{k} w_i m_i = 0. \tag{1.2}$$

EXAMPLE 1.6. *The tropical Bézout theorem.* Suppose $\dim M_{\mathbb{R}} = 2$, and let e_1, e_2 be a basis for M. Let Δ_d be the polytope which is the convex hull of $0, de_1$, and de_2. If $f = \sum_{n \in \Delta_d} a_n z^n$, then $V(f)$ is a tropical curve in $M_{\mathbb{R}}$, which we call a *degree d curve in the tropical projective plane*. For example, Figure 2 depicts a degree 1 curve, i.e., a tropical line, and Figures 3 and 4 depict degree 2 curves, i.e., tropical conics. These should be thought of as tropical analogues of ordinary lines and conics in $\mathbb{P}^2$. These tropical versions often share surprising properties in common with the usual algebraic versions. We give one example here.

Let $C, D \subseteq M_{\mathbb{R}}$ be two tropical curves in the tropical projective plane of degree d and e respectively. Suppose that C and D intersect at only a finite number of points; this can always be achieved by translating C or D. In fact, we can similarly assume that none of these intersection points are vertices of C or D. We can define a notion of multiplicity of an intersection point of these two curves. Suppose that a point $P \in C \cap D$ is contained in an edge E of C and an edge F of D, of weights $w(E)$ and $w(F)$ respectively. Let m_1 be a primitive tangent vector to E and m_2 be a primitive tangent vector to F. Then we define the *intersection multiplicity of C and D at P* to be the positive integer

$$i_P(C, D) := w(E)w(F)|m_1 \wedge m_2|.$$

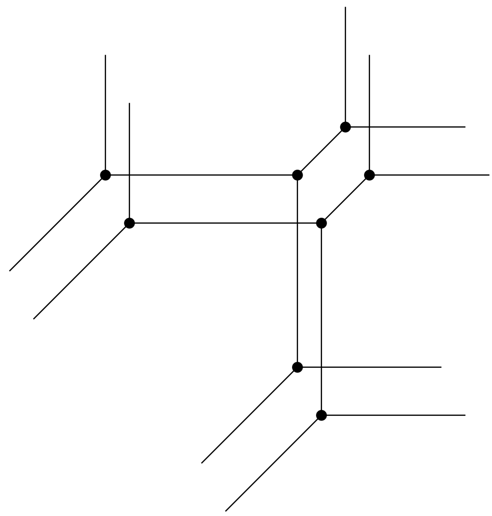

FIGURE 13. Two tropical conics meeting at four points.

Here $m_1, m_2 \in M \cong \mathbb{Z}^2$, and $\bigwedge^2 M \cong \mathbb{Z}$, so $|m_1 \wedge m_2|$ makes sense as a positive number no matter which isomorphism is chosen. We then have the tropical Bézout theorem, which states that

$$\sum_{P \in C \cap D} i_P(C, D) = d \cdot e.$$

This is exactly the expected result for ordinary algebraic curves in $\mathbb{P}^2$, of course. For a proof, see [**96**], §4. See Figure 13 for an example.

1.2. Some background on fans

We will collect here a number of standard notions concerning fans. We send the reader to [**27**] for more details.

DEFINITION 1.7. A *strictly convex rational polyhedral cone* in $M_{\mathbb{R}}$ is a lattice polyhedron in $M_{\mathbb{R}}$ with exactly one vertex, which is $0 \in M_{\mathbb{R}}$.

A *fan* Σ in $M_{\mathbb{R}}$ is a set of strictly convex rational polyhedral cones such that

(1) If $\sigma \in \Sigma$, and $\tau \subseteq \sigma$ is a face, then $\tau \in \Sigma$.
(2) If $\sigma_1, \sigma_2 \in \Sigma$, then $\sigma_1 \cap \sigma_2$ is a face of σ_1 and σ_2.

In other words, a fan Σ is a polyhedral decomposition of a set $|\Sigma| \subseteq M_{\mathbb{R}}$, called the *support* of Σ, with all elements of the polyhedral decomposition being strictly convex rational polyhedral cones.

A fan is *complete* if $|\Sigma| = M_{\mathbb{R}}$.

DEFINITION 1.8. Let Σ be a fan in $M_{\mathbb{R}}$. A PL (piecewise linear) function on Σ is a continuous function $\varphi : |\Sigma| \to \mathbb{R}$ which is linear when restricted to each cone of Σ.

The function φ is *strictly convex* if

(1) $|\Sigma|$ is a convex set in $M_{\mathbb{R}}$;
(2) For $m, m' \in |\Sigma|$, $\varphi(m) + \varphi(m') \geq \varphi(m + m')$, with equality holding if and only if m, m' lie in the same cone of Σ.

The function φ is *integral* if for each $\sigma \in \Sigma_{\max}$ there exists an $n_\sigma \in N$ such that n_σ and φ agree on σ.

The *Newton polyhedron* of a strictly convex PL function $\varphi : |\Sigma| \to \mathbb{R}$ is

$$\Delta_\varphi := \{n \in N_\mathbb{R} \mid \varphi(m) + \langle n, m\rangle \geq 0 \text{ for all } m \in |\Sigma|\}.$$

The Newton polyhedron of a function φ is unbounded if and only if Σ is not a complete fan. If Σ is complete, it is easy to see that

$$\Delta_\varphi = \mathrm{Conv}(\{-n_\sigma \mid \sigma \in \Sigma_{\max}\})$$

where $n_\sigma \in N_\mathbb{R}$ is the linear function $\varphi|_\sigma$. Note there is a one-to-one inclusion reversing correspondence between cones in Σ and faces of Δ_φ, with $\sigma \in \Sigma$ corresponding to

$$\{n \in \Delta_\varphi \mid \varphi(m) + \langle n, m\rangle = 0 \text{ for all } m \in \sigma\}.$$

Definition 1.9. If $\Delta \subseteq N_\mathbb{R}$ is a polyhedron, $\sigma \subseteq \Delta$ a face, the *normal cone to* Δ *along* σ is

$$N_\Delta(\sigma) = \{m \in M \mid m|_\sigma = \text{constant}, \langle m, n\rangle \geq \langle m, n'\rangle \text{ for all } n \in \Delta,\ n' \in \sigma\}.$$

If $\tau \subseteq \sigma$ is a subset, then $T_\tau\sigma$ denotes the *tangent wedge* to σ along τ, defined by

$$T_\tau\sigma = \{r(m - m') \mid m \in \sigma, m' \in \tau, r \geq 0\}.$$

The *normal fan* of Δ is

$$\check{\Sigma}_\Delta := \{N_\Delta(\sigma) \mid \ \sigma \text{ is a face of } \Delta\}.$$

One checks easily that

$$T_\sigma\Delta = (N_\Delta(\sigma))^\vee := \{n \in N \mid \langle n, m\rangle \geq 0 \quad \forall m \in N_\Delta(\sigma)\}. \tag{1.3}$$

The normal fan $\check{\Sigma}_\Delta$ to Δ carries a PL function $\varphi_\Delta : |\check{\Sigma}_\Delta| \to \mathbb{R}$ defined by

$$\varphi_\Delta(m) = -\inf\{\langle n, m\rangle \mid n \in \Delta\}.$$

It is easy to see that if $\varphi : |\Sigma| \to \mathbb{R}$ is strictly convex, then Σ is the normal fan to Δ_φ. This in fact gives a one-to-one correspondence between strictly convex PL functions φ on a fan Σ and polyhedra Δ with normal fan Σ. Note given Δ, $\Delta_{\varphi_\Delta} = \Delta$, and given $\varphi : |\Sigma| \to \mathbb{R}$, $\varphi_{\Delta_\varphi} = \varphi$.

Definition 1.10. If Σ is a fan in $M_\mathbb{R}$, $\tau \in \Sigma$, we define the *quotient fan* $\Sigma(\tau)$ *of* Σ *along* τ to be the fan

$$\Sigma(\tau) := \{(\sigma + \mathbb{R}\tau)/\mathbb{R}\tau \mid \sigma \in \Sigma, \tau \subseteq \sigma\}$$

in $M_\mathbb{R}/\mathbb{R}\tau$, where $\mathbb{R}\tau$ is the linear space spanned by τ.

If $\varphi : |\Sigma| \to \mathbb{R}$ is a PL function on Σ, then φ induces a function $\varphi(\tau) : \Sigma(\tau) \to \mathbb{R}$, well-defined up to linear functions, as follows. Choose $n \in N_\mathbb{R}$ such that $\varphi(m) = \langle n, m\rangle$ for $m \in \tau$. Then for any σ containing τ, $\varphi|_\sigma - n$ is zero on τ, hence descends to a linear function on $(\sigma + \mathbb{R}\tau)/\mathbb{R}\tau$. These piece together to give a PL function $\varphi(\tau)$ on $\Sigma(\tau)$, well-defined up to a linear function (determined by the choice of n).

Note that if φ is strictly convex, then so is $\varphi(\tau)$. If $M'_\mathbb{R} = M_\mathbb{R}/\mathbb{R}\tau$ and $N'_\mathbb{R} = \mathrm{Hom}(M'_\mathbb{R}, \mathbb{R})$, then $N'_\mathbb{R} = (\mathbb{R}\tau)^\perp$. It is then easy to see that $\Delta_{\varphi(\tau)}$ is just the translate by n of the face of Δ_φ corresponding to τ.

1.3. Parameterized tropical curves

We shall now use the discussion of the balancing condition in §1.1 to define tropical curves in a more abstract setting. In theory, similar definitions could be given for tropical varieties of higher dimension, but we will not do so here.

Let $\overline{\Gamma}$ be a connected graph with no bivalent vertices. Such a graph can be viewed in two different ways. First, it can be viewed as a purely combinatorial object, i.e., a set $\overline{\Gamma}^{[0]}$ of vertices and a set $\overline{\Gamma}^{[1]}$ of edges consisting of unordered pairs of elements of $\overline{\Gamma}^{[0]}$, indicating the endpoints of an edge.

We can also view $\overline{\Gamma}$ as the topological realization of the graph, i.e., a topological space which is the union of line segments corresponding to the edges. We shall confuse these two viewpoints at will, hopefully without any confusion.

Let $\overline{\Gamma}^{[0]}_\infty$ be the set of univalent vertices of $\overline{\Gamma}$, and write

$$\Gamma = \overline{\Gamma} \setminus \overline{\Gamma}^{[0]}_\infty.$$

Let $\Gamma^{[0]}, \Gamma^{[1]}$ denote the set of vertices and edges of Γ. Here we are thinking of Γ and $\overline{\Gamma}$ as topological spaces, so Γ now has some non-compact edges. Let $\Gamma^{[1]}_\infty$ be the set of non-compact edges of Γ. A *flag* of Γ is a pair (V, E) with $V \in \Gamma^{[0]}$ and $E \in \Gamma^{[1]}$ with $V \in E$.

In addition, all graphs will be weighted graphs, i.e., $\overline{\Gamma}$ comes along with a weight function

$$w : \overline{\Gamma}^{[1]} \to \mathbb{N} = \{0, 1, 2, \ldots\}.$$

We will often consider *marked graphs*, $(\Gamma, x_1, \ldots, x_k)$, where Γ is as above and $x_1, \ldots, x_k$ are labels assigned to non-compact edges of weight 0, i.e., we are given an inclusion

$$\begin{array}{rcl} \{x_1, \ldots, x_k\} & \hookrightarrow & \Gamma^{[1]}_\infty \\ x_i & \mapsto & E_{x_i} \end{array}$$

with $w(E_{x_i}) = 0$. We will use the convention in this book which is not actually quite standard in the tropical literature that $w(E) \neq 0$ unless $E = E_{x_i}$ for some x_i.

We can now define a marked parameterized tropical curve in $M_\mathbb{R}$, where as usual, $M = \mathbb{Z}^n$, $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$ and $N = \mathrm{Hom}_\mathbb{Z}(M, \mathbb{Z})$.

Definition 1.11. A *marked parameterized tropical curve*

$$h : (\Gamma, x_1, \ldots, x_k) \to M_\mathbb{R}$$

is a continuous map h satisfying the following two properties:

(1) If $E \in \Gamma^{[1]}$ and $w(E) = 0$, then $h|_E$ is constant; otherwise, $h|_E$ is a proper embedding of E into a line of rational slope in $M_\mathbb{R}$.

(2) *The balancing condition.* Let $V \in \Gamma^{[0]}$, and let $E_1, \ldots, E_\ell \in \Gamma^{[1]}$ be the edges adjacent to V. Let $m_i \in M$ be a primitive tangent vector to $h(E_i)$ pointing away from $h(V)$. Then

$$\sum_{i=1}^{\ell} w(E_i) m_i = 0.$$

If $h : (\Gamma, x_1, \ldots, x_n) \to M_\mathbb{R}$ is a marked parameterized tropical curve, we write $h(x_i)$ for $h(E_{x_i})$.

We will call two marked parameterized tropical curves $h : (\Gamma, x_1, \ldots, x_k) \to M_{\mathbb{R}}$ and $h' : (\Gamma', x'_1, \ldots, x'_k) \to M_{\mathbb{R}}$ *equivalent* if there is a homeomorphism $\varphi : \Gamma \to \Gamma'$ with $\varphi(E_{x_i}) = E_{x'_i}$ and $h = h' \circ \varphi$. We will define a marked tropical curve to be an equivalence class of parameterized marked tropical curves.

The *genus* of h is $b_1(\Gamma)$. □

We wish to talk about the *degree* of a tropical curve, and to do so, we need to fix a fan Σ. In fact, for the moment, we will only make use of the set of one-dimensional cones in Σ, $\Sigma^{[1]}$. Denote by T_Σ the free abelian group generated by $\Sigma^{[1]}$. For $\rho \in \Sigma^{[1]}$, denote by $t_\rho \in T_\Sigma$ the corresponding generator. We have a map

$$\begin{array}{rcl} r : T_\Sigma & \to & M \\ t_\rho & \mapsto & m_\rho \end{array}$$

where m_ρ is the primitive generator of the ray ρ.

Definition 1.12. A marked tropical curve h is *in* X_Σ if for each $E \in \Gamma^{[1]}_\infty$ which is not a marked edge, $h(E)$ is a translate of some $\rho \in \Sigma^{[1]}$.

If h is a curve in X_Σ, the *degree* of h is $\Delta(h) \in T_\Sigma$ defined by

$$\Delta(h) = \sum_{\rho \in \Sigma^{[1]}} d_\rho t_\rho$$

where d_ρ is the number of edges $E \in \Gamma^{[1]}_\infty$ with $h(E)$ a translate of ρ, counted with weight.

For $\Delta \in T_\Sigma$, $\Delta = \sum_{\rho \in \Sigma^{[1]}} d_\rho t_\rho$, define

$$|\Delta| := \sum_{\rho \in \Sigma^{[1]}} d_\rho.$$

The following lemma is a straightforward application of the balancing condition, obtained by summing the balancing conditions over all vertices of Γ:

Lemma 1.13. $r(\Delta(h)) = 0$.

Example 1.14. Let Σ be the fan for $\mathbb{P}^2$. This is the complete fan in $M_{\mathbb{R}} = \mathbb{R}^2$ whose one-dimensional rays are ρ_0, ρ_1, ρ_2 generated by $m_0 = (-1, -1)$, $m_1 = (1, 0)$ and $m_2 = (0, 1)$; see Figure 14. The two-dimensional cones are $\sigma_{i,i+1}$, with indices taken modulo 3 and where $\sigma_{i,i+1}$ is generated by m_i and m_{i+1}. We shall see in Chapter 3 that this fan defines $\mathbb{P}^2$ as a toric variety (Example 3.2). In particular, we shall give meaning to the symbol "X_Σ", which is actually a variety, and in the case of this particular Σ, $X_\Sigma = \mathbb{P}^2$. Then the examples of Figures 2, 3 and 4 are tropical curves in $X_\Sigma = \mathbb{P}^2$. The degree of Figure 2 is $t_{\rho_0} + t_{\rho_1} + t_{\rho_2}$, while the degree of Figures 3 and 4 is $2(t_{\rho_0} + t_{\rho_1} + t_{\rho_2})$. In general, a tropical curve in $\mathbb{P}^2$ will be, by the above lemma, of degree $d(t_{\rho_0} + t_{\rho_1} + t_{\rho_2})$, in which case we say the curve is *degree d in* $\mathbb{P}^2$ (compare with Example 1.6). So in particular, Figure 2 is a *tropical line*, and all degree one curves in $\mathbb{P}^2$ are just translates of this example. Figures 3 and 4 are tropical conics.

It is reasonable to ask what the relationship is between this new definition of tropical curve and the earlier notion of a tropical hypersurface in $M_{\mathbb{R}}$ with $\dim M_{\mathbb{R}} = 2$. In particular, one can ask whether or not $h(\Gamma)$ is a tropical hypersurface in $M_{\mathbb{R}}$. Of course, to pose this question, one must first define weights on $h(\Gamma)$, as h is in general not an embedding. Viewing $h(\Gamma)$ as a one-dimensional polyhedral complex,

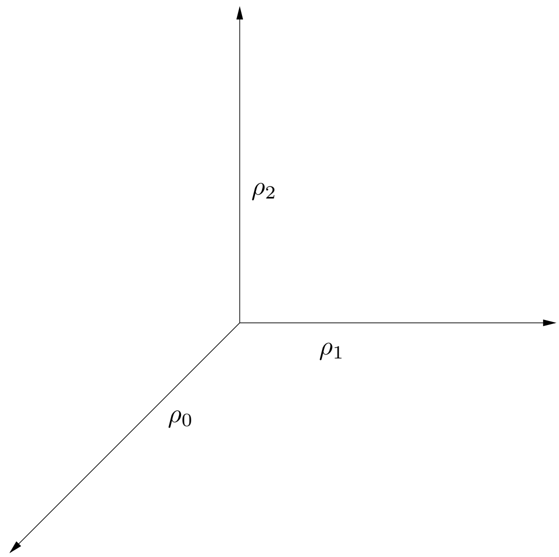

FIGURE 14. The fan for $\mathbb{P}^2$.

we need to assign a weight $w(E)$ to each edge E of $h(\Gamma)$. We define this as follows. Pick a point $m \in E$ which is not a vertex of $h(\Gamma)$ and is not the image of any vertex of Γ, and define

$$w(E) = \sum_{\substack{E' \in \Gamma^{[1]} \\ E' \cap h^{-1}(m) \neq \emptyset}} w(E'),$$

i.e., the weight of E is the sum of weights of edges of Γ whose image under h contains m. It is easy to check that the balancing condition on h implies firstly that this weight is well-defined, i.e., doesn't depend on the choice of m, and secondly that $h(\Gamma)$ satisfies the balancing condition.

PROPOSITION 1.15. *If $h : \Gamma \to M_{\mathbb{R}}$ is a tropical curve with $\dim M_{\mathbb{R}} = 2$, then there exists a tropical polynomial f such that $h(\Gamma) = V(f)$, as weighted one-dimensional polyhedral complexes.*

PROOF. We define f as follows. $h(\Gamma)$ yields a polyhedral decomposition $\check{\mathscr{P}}$ of $M_{\mathbb{R}}$ whose maximal cells are closures of connected components of $M_{\mathbb{R}} \setminus h(\Gamma)$.

Choose some cell $\sigma_0 \in \check{\mathscr{P}}_{\max}$ and define $f|_{\sigma_0} \equiv 0$. We then define f inductively. Suppose f is defined on $\sigma \in \check{\mathscr{P}}_{\max}$. If $\sigma' \in \check{\mathscr{P}}_{\max}$ and $E = \sigma \cap \sigma'$ satisfies $\dim E = 1$, then we can define f on σ' as follows. Extend $f|_{\sigma}$ to an affine linear function $f_\sigma : M_{\mathbb{R}} \to \mathbb{R}$. Let $n_E \in N$ be a primitive normal vector to E which takes a constant value on E and takes larger values on σ than on σ'. Denote by $\langle n_E, E\rangle$ the value n_E takes on E. Then we define f to be $f_\sigma + w(E)(n_E - \langle n_E, E\rangle)$ on σ'. It is an immediate consequence of the balancing condition that this is well-defined. Indeed, if we define f on a sequence of polygons with a common vertex V, starting at σ and passing successively through edges $E_1, \ldots, E_n$, then when we return to σ, we have constructed the function $f_\sigma + \sum_{i=1}^n w(E_i)(n_{E_i} - \langle n_{E_i}, E_i\rangle) = f_\sigma$ by the balancing condition.

Finally, we note that f is convex, i.e., given by a tropical polynomial. In addition, clearly $h(\Gamma) = V(f)$. □

We are ready to talk about moduli spaces of such curves. For this, we need to talk about the *combinatorial type* of a marked tropical curve $h : (\Gamma, x_1, \ldots, x_n) \to M_{\mathbb{R}}$. This is the data of the labelled graph $(\Gamma, x_1, \ldots, x_n)$, the weight function w, along with, for each flag (V, E) of Γ, the primitive tangent vector $m_{(V,E)} \in M$ to $h(E)$ pointing away from $h(V)$. A *combinatorial equivalence class* is the set of all tropical curves of the same combinatorial type. We denote by $[h]$ the combinatorial equivalence class of a curve h.

Definition 1.16. For $g, k \geq 0$, Σ a fan in $M_{\mathbb{R}}$, $\Delta \in T_\Sigma$ with $r(\Delta) = 0$, denote by $\mathcal{M}_{g,k}(\Sigma, \Delta)$ the set of tropical curves in X_Σ of genus g, degree Δ and with k marked points.

If $[h]$ is a combinatorial equivalence class of curves of genus g with k marked points of degree Δ in X_Σ, we denote by $\mathcal{M}^{[h]}_{g,k}(\Sigma, \Delta) \subseteq \mathcal{M}_{g,k}(\Sigma, \Delta)$ the set of all curves of combinatorial equivalence class $[h]$.

Proposition 1.17.

$$\mathcal{M}_{g,k}(\Sigma, \Delta) = \coprod \mathcal{M}^{[h]}_{g,k}(\Sigma, \Delta),$$

where the disjoint union is over all combinatorial equivalence classes of curves of degree Δ and genus g with k marked points. For a given combinatorial equivalence class $[h]$ of a curve $h : (\Gamma, x_1, \ldots, x_k) \to M_{\mathbb{R}}$, $\mathcal{M}^{[h]}_{g,k}(\Sigma, \Delta)$ is the interior of a polyhedron of dimension

$$\geq e + k + (3 - \dim M_{\mathbb{R}})(g - 1) - \mathrm{ov}(\Gamma),$$

where

$$\mathrm{ov}(\Gamma) = \sum_{V \in \Gamma^{[0]}} \big(\mathrm{Valency}(V) - 3\big)$$

is the overvalence *of Γ and e is the number of non-compact unmarked edges of Γ.*

Proof. First note the topological Euler characteristic $\chi(\overline{\Gamma}) = 1 - g$ satisfies

$$\begin{aligned} \chi(\overline{\Gamma}) &= \#\overline{\Gamma}^{[0]} - \#\overline{\Gamma}^{[1]} \\ &= \#\Gamma^{[0]} - \#(\Gamma^{[1]} \setminus \Gamma^{[1]}_\infty). \end{aligned}$$

On the other hand,

$$3(\#\Gamma^{[0]}) + \mathrm{ov}(\Gamma) = \sum_{V \in \Gamma^{[0]}} \mathrm{Valency}(V) = 2(\#\Gamma^{[1]} \setminus \Gamma^{[1]}_\infty) + \#\Gamma^{[1]}_\infty,$$

from which we conclude that

$$\begin{aligned} \#\text{ of compact edges of } \Gamma &= \#\text{ non-compact edges of } \Gamma + 3g - 3 - \mathrm{ov}(\Gamma) \\ &= e + k + 3g - 3 - \mathrm{ov}(\Gamma). \end{aligned}$$

Now to describe all possible tropical curves with the given topological type, we choose a reference vertex $V \in \Gamma^{[0]}$, and we need to choose $h(V) \in M_{\mathbb{R}}$ and affine lengths[1] ℓ_E of each bounded edge $h(E)$. However, these lengths cannot be chosen independently. Indeed, suppose we have a cycle $E_1, \ldots, E_m$ of edges in Γ, $\partial E_i =$

[1]The affine length of a line segment of rational slope in $M_{\mathbb{R}}$ with endpoints m_1, m_2 is the number $\ell \in \mathbb{R}_{>0}$ such that $m_1 - m_2 = \ell m_{\mathrm{prim}}$ for some primitive $m_{\mathrm{prim}} \in M$.

$\{V_{i-1}, V_i\}$ with $V_m = V_0$. We of course have by definition of $m_{(V_{i-1},E_i)}$ that $V_i = V_{i-1} + \ell_{E_i} m_{(V_{i-1},E_i)}$. Thus $V_0 = V_m = V_0 + \sum_{i=1}^{m} \ell_{E_i} m_{(V_{i-1},E_i)}$, or

$$\sum_{i=1}^{m} \ell_{E_i} m_{(V_{i-1},E_i)} = 0$$

in $M_{\mathbb{R}}$. So, for each cycle, we obtain the above linear equation, which imposes $\dim M_{\mathbb{R}}$ linear conditions on the ℓ_E's. Thus, given that there exists a tropical curve of the given combinatorial type, the set of all curves of this combinatorial type is

$$M_{\mathbb{R}} \times (\mathbb{R}_{>0}^{e+k+3g-3-\mathrm{ov}(\Gamma)} \cap L)$$

where $L \subseteq \mathbb{R}^{e+k+3g-3-\mathrm{ov}(\Gamma)}$ is a linear subspace of codimension $\leq g \cdot \dim M_{\mathbb{R}}$ and hence the whole cell is of dimension

$$\geq e + k + (3 - \dim M_{\mathbb{R}})(g - 1) - \mathrm{ov}(\Gamma).$$

□

REMARK 1.18. One should view the case where all vertices of Γ are trivalent as a generic situation. However, there are tropical curves of genus $g \geq 1$ in $M_{\mathbb{R}}$ for $\dim M_{\mathbb{R}} \geq 3$ which are not trivalent and cannot be viewed as limits of trivalent curves.

Of course, for $g = 0$, equality always holds for the dimension, but for $g \geq 1$ equality need not hold. A curve of a given combinatorial type is said to be *superabundant* if the moduli space of curves of that type is larger than

$$e + k + (3 - \dim M_{\mathbb{R}})(g - 1) - \mathrm{ov}(\Gamma).$$

Otherwise a curve is called *regular*. Superabundant curves cause a great deal of difficulty for tropical geometry, and we shall handle this by restricting further to plane curves, i.e., $\dim M_{\mathbb{R}} = 2$. Furthermore, as we shall only need the genus zero case for our discussion, we shall often restrict our attention to this case also.

Restricting to the case that $\dim M_{\mathbb{R}} = 2$, we define

DEFINITION 1.19. A marked tropical curve $h : \Gamma \to M_{\mathbb{R}}$ for $\dim M_{\mathbb{R}} = 2$ is *simple* if

(1) Γ is trivalent;
(2) h is injective on the set of vertices and there are no disjoint edges E_1, E_2 with a common vertex V for which $h|_{E_1}$ and $h|_{E_2}$ are non-constant and $h(E_1) \subseteq h(E_2)$;
(3) Each unbounded unmarked edge of Γ has weight one.

By our discussion above, simple curves move in a family of dimension at least $|\Delta| + k + g - 1$, as now $e = |\Delta|$. However, one can show that simple curves in dimension two are always regular, see [**80**], Proposition 2.21, so in fact simple curves move in $(|\Delta| + k + g - 1)$-dimensional families. We know this for $g = 0$ already, but since we shall not be focussing on higher genus curves, we omit a proof of this fact.

LEMMA 1.20. *Fix Σ a fan in $M_{\mathbb{R}}$, $\dim M_{\mathbb{R}} = 2$, and a degree $\Delta \in T_{\Sigma}$. Let $P_1, \ldots, P_{|\Delta|-1} \in M_{\mathbb{R}}$ be general points.*[2] *Then there are a finite number of marked*

[2] By general, we mean that $(P_1, \ldots, P_{|\Delta|-1}) \in M_{\mathbb{R}}^{|\Delta|-1}$ lies in some dense open subset of $M_{\mathbb{R}}^{|\Delta|-1}$.

genus zero tropical curves $h : (\Gamma, x_1, \ldots, x_{|\Delta|-1}) \to M_{\mathbb{R}}$ *in* X_Σ *with* $h(x_i) = P_i$ *for all* i*. Furthermore, these curves are simple, and there is at most one such curve of any given combinatorial type.*

Proof. First note there are only a finite number of combinatorial types of curves of degree Δ in X_Σ. Indeed, given a curve $h : \Gamma \to M_{\mathbb{R}}$, we know by Proposition 1.15 that $h(\Gamma)$ is in fact a tropical hypersurface in $M_{\mathbb{R}}$. The degree Δ in fact determines the Newton polytope (up to translation) of a defining equation for $h(\Gamma)$. Furthermore, specifying a regular subdivision of the Newton polytope is equivalent to specifying the combinatorial type of $h(\Gamma)$. For each possible combinatorial type of $h(\Gamma)$, there are only a finite number of ways of parameterizing such a curve. Since there are a finite number of lattice subdivisions of the Newton polytope, this implies there are only a finite number of combinatorial types.

So in fact we can prove the result just by fixing one combinatorial type of curve, $[h]$. This gives the description as in the proof of Proposition 1.17,

$$\mathcal{M}^{[h]}_{0,|\Delta|-1}(\Sigma, \Delta) \cong M_{\mathbb{R}} \times \mathbb{R}^{e+|\Delta|-4-\mathrm{ov}(\Gamma)}_{>0},$$

obtained after choosing a reference vertex $V \in \Gamma^{[0]}$. We have an evaluation map

$$\mathrm{ev} : \mathcal{M}^{[h]}_{0,|\Delta|-1}(\Sigma, \Delta) \to (M_{\mathbb{R}})^{|\Delta|-1}$$

sending $h : (\Gamma, x_1, \ldots, x_{|\Delta|-1}) \to M_{\mathbb{R}}$ to

$$\mathrm{ev}(h) = (h(x_1), \ldots, h(x_{|\Delta|-1})).$$

Note that in fact ev is an affine linear map. Indeed, to compute $h(x_i)$ given h corresponding to a point in $M_{\mathbb{R}} \times \mathbb{R}^{e+|\Delta|-4-\mathrm{ov}(\Gamma)}_{>0}$, let $E_1, \ldots, E_n$ be the sequence of edges traversed from the reference vertex V to the vertex adjacent to E_{x_i}, with $\partial E_i = \{V_{i-1}, V_i\}$, $V_0 = V$. Then

$$h(x_i) = h(V) + \sum_{i=1}^{n} \ell_{E_i} m_{(V_{i-1}, E_i)}$$

where ℓ_{E_i} is the affine length of E_i. This shows that $h(x_i)$ depends affine linearly on $h(V)$ and the length of the edges.

Thus, unless $\dim \mathcal{M}^{[h]}_{0,|\Delta|-1}(\Sigma, \Delta) \geq \dim((M_{\mathbb{R}})^{|\Delta|-1})$, there is no curve of combinatorial type $[h]$ through general $(P_1, \ldots, P_{|\Delta|-1}) \in (M_{\mathbb{R}})^{|\Delta|-1}$. This inequality of dimensions only holds if

$$e + |\Delta| - 2 - \mathrm{ov}(\Gamma) \geq 2(|\Delta| - 1),$$

or

$$e - \mathrm{ov}(\Gamma) \geq |\Delta|.$$

Since $e \leq |\Delta|$ and $\mathrm{ov}(\Gamma) \geq 0$, strict inequality never holds and equality only holds if $e = |\Delta|$ and $\mathrm{ov}(\Gamma) = 0$, i.e., all unbounded edges of Γ are weight 1 and Γ is trivalent. If this equality holds, then either the image of ev is codimension ≥ 1, in which case again there are no curves of combinatorial type $[h]$ through general $(P_1, \ldots, P_{|\Delta|-1})$, or else ev is a local isomorphism and then there is at most one curve of combinatorial type $[h]$ passing through any $P_1, \ldots, P_{|\Delta|-1} \in M_{\mathbb{R}}$.

Finally, it is easy to see that the general curve in $\mathcal{M}^{[h]}_{0,|\Delta|-1}(\Sigma, \Delta)$ is injective on the set of vertices. Also, if there is a vertex V with attached edges E_1, E_2 with $h|_{E_1}$, $h|_{E_2}$ non-constant and $h(E_1) \subseteq h(E_2)$, then we are free to move $h(V)$ along the affine line containing $h(E_i)$, violating the fact that there is only one such curve. So a curve of type $[h]$ passing through general points $P_1, \ldots, P_{|\Delta|-1}$ is simple, as desired. $\square$

This result allows us to count tropical curves passing through a general set of points. However, as Mikhalkin showed, to get a meaningful result these must be counted with a suitable multiplicity, which we now define.

Definition 1.21. Let $h : \Gamma \to M_{\mathbb{R}}$ be a simple tropical curve, with $\dim M_{\mathbb{R}} = 2$. We define for $V \in \Gamma^{[0]}$ with adjacent edges E_1, E_2 and E_3,

$$\begin{aligned} \mathrm{Mult}_V(h) &= w_\Gamma(E_1)w_\Gamma(E_2)|m_{(V,E_1)} \wedge m_{(V,E_2)}| \\ &= w_\Gamma(E_2)w_\Gamma(E_3)|m_{(V,E_2)} \wedge m_{(V,E_3)}| \\ &= w_\Gamma(E_3)w_\Gamma(E_1)|m_{(V,E_3)} \wedge m_{(V,E_1)}| \end{aligned}$$

if none of E_1, E_2, E_3 are marked, and otherwise $\mathrm{Mult}_V(h) = 1$. Here for $m_1, m_2 \in M$, we identify $\bigwedge^2 M$ with $\mathbb{Z}$ so that $|m_1 \wedge m_2|$ makes sense. The equalities follow from the balancing condition.

We then define the *(Mikhalkin) multiplicity of* h to be

$$\mathrm{Mult}(h) = \prod_{V \in \Gamma^{[0]}} \mathrm{Mult}_V(h).$$

Finally, for a given fan Σ and degree Δ, we write

$$N^{0,\mathrm{trop}}_{\Delta,\Sigma} = \sum_h \mathrm{Mult}(h)$$

where the sum is over all $h \in \mathcal{M}_{0,|\Delta|-1}(\Sigma, \Delta)$ passing through $|\Delta|-1$ general points in $M_{\mathbb{R}}$.

While the generality of these points guarantees that the sum makes sense, it is not obvious that $N^{0,\mathrm{trop}}_{\Delta,\Sigma}$ doesn't depend on the choice of these points. This will be shown later, twice, once in Chapter 4 and once in Chapter 5.

In fact, the same definition can be made for curves of genus g. Indeed, one can show that, for a choice of $|\Delta|+g-1$ general points in $M_{\mathbb{R}}$, there are a finite number of simple genus g curves passing through these points (see [**80**], Proposition 2.23). Using the same definition of multiplicity, one obtains numbers $N^{g,\mathrm{trop}}_{\Delta,\Sigma}$.

If $\dim M_{\mathbb{R}} > 2$, there are similar definitions for counting formulas for genus zero curves: see [**86**] for precise statements. However, because of superabundant families of $g > 0$ curves, there are more serious issues in higher genus.

Mikhalkin's main result in [**80**] relates the numbers $N^{g,\mathrm{trop}}_{\Delta,\Sigma}$, which are of course purely combinatorial, to counts of holomorphic curves in the toric variety X_Σ. Stating this result rather imprecisely here, he shows that the numbers $N^{g,\mathrm{trop}}_{\Delta,\Sigma}$ coincide with the numbers $N^{g,\mathrm{hol}}_{\Delta,\Sigma}$ of holomorphic curves of genus g in X_Σ passing through $|\Delta|+g-1$ points in general position. The fact that this count can be computed in this purely combinatorial fashion was the first significant result in tropical geometry. We shall give a proof of this result for genus zero in Chapter 4.

1.4. Affine manifolds with singularities

We shall now discuss possible generalizations of the discussion of tropical curves. The question we want to pose here is: all our curves have lived in $M_{\mathbb{R}}$, a real affine space. Are there interesting choices for more general target manifolds?

One could, for example, study tropical curves inside tropical hypersurfaces, as has been done, say, in [**111**]. However, this is not the point of view we want to take here. Instead, we want to look at target spaces which "locally look like $M_{\mathbb{R}}$," in such a way that we can still talk about tropical curves. The main point is that to talk about tropical curves, one needs the structure of $M_{\mathbb{R}}$ as an affine space, but one also needs to know about the integral structure $M \subseteq M_{\mathbb{R}}$.

In what follows, we consider the group $\mathrm{Aff}(M_{\mathbb{R}}) = M_{\mathbb{R}} \rtimes \mathrm{GL}(M_{\mathbb{R}})$ of affine linear automorphisms of $M_{\mathbb{R}}$, given by $m \mapsto Am + b$, where $A \in GL_n(\mathbb{R})$ and $b \in M_{\mathbb{R}}$, and its subgroups

$$M_{\mathbb{R}} \rtimes \mathrm{GL}(M) \subseteq \mathrm{Aff}(M_{\mathbb{R}})$$

and

$$\mathrm{Aff}(M) := M \rtimes \mathrm{GL}(M) \subseteq \mathrm{Aff}(M_{\mathbb{R}}).$$

Definition 1.22. A *tropical affine manifold* is a real topological manifold B (possibly with boundary) with an atlas of coordinate charts $\psi_i : U_i \to M_{\mathbb{R}}$ with transition functions $\psi_i \circ \psi_j^{-1} \in M_{\mathbb{R}} \rtimes \mathrm{GL}(M) \subseteq \mathrm{Aff}(M_{\mathbb{R}})$.

An *integral affine manifold* is a tropical manifold with transition functions in $\mathrm{Aff}(M)$.

We will often make use of two local systems on a tropical affine manifold, defining $\Lambda \subseteq \mathcal{T}_B$ to be the family of lattices locally generated by $\partial/\partial y_1, \ldots, \partial/\partial y_n$ for $y_1, \ldots, y_n$ local affine coordinates. The sheaf $\check{\Lambda} \subseteq \mathcal{T}_B^*$ is the dual local system locally generated by $dy_1, \ldots, dy_n$. The point is these families of lattices are well-defined on tropical manifolds because of the restriction on the transition maps. Note that $\mathcal{T}_B$ carries a natural flat connection, ∇_B, with flat sections being $\mathbb{R}$-linear combinations of $\partial/\partial y_1, \ldots, \partial/\partial y_n$.

It is easy to generalize the notion of parameterized tropical curve with target a tropical affine manifold, as locally the notion of a line segment with rational slope and the balancing condition make sense.

Example 1.23. $B = M_{\mathbb{R}}/\Gamma$ for a lattice $\Gamma \subseteq M_{\mathbb{R}}$ gives an example of a compact tropical affine manifold. In [**82**], such spaces arise naturally as tropical Jacobians of tropical curves. Figure 15 gives an example of a two-dimensional torus containing a genus 2 tropical curve.

Unfortunately, tori are not particularly useful for the applications we have in mind, so we shall generalize the notion of tropical affine manifold as follows.

Definition 1.24. A *tropical affine manifold with singularities* is a topological manifold B along with data

- a subset $\Delta \subseteq B$ which is a locally finite union of codimension ≥ 2 locally closed submanifolds of B;
- a tropical affine structure on $B_0 := B \setminus \Delta$.

We say B is an *integral affine manifold with singularities* if the affine structure on B_0 is integral. The set Δ is called the *singular locus* or *discriminant locus* of B.

Note that we are assuming that B is still a topological manifold even at the singular points: the singularities lie in the affine structure in the sense that, in general, the affine structure cannot be extended across Δ. We shall see some examples later, but for the moment one can imagine a two-dimensional cone as an

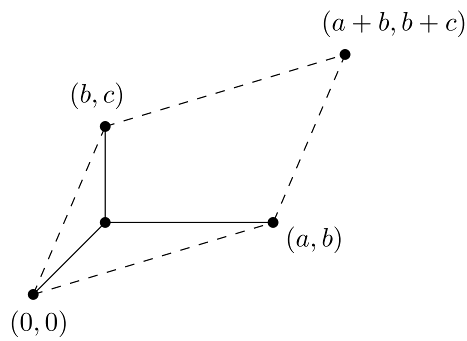

FIGURE 15. A tropical curve of genus two in $\mathbb{R}^2/\Gamma$, where Γ is generated by (a, b) and (b, c). The dotted lines give a fundamental domain, and the three external vertices of the curve are in fact identified, so these vertices represent a single trivalent vertex.

example, obtained by cutting an angular sector out of a piece of paper and then gluing together the two edges of the sector. (However, in fact we will not ultimately allow this particular example.)

A priori, the singularities of the affine structure can be arbitrarily complicated, and there are many reasonable examples which we shall not wish to consider. In addition, it is often convenient to consider restrictions on the nature of the boundary of B. To control the singularities and the boundary, we introduce a refined notion which arises from the following construction.

CONSTRUCTION 1.25. Let B be a topological manifold (possibly with boundary) equipped with a polyhedral decomposition $\mathscr{P}$, i.e.,

$$B = \bigcup_{\sigma \in \mathscr{P}} \sigma$$

where

- $\sigma \in \mathscr{P}$ is a subset of B equipped with a homeomorphism to a (not necessarily compact) polyhedron in $M_{\mathbb{R}}$ with faces of rational slope and at least one vertex. Thus in particular any $\sigma \in \mathscr{P}$ has a set of faces: these faces are inverse images of faces of the polyhedron in $M_{\mathbb{R}}$.
- If $\sigma \in \mathscr{P}$ and $\tau \subseteq \sigma$ is a face, then $\tau \in \mathscr{P}$.
- If $\sigma_1, \sigma_2 \in \mathscr{P}$, $\sigma_1 \cap \sigma_2 \neq \emptyset$, then $\sigma_1 \cap \sigma_2$ is a face of σ_1 and σ_2.

For each $\sigma \in \mathscr{P}$, viewing $\sigma \subseteq M_{\mathbb{R}}$ yields a tangent space $\Lambda_{\sigma,\mathbb{R}} \subseteq M_{\mathbb{R}}$ to σ, and we can set

$$\Lambda_\sigma := \Lambda_{\sigma,\mathbb{R}} \cap M.$$

The assumption that σ has faces of rational slope implies in particular that if σ is of codimension at least one in $M_{\mathbb{R}}$, then the affine space spanned by σ has rational slope. Thus Λ_σ generates $\Lambda_{\sigma,\mathbb{R}}$ as an $\mathbb{R}$-vector space.

Now the interior of each $\sigma \in \mathscr{P}$ carries a natural affine structure. Indeed, σ is equipped with a homeomorphism with a polyhedron in $M_{\mathbb{R}}$, which is embedded in the affine space it spans in $M_{\mathbb{R}}$. This gives an affine coordinate chart on $\mathrm{Int}(\sigma)$. However, this doesn't define an affine structure on B, but only an affine structure

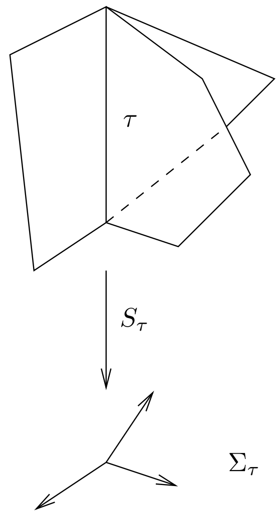

FIGURE 16. A fan structure given by the map S_τ in a neighbourhood of the one-dimensional cell τ. This can be viewed as describing an affine structure in a direction transverse to τ.

on the subset of B given by

$$\bigcup_{\sigma\in\mathscr{P}_{\max}} \mathrm{Int}(\sigma),$$

where $\mathscr{P}_{\max}$ is the set of maximal cells in $\mathscr{P}$. This is insufficient for giving a structure of tropical affine manifold with singularities to B, so we need to extend this affine structure. To do so requires the choice of some extra structure, known as a fan structure.

DEFINITION 1.26. Let $\tau \in \mathscr{P}$. The *open star* of τ is

$$U_\tau := \bigcup_{\sigma \in \mathscr{P} \text{ s.t. } \tau\subseteq\sigma} \mathrm{Int}(\sigma).$$

A *fan structure* along $\tau \in \mathscr{P}$ is a continuous map $S_\tau : U_\tau \to \mathbb{R}^k$ where $k = \dim B - \dim\tau$, satisfying

(1) $S_\tau^{-1}(0) = \mathrm{Int}(\tau)$.
(2) If $\tau \subseteq \sigma$, then $S_\tau|_{\mathrm{Int}(\sigma)}$ is an integral affine submersion onto its image, with $\dim S_\tau(\sigma) = \dim\sigma - \dim\tau$. By integral affine submersion we mean the following. We can think of σ as a lattice polytope in $\Lambda_{\sigma,\mathbb{R}}$. Then the map $S_\tau|_\sigma$ is induced by a surjective affine map $\Lambda_\sigma \to W \cap \mathbb{Z}^k$, for some vector subspace $W \subseteq \mathbb{R}^k$ of codimension equal to the codimension of σ in B.
(3) For $\tau\subseteq\sigma$, define $K_{\tau,\sigma}$ to be the cone generated by $S_\tau(\sigma\cap U_\tau)$. Then

$$\Sigma_\tau := \{K_{\tau,\sigma} \mid \tau\subseteq\sigma\in\mathscr{P}\}$$

is a fan with $|\Sigma_\tau|$ convex.

Two fan structures S_τ, S'_τ are considered *equivalent* if $S_\tau = \alpha\circ S_{\tau'}$ for some $\alpha \in \mathrm{GL}_k(\mathbb{Z})$.

If $S_\tau : U_\tau \to \mathbb{R}^k$ is a fan structure along $\tau \in \mathscr{P}$ and $\sigma \supseteq \tau$, then $U_\sigma \subseteq U_\tau$. We then obtain a fan structure along σ induced by S_τ given by the composition

$$U_\sigma \hookrightarrow U_\tau \xrightarrow{S_\tau} \mathbb{R}^k \to \mathbb{R}^k/L_\sigma \cong \mathbb{R}^l$$

where $L_\sigma \subseteq \mathbb{R}^k$ is the linear span of $K_{\tau,\sigma}$. This is well-defined up to equivalence. It is easy to see that with the induced fan structure on σ, $\Sigma_\sigma = \Sigma_\tau(K_{\tau,\sigma})$ in the notation of Definition 1.10.

See Figure 16 for a picture of a fan structure. The most important case is when $\tau = v$ a vertex of $\mathscr{P}$. Then a fan structure is an identification of a neighbourhood of v in B with a neighbourhood of the origin in $\mathbb{R}^n$ ($n = \dim B$). This identification locally near v identifies $\mathscr{P}$ with a fan Σ_v in $\mathbb{R}^n$.

Given a fan structure S_v at each vertex $v \in \mathscr{P}$, we can construct a tropical structure on B as follows. We first need to choose a discriminant locus $\Delta \subseteq B$. The precise details of this choice of discriminant locus in fact turn out not to be so important, and it can be chosen fairly arbitrarily, subject to certain constraints:

(1) Δ does not contain any vertex of $\mathscr{P}$.
(2) Δ is disjoint from the interior of any maximal cell of $\mathscr{P}$.
(3) For any $\rho \in \mathscr{P}$ which is a codimension one cell not contained in ∂B, the connected components of $\rho \setminus \Delta$ are in one-to-one correspondence with vertices of ρ, with each vertex contained in the corresponding connected component.

For example, if $\dim B = 2$, we simply choose one point in the interior of each compact edge of $\mathscr{P}$ not contained in ∂B, and take Δ to be the set of these chosen points. If B is compact without boundary of any dimension, we can take Δ to be the union of all simplices in the first barycentric subdivision of $\mathscr{P}$ which neither contain a vertex of $\mathscr{P}$ nor intersect the interior of a maximal cell of $\mathscr{P}$. For the general case, see [**49**], §1.1. We should also note that, for us, this is a maximal choice of discriminant locus, and if there is a subset $\Delta' \subseteq \Delta$ such that the affine structure on $B \setminus \Delta$ extends to an affine structure across $B \setminus \Delta'$, we will replace Δ with Δ' without comment.

For a vertex v of $\mathscr{P}$, let W_v denote a choice of open neighbourhood of v with $W_v \subseteq U_v$ satisfying the condition that if $v \in \rho$ with ρ a codimension one cell, then $W_v \cap \rho$ is the connected component of $\rho \setminus \Delta$ containing v. Then

$$\{\mathrm{Int}(\sigma) \,|\, \sigma \in \mathscr{P}_{\max}\} \cup \{W_v \,|\, v \in \mathscr{P}^{[0]}\}$$

form an open cover of $B_0 := B \setminus \Delta$. We define an affine structure on B_0 via the already given affine structure on $\mathrm{Int}(\sigma)$ for $\sigma \in \mathscr{P}_{\max}$,

$$\psi_\sigma : \mathrm{Int}(\sigma) \hookrightarrow M_{\mathbb{R}}$$

and the composed maps

$$\psi_v : W_v \hookrightarrow U_v \xrightarrow{S_v} \mathbb{R}^{\dim B}$$

where the first map is the inclusion.

It is easy to see that this produces the structure of a tropical affine manifold with singularities on B. Indeed, the crucial point is that the affine charts ψ_v induced by the choice of fan structure are compatible with the charts ψ_σ on the interior of maximal cells of $\mathscr{P}$, but this follows precisely from item (2) in the definition of a fan structure.

If furthermore all polyhedra in $\mathscr{P}$ are lattice polytopes, then in fact the affine structure is integral.

This construction provides a wide class of examples. However, these examples are still too general. We will impose one additional condition.

We say a collection of fan structures $\{S_v \mid v \in \mathscr{P}^{[0]}\}$ is *compatible* if, for any two vertices v, w of $\tau \in \mathscr{P}$, the fan structures induced on τ by S_v and S_w are equivalent. Note that given such a compatible set of fan structures, we obtain well-defined fan structures along every $\tau \in \mathscr{P}$.[3]

DEFINITION 1.27. A *tropical manifold* is a pair $(B, \mathscr{P})$ where B is a tropical affine manifold with singularities obtained from the polyhedral decomposition $\mathscr{P}$ of B and a compatible collection $\{S_v \mid v \in \mathscr{P}^{[0]}\}$ of fan structures.

$(B, \mathscr{P})$ is an *integral tropical manifold* if in addition all polyhedra in $\mathscr{P}$ are lattice polyhedra.

EXAMPLES 1.28. (1) Any lattice polyhedron σ with at least one vertex supplies an example of an integral tropical manifold, either bounded or unbounded, with $B = \sigma$ and $\mathscr{P}$ the set of faces of σ. In this case the affine structure on $\mathrm{Int}(\sigma)$ extends to give the structure of an affine manifold (with boundary) on σ. Here $\Delta = \emptyset$.

(2) The polyhedral decomposition of $B = M_{\mathbb{R}}$ given in Definition 1.4 is also an example of a tropical manifold (provided the tropical hypersurface in question has at least one vertex).

(3) Let $\Xi \subseteq M_{\mathbb{R}}$ be a reflexive lattice polytope, i.e., $0 \in \mathrm{Int}(\Xi)$ and the polytope

$$\Xi^* := \{n \in N_{\mathbb{R}} \mid \langle n, m\rangle \geq -1 \quad \forall m \in \Xi\}$$

is also a lattice polytope.

Then $B = \partial\Xi$ carries the obvious polyhedral decomposition $\mathscr{P}$ consisting of the proper faces of Ξ. These faces are lattice polytopes. So, to specify an integral tropical manifold structure on B, we need only specify a fan structure at each vertex v of Ξ. This is done via the projection $S_v : U_v \to M_{\mathbb{R}}/\mathbb{R}v$. Compatibility is easily checked, as the induced fan structure on a cell $\omega \in \mathscr{P}$ containing v is the projection $U_\omega \to M_{\mathbb{R}}/\mathbb{R}\omega$, where $\mathbb{R}\omega$ now denotes the vector subspace of $M_{\mathbb{R}}$ spanned by ω.

There are a number of refinements of this construction. For example, if we take a refinement $\mathscr{P}'$ of $\mathscr{P}$ by integral lattice polytopes, we can use the same prescription above for the fan structure at the vertices.

Note that, in this case, the description of $\Delta' \subseteq B$ of the singular locus as determined by the refinement $\mathscr{P}'$ may give a much bigger discriminant locus, with $\Delta' \cap \mathrm{Int}(\sigma) \neq \emptyset$ for σ a maximal proper face of Ξ. However, the affine structure induced by $\mathscr{P}'$ on $\mathrm{Int}(\sigma)$ is compatible with the obvious affine structure on $\mathrm{Int}(\sigma)$, so in fact the affine structure extends across points of $\Delta' \cap \mathrm{Int}(\sigma)$. Thus we can replace Δ' with

$$\Delta' \cap \bigcup_{\substack{\tau \in \mathscr{P} \\ \dim\tau = \dim\Xi - 2}} \tau.$$

For example, let

$$\Xi_1 := \mathrm{Conv}\{(-1,-1,-1), (3,-1,-1), (-1,3,-1), (-1,-1,3)\}.$$

[3]In the case we will focus on in this book, $\dim B = 2$, this compatibility condition is in fact trivial, since, provided $v \neq w$, τ is dimension one or two and there are not many choices for zero or one-dimensional fans! So, for the most part, the reader can ignore this condition.

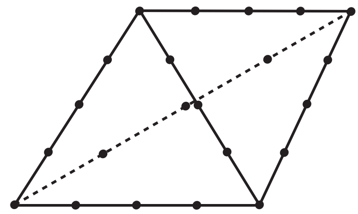

Figure 17

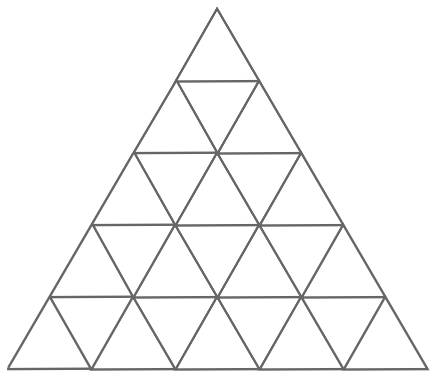

Figure 18

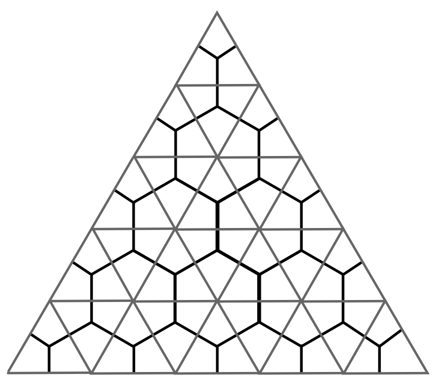

Figure 19

Then choose $\mathscr{P}'$ so that each edge of $\mathscr{P}$ is subdivided into four line segments of affine length 1, as in Figure 17. Then Δ' consists of 24 points, one in the interior of each of these line segments.

We can repeat this in higher dimension, say taking

$$\Xi_2 := \mathrm{Conv}\{(-1,-1,-1,-1),(4,-1,-1,-1),\ldots,(-1,-1,-1,4)\}.$$

Suppose $\mathscr{P}'$ is chosen so that each 2-face of Ξ is triangulated by $\mathscr{P}'$ as in Figure 18. Then Δ' restricted to such a two-face is depicted in Figure 19.

So far, these examples are all integral tropical manifolds. To obtain examples which are not integral, one can deform one of the above examples continuously.

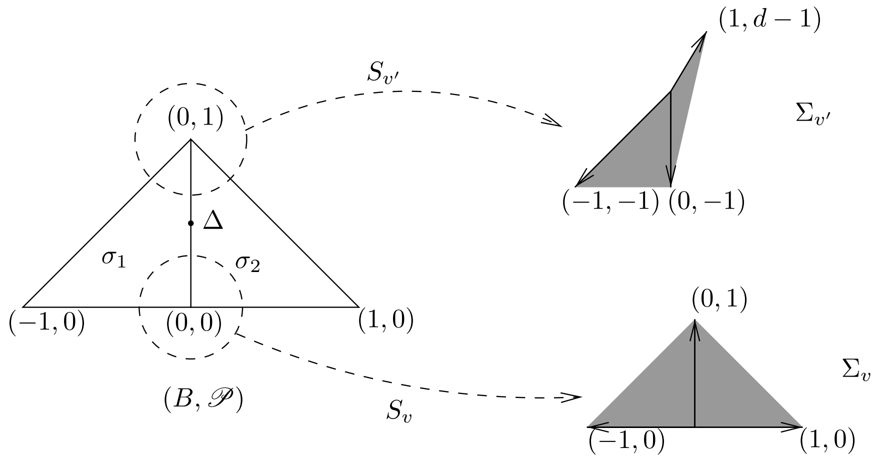

FIGURE 20. $(B, \mathscr{P})$ is shown on the left, with the embedding in $\mathbb{R}^2$ shown giving the correct affine structures on σ_1 and σ_2. The fan structures S_v and $S_{v'}$ depicted then turn $(B, \mathscr{P})$ into a tropical manifold.

Suppose $\Xi' \subseteq M_{\mathbb{R}}$ is a deformation of Ξ which has the same normal fan. For example, Ξ_1 and Ξ_2 can just be rescaled, but

$$\Xi_3 := \{(x, y, z) \mid -1 \leq x, y, z \leq 1\}$$

can be deformed to

$$\Xi_3' := \{(x, y, z) \mid -a \leq x \leq a, -b \leq y \leq b, -c \leq z \leq c\}.$$

As before, take the polyhedral decomposition $\mathscr{P}'$ of $B' = \partial\Xi'$ to consist of all proper faces of Ξ'. Then, for each vertex $v' \in \mathscr{P}'$ corresponding to a vertex v of Ξ, define a fan structure $S_{v'} : W_{v'} \to M_{\mathbb{R}}/\mathbb{R}v$ via

$$S_{v'}(m) = m - v' \mod \mathbb{R}v.$$

One checks easily again that this defines a fan structure, and hence a tropical manifold $(B', \mathscr{P}')$ which is not, in general, integral, as the elements of $\mathscr{P}'$ are not lattice polytopes.

These are all special cases of much more general constructions given in [**40, 54, 55**]. The Batyrev-Borisov [**7**] construction, in particular, for complete intersection Calabi-Yau varieties in toric varieties yields vast numbers of examples, as discussed in [**40, 55**].

(4) In Chapter 6, we will focus largely on the two-dimensional case. In this case, the classification of singularities which may occur in tropical manifolds is particularly straightforward. We now give a simple model for the singularities which can occur.

Consider $(B, \mathscr{P})$ as depicted in Figure 20, with two maximal cells σ_1, σ_2 and one one-dimensional cell $\tau = \sigma_1 \cap \sigma_2$, and four vertices, including $v = (0,0)$ and $v' = (0,1)$, as depicted. The picture gives the affine structures on σ_1 and σ_2, and to specify the full affine structure we need to specify fan structures at the vertices v and v', which are shown on the right in Figure 20. We note that we are violating the condition that $|\Sigma_{v'}|$ be convex, if $d > 2$, but this can be rectified by embedding

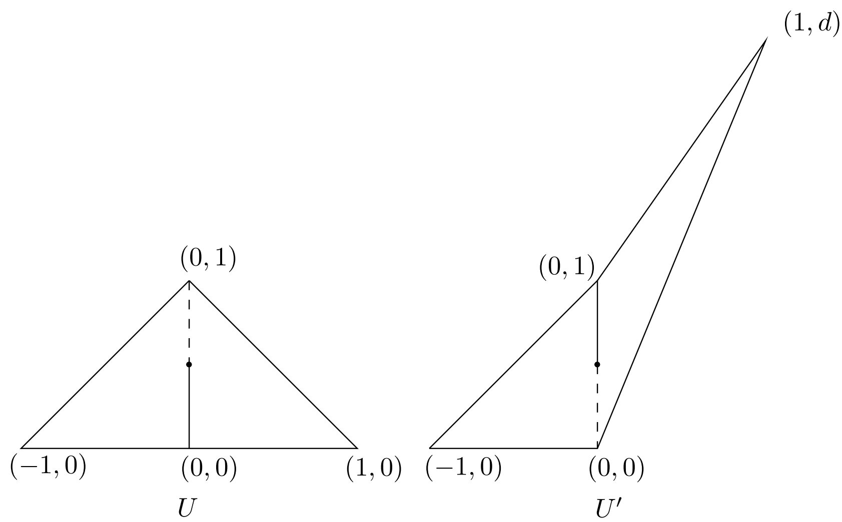

FIGURE 21

this local example in a larger B. One can think of the induced affine structure as coming from two charts on a cover of $B_0 = B \setminus \Delta$ consisting of the open sets

$$\begin{aligned} U &= B \setminus \{(0,x) \in B \,|\, x \geq 1/2\} \\ U' &= B \setminus \{(0,x) \in B \,|\, x \leq 1/2\}. \end{aligned}$$

Figure 21 then shows the embeddings of U and U' into $\mathbb{R}^2$. The best way to describe the singularity is to describe the monodromy of the local system Λ around, say, a counterclockwise loop $\gamma : [0,1] \to B_0$ about Δ. Suppose $\gamma(0)$ is a point right below Δ. Using the chart on U, we identify the integral tangent vectors at $\gamma(0)$ with $\mathbb{Z}^2 \subseteq \mathbb{R}^2$, the latter being the tangent space of the chart on U. We then move into σ_2 along γ, and to cross back into σ_1 above Δ we need to switch to the chart on U', which requires applying on σ_2 the linear transformation $\left(\begin{smallmatrix} 1 & 0 \\ d & 1 \end{smallmatrix}\right)$; this acts the same way on tangent vectors. We can then complete our loop γ through U', and switching back to U requires no further change of coordinates. Hence the monodromy of Λ around γ is given in the standard basis by $\left(\begin{smallmatrix} 1 & 0 \\ d & 1 \end{smallmatrix}\right)$.

In fact, this example describes the local structures of all possible singularities which can occur in two-dimensional tropical manifolds.

An important feature of these singularities is that tangent vectors to the edge containing the singularity are left invariant by monodromy.

1.5. The discrete Legendre transform

We now wish to generalize the discrete Legendre transform that we saw in §1.1 to tropical manifolds. As we shall describe in Chapter 6, the motivation for this is that the discrete Legendre transform in fact describes a form of mirror symmetry. This section may be skipped on a first reading.

To define the discrete Legendre transform, we first need to generalize the notion of a convex piecewise linear function. Even if there are no singularities, a tropical manifold B may carry no convex functions, e.g., $B = M_{\mathbb{R}}/\Gamma$ a torus. Instead, we use multi-valued functions.

To begin, we need

Definition 1.29. A function $f : M_{\mathbb{R}} \to \mathbb{R}$ is *affine linear* if it is of the form $m \mapsto \langle n, m\rangle + b$ for some $n \in N_{\mathbb{R}}$, $b \in \mathbb{R}$. It is *integral affine linear* if $n \in N$, $b \in \mathbb{Z}$.

Definition 1.30. If B is an affine manifold, an *affine linear function* $f : B \to \mathbb{R}$ is a continuous function given on a coordinate chart $\psi_i : U_i \to M_{\mathbb{R}}$ by the composition of ψ_i with an affine linear function $M_{\mathbb{R}} \to \mathbb{R}$. Furthermore, if B is integral, we say $f : B \to \mathbb{R}$ is *integral affine* if it is locally given by $M_{\mathbb{R}} \to \mathbb{R}$ integral affine linear.

If $(B, \mathscr{P})$ is an (integral) tropical manifold, an (integral) affine function on an open set $U \subseteq B$ is a continuous map $\varphi : U \to \mathbb{R}$ that is (integral) affine on $U \setminus \Delta$.

An (integral) PL function on U is a continuous map $\varphi : U \to \mathbb{R}$ such that if $S_\tau : U_\tau \to \mathbb{R}^k$ is the fan structure along $\tau \in \mathscr{P}$, then

$$\varphi|_{U\cap U_\tau} = \lambda + \varphi_\tau \circ S_\tau, \tag{1.4}$$

where

$$\varphi_\tau : |\Sigma_\tau| \to \mathbb{R}$$

is an (integral) PL function on the fan Σ_τ determined by the fan structure S_τ along τ, and λ is (integral) affine linear.

Denote by $\mathcal{A}\!f\!f(B, \mathbb{R})$ the sheaf of affine linear functions on B (and denote by $\mathcal{A}\!f\!f(B, \mathbb{Z})$ the sheaf of integral affine linear functions if B is integral), and similarly by $\mathcal{PL}_{\mathscr{P}}(B, \mathbb{R})$ (or $\mathcal{PL}_{\mathscr{P}}(B, \mathbb{Z})$) the sheaf of (integral) PL functions.

A *multi-valued (integral) PL function* is a section of $\mathcal{PL}_{\mathscr{P}}(B, \mathbb{R})/\mathcal{A}\!f\!f(B, \mathbb{R})$ (respectively $\mathcal{PL}_{\mathscr{P}}(B, \mathbb{Z})/\mathcal{A}\!f\!f(B, \mathbb{Z})$). In other words, it is a collection $\{(U_i, \varphi_i)\}$ of PL functions with $\varphi_i - \varphi_j$ affine linear on $U_i \cap U_j$.

We say a multi-valued PL function on U is *convex* if it is locally represented on $U \cap U_\tau$ by $\varphi_\tau \circ S_\tau$ with φ_τ a strictly convex function.

We can now generalize the discrete Legendre transform given in §1.1. Given a triple $(B, \mathscr{P}, \varphi)$, where $(B, \mathscr{P})$ is a tropical manifold and φ is a multi-valued strictly convex PL function on B, we will define the discrete Legendre transform of $(B, \mathscr{P}, \varphi)$, denoted by $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$.

For any $\tau \in \mathscr{P}$, we have a representative $\varphi_\tau : |\Sigma_\tau| \to \mathbb{R}$ of φ as in (1.4), and set $\check{\tau} := \Delta_{\varphi_\tau}$, the Newton polyhedron of φ_τ. If $\tau \subseteq \sigma$ for $\tau, \sigma \in \mathscr{P}$, then Σ_σ is the quotient fan of Σ_τ by the cone in Σ_τ corresponding to σ and, up to a linear map, $\varphi_\sigma : |\Sigma_\sigma| \to \mathbb{R}$ is induced by $\varphi_\tau : |\Sigma_\tau| \to \mathbb{R}$ as in Definition 1.10. Hence $\check{\sigma}$ can be identified with a face of $\check{\tau}$.

This gives us the pair $(\check{B}, \check{\mathscr{P}})$, where $\check{\mathscr{P}} = \{\check{\sigma} \,|\, \sigma \in \mathscr{P}\}$, and $\check{B}$ is obtained by identifying $\check{\tau}_1$ and $\check{\tau}_2$ along the common face $\check{\sigma}$ if $\sigma \in \mathscr{P}$ is the smallest cell of $\mathscr{P}$ containing τ_1 and τ_2.

It is easy to see $\check{B}$ constructed in this way is a topological manifold, with $\check{\mathscr{P}}$ the dual polyhedral complex to $\mathscr{P}$, and in fact $B \setminus \partial B$ and $\check{B} \setminus \partial \check{B}$ are homeomorphic.

To complete the description of $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$, we need to define the fan structure on $(\check{B}, \check{\mathscr{P}})$ and the function $\check{\varphi}$.

For $\sigma \in \mathscr{P}_{\max}$, $\check{\sigma}$ is a vertex of $\check{\mathscr{P}}$, and let $\check{\Sigma}_\sigma$ denote the normal fan (Definition 1.9) to σ. The cones of $\check{\Sigma}_\sigma$ are in one-to-one inclusion reversing correspondence with faces of σ, and for $\tau \subseteq \sigma$ the tangent wedge (Definition 1.9) to $\check{\tau}$ at $\check{\sigma}$ is $N_\sigma(\tau)$. Thus there is a natural fan structure

$$\check{S}_{\check{\sigma}} : U_{\check{\sigma}} \to N_{\mathbb{R}}.$$

Indeed, locally $U_{\check{\sigma}} \cap \check{\tau}$ looks like the tangent wedge to $\check{\tau}$ at $\check{\sigma}$, which is identified via $\check{S}_{\check{\sigma}}$ with the normal cone $N_\sigma(\tau)$. Finally, we define $\check{\varphi}$ by taking $\check{\varphi}_\tau$ on $\Sigma_{\check{\tau}} = \check{\Sigma}_\tau$ to be the function induced by τ as in Definition 1.9.

EXAMPLES 1.31. (1) Let $B = \sigma \subseteq M_{\mathbb{R}}$ be a strictly convex rational polyhedral cone, $\mathscr{P}$ the set of faces of σ, and $\varphi \equiv 0$. Then $\check{B}$ is simply the dual cone

$$\sigma^\vee := \{n \in N_{\mathbb{R}} \,|\, \langle n, m\rangle \geq 0 \text{ for all } m \in \sigma\}$$

and $\check{\varphi} \equiv 0$.

(2) Let $B \subseteq M_{\mathbb{R}}$ be a polyhedron, $\mathscr{P}$ its set of faces, $\varphi \equiv 0$. Then $\check{\mathscr{P}}$ is the normal fan to B in $N_{\mathbb{R}}$, $\check{B}$ is the support of the normal fan, and

$$\check{\varphi}(n) := -\inf\{\langle n, m\rangle \,|\, m \in B\}$$

is the PL function on the normal fan induced by B, as in Definition 1.9.

(3) Let $B \subseteq N_{\mathbb{R}}$ be a (compact) polytope, $\mathscr{P}$ a polyhedral decomposition of B, φ a strictly convex PL function on B. Then $\check{B} = M_{\mathbb{R}}$, and $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ agrees with the discrete Legendre transform defined in §1.1, up to a change of sign of $\check{\varphi}$.

(4) Let $\Xi \subseteq M_{\mathbb{R}}$ be a reflexive polytope, $B = \partial\Xi$, $\mathscr{P}$ the set of proper faces of B, so the construction of Example 1.28, (3), yields an integral tropical manifold $(B, \mathscr{P})$. Define φ as follows. First, let Σ be the fan defined by $\Sigma = \{\mathbb{R}_{\geq 0}\sigma \,|\, \sigma \in \mathscr{P}\}$. The fact that Ξ is reflexive implies there is a strictly convex integral PL function $\psi : |\Sigma| \to \mathbb{R}$ which takes the value 1 on each vertex of Ξ. For $\tau \in \mathscr{P}$, choose $n_\tau \in N$ such that $n_\tau = \psi|_\tau$ on τ, and define φ_τ on the quotient fan $\Sigma(\mathbb{R}_{\geq 0}\tau)$ to be defined by the function induced by $\psi - n_\tau$, as in Definition 1.10. From the way the fan structure was defined on B, $\Sigma(\mathbb{R}_{\geq 0}\tau)$ is precisely the fan Σ_τ associated to $\tau \in \mathscr{P}$. Hence the collection of PL functions $\{\varphi_\tau \,|\, \tau \in \mathscr{P}\}$ defines a convex multi-valued integral PL function on B.

As an exercise in the definitions, check that $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ is the integral tropical manifold defined by the same construction using the dual reflexive polytope Ξ^*.

(5) Consider the surface depicted in Figure 22. This figure depicts a surface B which is the union of three two-dimensional simplices, each isomorphic to the standard two-simplex. Furthermore, there is one point of the discriminant locus along each interior edge. The figure shows an open subset of B obtained by removing the dotted lines along the edges. This open set is embedded in $\mathbb{R}^2$ as depicted by an affine coordinate chart. In this embedding, the vertices are $v_0 = (0, 0)$, $v_1 = (1, 0)$, $v_2 = (0, 1)$ and $v_3 = (-1, -1)$. This embedding defines a fan structure at the internal vertex, and we give fan structures at the three boundary vertices as depicted. As a result, the boundary is actually a straight line in the affine structure. This gives B and $\mathscr{P}$, and in addition, we can choose a strictly convex PL function φ. On the open set depicted in the figure, φ will be given by $(0, 0)$ on the upper right-hand triangle, $(-1, 0)$ on the left-hand triangle, and $(0, -1)$ on the lower left-hand triangle. In other words, φ is completely determined by the fact that it is zero at all vertices, except for the vertex v_3, where φ is 1.

What is the discrete Legendre transform of $(B, \mathscr{P}, \varphi)$? First, the Newton polytope of φ at v_0 is the standard simplex, and the Newton polyhedra of φ at v_1, v_2 and v_3 are all isomorphic to $[0, 1] \times [0, \infty)$. This gives a picture roughly as in Figure 23. That figure is misleading though: all three unbounded edges are in fact parallel! Furthermore, the fan structure at each vertex is the normal fan to the standard simplex, i.e., the fan for $\mathbb{P}^2$. This gives us $\check{B}$ and $\check{\mathscr{P}}$. Finally, $\check{\varphi}$ can

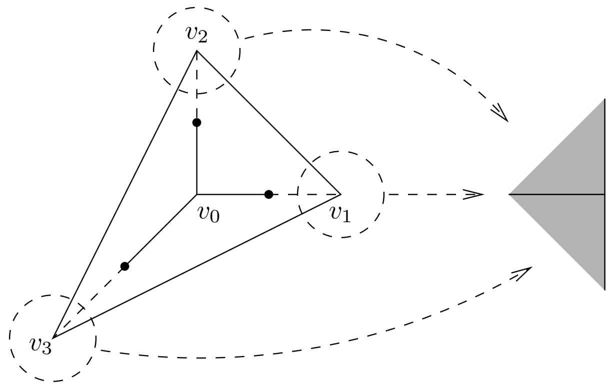

Figure 22

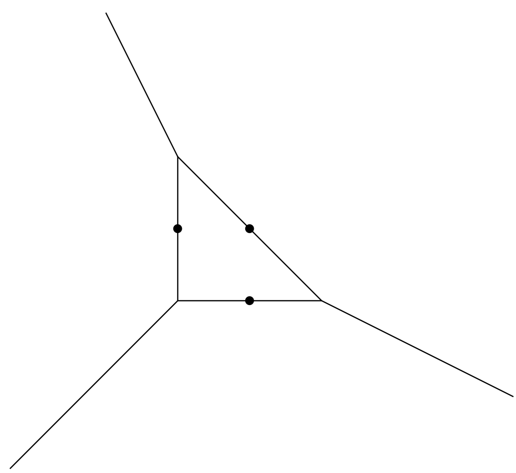

Figure 23

still be taken to be single-valued, taking the value 0 on the compact maximal cell, and on each unbounded cell $[0, 1] \times [0, +\infty)$, $\check{\varphi}$ is just the second coordinate in this representation.

One can in fact compute the monodromy of Λ around the three singular points of $\check{B}$, and one finds at each point that it is given by $\left(\begin{smallmatrix} 1 & 0 \\ 3 & 1 \end{smallmatrix}\right)$ in a suitable basis. As a result, it is actually possible to pull apart each singular point into three singular points, each with monodromy $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, as in Figure 24. We will use this deformation as an example in the next section.

1.6. Tropical curves on tropical surfaces

Briefly we show how to define tropical curves on two-dimensional tropical manifolds. The reason for restricting to dimension two is that it is not yet clear what the correct definitions in higher dimensions should be.

So fix a tropical manifold $(B, \mathscr{P})$ with $\dim B = 2$. Let $\overline{\Gamma}$ be a connected graph with no bivalent vertices. Let $\overline{\Gamma}^{[0]}_{\infty}$ be a *subset* of the set of univalent vertices (in

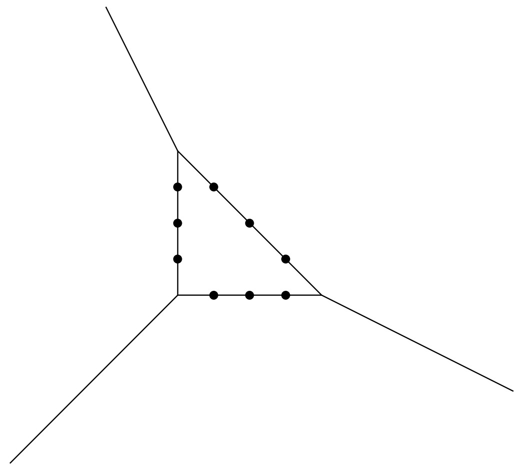

FIGURE 24

distinction from §1.3, where $\overline{\Gamma}^{[0]}_{\infty}$ denoted the set of *all* univalent vertices). Let $\Gamma = \overline{\Gamma} \setminus \overline{\Gamma}^{[0]}_{\infty}$. In addition, assume $\overline{\Gamma}$ has a weighting $w : \overline{\Gamma}^{[1]} \to \mathbb{N}$.

We will not bother with marked tropical curves here; one can fill in the details for this if the reader wishes. So we assume all weights are positive.

In what follows, recall that with $B_0 := B \setminus \Delta$, $B_0 \setminus \partial B_0$ is an honest tropical affine manifold, and hence carries a local system Λ. Let $i : B_0 \setminus \partial B_0 \hookrightarrow B$ be the inclusion. We will use below the sheaf $i_*\Lambda$. For U a contractible open set in B_0, $\Gamma(U, i_*\Lambda) \cong \mathbb{Z}^2$, i.e., $i_*\Lambda$ is locally constant on B_0. But if U is a small neighbourhood of a point of Δ, and the affine structure can't be extended across this point, then

$$\Gamma(U, i_*\Lambda) \cong \mathbb{Z}, \tag{1.5}$$

the monodromy invariant part of the local system on U. Here, recall that the monodromy around Δ takes the form $\left(\begin{smallmatrix} 1 & 0 \\ d & 1 \end{smallmatrix}\right)$ for some $d \geq 0$ in a suitable basis.

DEFINITION 1.32. A continuous map $h : \Gamma \to B$ is a *parameterized tropical curve* if it is proper and satisfies the following two conditions:

(1) For each edge E of Γ, $h|_E$ is an immersion (the image can self-intersect). Furthermore, there is a section $u \in \Gamma(E, h^{-1}(i_*\Lambda))$ which is tangent to every point of $h(E)$.
(2) For every vertex V of Γ, let $E_1, \ldots, E_m \in \Gamma^{[1]}$ be the edges adjacent to V. If $h(V) \in \Delta$, there is no further condition. Otherwise, let $u_1, \ldots, u_m$ be integral tangent vectors at $h(V)$, i.e., elements of the stalk $(i_*\Lambda)_{h(V)}$, with u_i primitive, tangent to $h(E_i)$, and pointing away from $h(V)$. Then

$$\sum_{j=1}^{m} w(E_j) u_j = 0.$$

Let's clarify what these conditions mean. (1) tells us that locally $h(E)$ is a line of rational slope; this is a well-defined notion in a tropical affine manifold. However, if $h(E)$ contains a point of Δ with non-trivial monodromy, the tangent direction to $h(E)$ near this point is completely determined, by (1.5). In other words, there is a

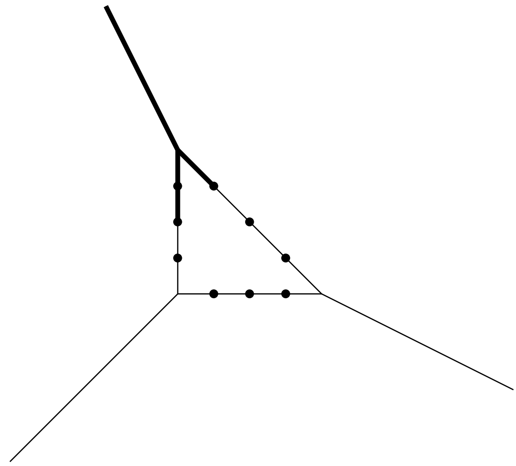

FIGURE 25. The darker lines give a tropical curve.

unique invariant direction at each point of Δ, and $h(E)$ must be tangent to that direction. Note that, in the analysis of two-dimensional singularities given in §1.4, this invariant direction is precisely the direction of the edge of $\mathscr{P}$ passing through the point of Δ.

The second condition tells us that, besides the usual balancing condition, we can have edges terminating at singular points. Even if an edge terminates at a singular point, however, it still must be tangent to the invariant direction at the singular point.

EXAMPLE 1.33. Consider the surface $\check{B}$ given in Example 1.31, (5), or rather, the variant with 9 singular points rather than 3. Figure 25 shows a tropical curve. In fact, there are 27 such tropical curves: three choices of unbounded edge, and 3^2 choices for the endpoints. In Example 6.2, we shall see that this surface corresponds to the cubic surface in $\mathbb{P}^3$. Morally, these tropical curves correspond to the 27 lines on the cubic surface.

1.7. References and further reading

The material of §1.1 largely follows the more in depth introductory paper [**96**]. Standard references for fans are the books on toric varieties by Fulton [**27**] and Oda [**87**]. The material on parameterized tropical curves originates in Mikhalkin's work [**80**]. See also the book of Itenberg, Mikhalkin and Shustin, [**62**]. The material on affine manifolds with singularities and the discrete Legendre transform forms part of the Gross-Siebert program: see [**48**], [**47**], [**49**] and [**41**].