Preface

This monograph covers ten lectures given by the author at the North Carolina State University at Raleigh, NC, during the week May 16–20, 2011. The choice of topics was a result of a compromise, given by the fact that the audience consisted of both graduate students and specialists in the field. The author resisted the temptation to devote the talks to his own results and attempted to present the material scattered in the literature in a compact fashion.

Also, the level of generality was determined by the purpose of the book. We decided to focus on deformations over local complete Noetherian rings (which of course includes the Artin case), though more general bases, as formal dg-commutative algebras or formal dg-schemes, can be considered. The complete Noetherian case covers most types of deformations of algebraic structures a working mathematician meets in his/her professional life.

The place for the conference was sensibly chosen, because the birth of deformation theory as we understand it today is related to this part of the globe. Mike Schlessinger and Jim Stasheff worked at Chapel Hill, only 28 miles from Raleigh—Mike contributed the Artin ring approach to deformation theory and, together with Jim, introduced the intrinsic bracket. They also wrote a seminal paper that predated modern deformation theory with its emphasis on the moduli space of the Maurer-Cartan equation. Tom Lada, who has spent most of his professional career at Raleigh, together with Jim, introduced L_∞-algebras. And, of course, the life of Murray Gerstenhaber, the founding father of algebraic deformation theory, is connected to Philadelphia, a 7 hour drive from Raleigh. Moreover, the author of this monograph worked as a Fulbright fellow for two fruitful semesters in Chapel Hill with Jim and Tom.

The book consists of 10 chapters which more or less correspond to the material of the respective talks. Chapters 1–3 review classical Gerstenhaber's deformation theory of associative algebras over a local complete Noetherian ring. In chapters 4 and 5, which are devoted to Maurer-Cartan elements in differential graded Lie algebras, the moduli space point of view begins to prevail. In chapter 6 we recall L_∞-algebras and, in chapter 7, the related simplicial version of the Maurer-Cartan moduli space, and prove its homotopy invariance. As an application, we review the main features of Kontsevich's approach to deformation quantization of Poisson manifolds. In chapters 8–9 we describe a construction of an L_∞-algebra governing deformations of a given class of (diagrams of) algebras. The last chapter contains a couple of explicit examples and indicates possible generalizations.

Our intention was to make the presentation self-contained, assuming only basic knowledge of commutative algebra, homological algebra and category theory. Suitable references are [**AM69**, **HS71**, **ML63a**, **ML71**]. We sometimes omitted technically complicated proofs when a suitable reference was available. Namely, we did not prove Theorem 6.13 about the Kan property of the induced map between the simplicial Maurer-Cartan spaces. Since operads are not the central topic of this book, we also omitted proofs of statements from operad theory used in chapters 8 and 9. On the other hand, we explained in detail the relation between the uniform continuity of algebraic maps and topologized tensor products and included proofs of the related statements, as this subject does not seem to be commonly known and the literature is scarce.

This monograph is not the first text attempting to present algebraic deformation theory. The classical theory is the subject of [**GS88**], there are lecture notes [**DMZ07**] and very recent shorter accounts [**Fia08**, **Gia11**]. Useful historical remarks can be found in [**Pfl06**]; an annotated historical bibliography is contained in [**DW**], perturbations, deformations, variations and "near misses" are treated in [**Maz04**]. There is also the influential though still unfinished book [**KS**].

Acknowledgments. I would like to express my thanks to the organizers of the conference, namely to Tom Lada and Kailash Mishra, for the gigantic work they have done. I am indebted also to the audience, which demonstrated striking tolerance to my halting English. During my work on the manuscript, I enjoyed the stimulating atmosphere of the Universidad de Talca and of the Max-Planck-Institut für Mathematik in Bonn.

In formulating my definition of $A_{\infty\infty}$-algebras I profited from conversations with M. Doubek and M. Livernet. Also, comments and suggestions from T. Giaquinto, A. Lazarev, M. Manetti, J. Stasheff and D. Yau were very helpful. I wish to thank, in particular, Tom Lada for reading the drafts of the manuscript and correcting typos and the worst of my language insufficiency.

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Conventions. Most of the algebraic objects will be considered over a fixed field $\mathbb{k}$ of characteristic zero, although some results remain true in arbitrary characteristic, or even over the ring of integers. The symbol $\otimes$ will denote the tensor product over $\mathbb{k}$ and $Lin(-,-)$ the space of $\mathbb{k}$-linear maps. By $\mathrm{Span}(S)$ we denote the $\mathbb{k}$-vector space spanned by the set S. The arity of a multilinear map is the number of its arguments. For instance, a bilinear map has arity two. We will denote by $\mathbb{1}_X$ or simply by $\mathbb{1}$ when X is understood, the identity endomorphism of an object X (set, vector space, algebra, &c.). The symbol $\mathfrak{S}_n$ will refer to the symmetric group on n elements. We will observe the usual convention of calling a commutative associative algebra simply a commutative algebra.

We denote by $\mathbb{Z}$ the ring of integers and by $\mathbb{N}$ the set $\{1, 2, 3, \ldots\}$ of natural numbers. The adjective "graded" will usually mean $\mathbb{Z}$-graded, though we will also consider non-negatively or non-positively graded objects; the actual meaning will always be clear from the context. The degree of an homogeneous element a will be denoted by $\deg(a)$ or by $|a|$.

The grading will sometimes be indicated by $*$ in sub- or superscript, the simplicial and cosimplicial degrees by $\bullet$. If we write $v \in V^*$ for a graded vector space V^*, we automatically assume that v is homogeneous, i.e. it belongs to a specific component of V.

The abbreviation 'dg' will mean 'differential graded.' Since we decided to require that the Maurer-Cartan elements are placed in degree $+1$, our preferred degree of differentials is $+1$. As a consequence, resolutions are non-positively graded.

An unpleasant feature of the graded word is the necessity to keep track of complicated signs. We will always use the Koszul sign convention requiring that, whenever we interchange two graded objects of degrees p and q, respectively, we change the overall sign by $(-1)^{pq}$. This rule however does not determine the signs uniquely. For instance, in Remark 3.51 we explain that the signs in the definition of strongly homotopy algebras depend on the preference for the inversion of the tensor power of the suspension. We will use the sign convention determined by requiring that

(i) all terms in the L_∞-Maurer-Cartan equation come with the $+$ sign,[1] and that

(ii) the intrinsic bracket (9.22) agrees, in the associative algebra case, with the one of [**Ger63**].

Requirement (i) fixes the signs in L_∞-algebras. Requirement (ii) introduces the correction $(-1)^{k+1}$ to formula (9.4) and affects the sign in (9.13). Since A_∞-algebras are Maurer-Cartan elements in the extended Gerstenhaber-Hochschild dg-Lie algebra (3.24), (ii) in turn determines the convention for A_∞-algebras. An unfortunate but necessary consequence is the minus sign in the expression (9.15) for the curvature and in the related formulas.

[1] This convention is used in [**Get09a**].