CHAPTER 1

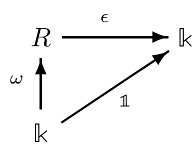
Basic notions

Augmented rings. By a *ring* we understand a commutative associative unital algebra[1] over a basic field $\Bbbk$, that is, the word "ring" shall mean a ring in the sense of Chapter 1 in [**AM69**] which is simultaneously a $\Bbbk$-vector space and whose structure operations are compatible with the scalar multiplication, i.e.

$$\alpha(r' + r'') = \alpha r' + \alpha r'', \quad \text{and} \quad \alpha(r'r'') = (\alpha r')r'' = r'(\alpha r''),$$

for $\alpha \in \Bbbk$, $r', r'' \in R$.

Definition 1.1. Let R be a ring with unit e and $\omega : \Bbbk \to R$ the morphism given by $\omega(1) := e$. A morphism $\epsilon : R \to \Bbbk$ is an *augmentation* of R if $\epsilon\,\omega = \mathbb{1}_{\Bbbk}$ or, diagrammatically,



A ring with an augmentation is an *augmented ring*.

The subspace $\overline{R} := \operatorname{Ker} \epsilon$ is called the *augmentation ideal* of R. Since the quotient $R/\overline{R}$ is isomorphic to the field $\Bbbk$, the augmentation ideal is always maximal. In this way, each augmentation determines a maximal ideal in R. Vice versa, each maximal ideal $\mathfrak{m} \subset R$ defines an augmentation $R \to R/\mathfrak{m}$ over the field $R/\mathfrak{m}$ which however, as we will see in Example 1.3 below, need not be isomorphic to the basic field $\Bbbk$.

Example 1.2. The unital ring $\Bbbk[[t]]$ of formal power series with coefficients in $\Bbbk$ is augmented, with the augmentation $\epsilon : \Bbbk[[t]] \to \Bbbk$ given by

$$\epsilon\Big(\sum_{i \geq 0} a_i t^i\Big) := a_0.$$

It turns out that $\Bbbk[[t]]$ is a local Noetherian ring, with the unique maximal ideal (t) and residue field $\Bbbk$, see [**AM69**, Chapter 1] for the terminology.

Example 1.3. Every $\alpha \in \Bbbk$ determines an augmentation $\epsilon_\alpha : \Bbbk[t] \to \Bbbk$ of the polynomial ring $\Bbbk[t]$ given by $\epsilon_\alpha(f) := f(\alpha)$, for $f \in \Bbbk[t]$. On the other hand, given an augmentation $\epsilon : \Bbbk[t] \to \Bbbk$, take $\alpha := \epsilon(t)$. It is clear that, for this α, $\epsilon = \epsilon_\alpha$. There is therefore a one-to-one correspondence between augmentations of $\Bbbk[t]$ and points in the affine plane $\Bbbk$.

The augmentation ideal of $\epsilon_\alpha : \Bbbk[t] \to \Bbbk$ is the maximal ideal generated by $(t - \alpha)$. If $\Bbbk$ is algebraically closed, then this assignment is one-to-one, i.e. there

[1] See Definition 1.15 below.

is a correspondence between augmentations of $\Bbbk[t]$ and maximal ideals in $\Bbbk[t]$, see [**Har77**, Example 1.4.4].

For a general field $\Bbbk$, there may be more maximal ideals in $\Bbbk[t]$ than augmentations $\epsilon : \Bbbk[t] \to \Bbbk$. For instance, the ring $\mathbb{R}[t]$ of polynomials with real coefficients contains the maximal principal ideal generated by $(1 + t^2)$, but the quotient $\mathbb{R}[t]/(1 + t^2)$ is isomorphic to the field of complex numbers $\mathbb{C}$. This isomorphism is induced by the ring morphism (augmentation over $\mathbb{C}$) $\epsilon : \mathbb{R}[t] \to \mathbb{C}$ given by $\epsilon(t) = -i$, where $i := \sqrt{-1}$ is the imaginary unit.

EXAMPLE 1.4. The truncated polynomial ring $\Bbbk[t]/(t^{n+1})$, $n \geq 1$, is augmented, with the augmentation

$$\epsilon\Big(\sum_{0 \leq i \leq n} a_i t^i \Big) := a_0.$$

It turns out that $\Bbbk[t]/(t^{n+1})$ is a local Artin ring, with the unique maximal ideal (t) and residue field $\Bbbk$ – for the terminology see again [**AM69**, Chapter 1]. The particular case $n = 1$ leads to the ring

$$D := \Bbbk[t]/(t^2)$$

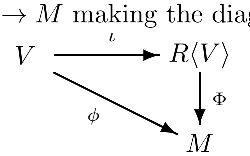
of *dual numbers*.

In the rest of this chapter, R will be an augmented ring, with the augmentation $\epsilon : R \to \Bbbk$ and the unit map $\omega : \Bbbk \to R$.

Modules over augmented rings. By an *R-module* we will understand a *left* R-module, i.e. a module in the sense of [**AM69**, Chapter 2] or [**ML63a**, §I.1]. As usual, a *vector space* is a module over a field, in most cases over our basic field $\Bbbk$. Bimodules are defined in [**ML63a**, §V3]. Let us formulate a couple of useful remarks.

A unital augmented ring R is a $\Bbbk$-$\Bbbk$-bimodule (that is, left $\Bbbk$- right $\Bbbk$- bimodule), with the bimodule structure induced by the unit map ω in the obvious manner. Likewise, $\Bbbk$ is an R-R bimodule, with the structure induced by the augmentation ϵ.

For a $\Bbbk$-vector space V and a unital augmented ring R we denote by $R\langle V\rangle$ the tensor product $R \otimes V$, with the left R-module action $r'(r'' \otimes v) := r'r'' \otimes v$, for $r', r'' \in R$ and $v \in V$. It is clear that $R\langle V\rangle$ together with the natural $\Bbbk$-linear inclusion $\iota : V \cong 1 \otimes V \hookrightarrow R\langle V\rangle$ is the *free R-module* generated by V. This means that, for every R-module M and a $\Bbbk$-linear map $\phi : V \to M$, there exists a unique R-module morphism $\Phi : R\langle V\rangle \to M$ making the diagram



commutative. We will use both notations for $R\langle V\rangle = R \otimes V$. The advantage of $R\langle V\rangle$ is that it is shorter and that it emphasizes the left R-module action, while $R \otimes V$ refers directly to the tensor product structure.

Topologies and completions. To include formal deformations[2] into our general setup, it will be necessary to introduce a completed version of the free R-module $R\langle V\rangle$. To this end we recall some basic facts from Chapter 10 of [**AM69**], which, along with [**Lef42**, Chapter II], should serve as the basic reference for this

[2]See Definition 1.24 on page 14.

subsection. Some relevant notes on topologies can also be found in Section 4.1 of [**CP94**].

Suppose we are given a descending sequence $G_1 \supset G_2 \supset G_3 \supset \cdots$ of subgroups of an Abelian group G. Then G has the unique linear topology with $\{G_i\}_{i\geq 1}$ a fundamental system of neighborhoods of 0. The *completion* with respect to this topology equals the inverse limit

$$\widehat{G} := \lim_{\longleftarrow i} G/G_i,$$

with the topology given by the fundamental system $\{\widehat{G}_i\}_{i\geq 1}$, where

$$\widehat{G}_i := \lim_{\longleftarrow j} G_i/(G_i \cap G_j).$$

The completion $\widehat{G}$ can also be described explicitly as

$$\widehat{G} = \big\{(g_1, g_2, g_3, \ldots);\ g_i \in G/G_i,\ g_i = \pi_{ij}(g_j),\ \forall\ i \leq j\big\}, \tag{1.1}$$

where $\pi_{ij} : G/G_j \to G/G_i$ is the canonical projection. So the completion $\widehat{G}$ consists of sequences $(g_1, g_2, g_3, \ldots)$ of elements which are *compatible* in that $g_i = \pi_{ij}(g_j)$, for all $i \leq j$. In this description, $\widehat{G}$ appears a subspace of the Cartesian product $\prod_{i=1}^{\infty} G/G_i$ of discrete spaces, with the induced topology. It is a standard fact that the completion $\widehat{G}$ is a Hausdorff space. We will often tacitly use the isomorphism of the (discrete) quotients [**AM69**, Corollary 10.4]:

$$G/G_i \cong \widehat{G}/\widehat{G}_i,\ i \geq 1, \tag{1.2}$$

An important special case of the above situation is provided by a ring R with a distinguished ideal $\mathfrak{a}$, together with an R-module M. The descending sequence $M_n := \mathfrak{a}^n M$, $n \geq 1$, determines the $\mathfrak{a}$-*adic topology* of M and one can form the completion $\widehat{M}$ with respect to this topology. In particular, one can consider a local ring $R = (R, \mathfrak{m})$ as a module over itself, and take its completion $\widehat{R}$ with respect to the $\mathfrak{m}$-adic topology. Recall that R is *complete* if the canonical map $R \to \widehat{R}$ is an isomorphism.

EXAMPLE 1.5. It is easy to prove that the completion of the ring of polynomials $\Bbbk[t]$ with respect to the (t)-adic topology equals the local ring $\Bbbk[[t]]$ of formal power series. Since the completion of any ring is complete, $\Bbbk[[t]]$ is a complete ring. Moreover, the completion of a Noetherian ring is Noetherian and $\Bbbk[t]$ is Noetherian by the Hilbert basis theorem, so $\Bbbk[[t]]$ is also Noetherian [**AM69**, Corollary 10.27].

EXAMPLE 1.6. In an Artin local ring $(R, \mathfrak{m})$, $\mathfrak{m}^n = 0$ for n sufficiently large. Therefore the $\mathfrak{m}$-adic topology of R is discrete, so $\widehat{R} = R$ and R is complete. It is a standard fact that each Artin ring is Noetherian.

Suppose that $R = (R, \mathfrak{m})$ is a compete local Noetherian ring with residue field $\Bbbk$. Denote by $\widehat{R\langle V\rangle} = R \mathbin{\widehat{\otimes}} V$ the $\mathfrak{m}$-adic completion of the R-module $R\langle V\rangle$,

$$\widehat{R\langle V\rangle} := \lim_{\longleftarrow i} R/\mathfrak{m}^i \otimes V. \tag{1.3}$$

The linear topology of $\widehat{R\langle V\rangle}$ is given by by the fundamental system

$$\widehat{R\langle V\rangle} = R \mathbin{\widehat{\otimes}} V \supset \mathfrak{m} \mathbin{\widehat{\otimes}} R \supset \mathfrak{m}^2 \mathbin{\widehat{\otimes}} R \supset \mathfrak{m}^3 \mathbin{\widehat{\otimes}} R \supset \cdots \supset \{0\}$$

where

$$\mathfrak{m}^i \mathbin{\widehat{\otimes}} V := \lim_{\longleftarrow j} \mathfrak{m}^i/\mathfrak{m}^j \otimes V, \quad \text{for } i \geq 0.$$

Isomorphism (1.2) describes the quotients as

$$\widehat{R\langle V\rangle}/(\mathfrak{m}^i \mathbin{\widehat{\otimes}} V) \cong R/\mathfrak{m}^i \otimes V, \quad \text{for each } i \geq 0.$$

By (1.1), the inverse limit in (1.3) equals

$$\widehat{R\langle V\rangle} = \big\{(x_1, x_2, x_3, \ldots);\ x_i \in R/\mathfrak{m}^i \otimes V,\ \pi_{ij}(x_j) = x_i,\ \forall\, i \leq j\big\} \tag{1.4}$$

with the component-wise R-module structure. In this description, the sub-module $\mathfrak{m}^i \mathbin{\widehat{\otimes}} V$ of $R \mathbin{\widehat{\otimes}} V$ consists of sequences $(x_1, x_2, x_3, \ldots)$ such that $x_j = 0$ for $j \leq i$. One has, for each $i \geq 0$, the inclusion

$$\mathfrak{m}^i (R \mathbin{\widehat{\otimes}} V) \subset \mathfrak{m}^i \mathbin{\widehat{\otimes}} V \tag{1.5}$$

which may, in general, be a proper one. There is a natural map $i : R \otimes V \to R \mathbin{\widehat{\otimes}} V$ given by

$$i(a) := ([a]_1, [a]_2, [a]_3, \ldots),\ a \in R \otimes V,$$

where $[a]_n$ is, for $n \geq 1$, the equivalence class of a in $R/\mathfrak{m}^n \otimes V$. Clearly, $i(a) = 0$ means that $a \in \mathfrak{m}^n \otimes V$ for each $n \geq 1$. Since $\bigcap_{n \geq 1} (\mathfrak{m}^n \otimes V) = \emptyset$ (we assume that R is complete), the map i is a monomorphism. We may use it to identify $R \otimes V$ with a subspace of $R \mathbin{\widehat{\otimes}} V$. It is a standard fact that $R \otimes V$ is dense in $R \mathbin{\widehat{\otimes}} V$.

Finally, one has the composed inclusion of $\Bbbk$-vector spaces

$$\iota : V \hookrightarrow R \otimes V \hookrightarrow R \mathbin{\widehat{\otimes}} V$$

given, in the language of (1.4), by

$$\iota(v) := (1 \otimes v, 1 \otimes v, 1 \otimes v, \ldots), \quad \text{for } v \in V.$$

It is easy to show that the object $V \stackrel{\iota}{\hookrightarrow} \widehat{R\langle V\rangle}$ is the free complete topological R-module generated by V – it has a universal property in the category of complete topological R-modules similar to that of $R\langle V\rangle$.

Example 1.7. The difference between $R\langle V\rangle$ and $\widehat{R\langle V\rangle}$ is best explained when we take as R the power series ring $\Bbbk[[t]]$ recalled in Example 1.2. The module $\widehat{R\langle V\rangle} = \Bbbk[[t]] \mathbin{\widehat{\otimes}} V$ then consists of expressions

$$v_0 + v_1 t + v_2 t^2 + v_3 t^3 + \cdots,\ v_0, v_1, v_2, \ldots \in V, \tag{1.6}$$

which can be understood as power series with coefficients in V. For this reason, one sometimes denotes $\Bbbk[[t]] \mathbin{\widehat{\otimes}} V$ by $V[[t]]$. The $\Bbbk[t]$-module $\widehat{R\langle V\rangle}$ is, up to isomorphism, characterized by the property that it is flat, (t)-adically complete, and

$$\widehat{R\langle V\rangle}/t\widehat{R\langle V\rangle} \cong V.$$

The uncompleted $R\langle V\rangle = \Bbbk[[t]] \otimes V$ is the subspace of $\Bbbk[[t]] \mathbin{\widehat{\otimes}} V$ consisting of expressions (1.6) such that the coefficients $v_0, v_1, v_2, \ldots$ span a *finite-dimensional* subspace of V. In particular, for V finite-dimensional, one has a $\Bbbk[[t]]$-module isomorphism $\Bbbk[[t]] \mathbin{\widehat{\otimes}} V \cong \Bbbk[[t]] \otimes V$.

The observation made in Example 1.7 is a particular case of:

Proposition 1.8. *Suppose that either V is a finite dimensional $\Bbbk$-vector space and R a local complete Noetherian ring, or V is arbitrary and R is Artin. Then one has an isomorphism*

$$\widehat{R\langle V\rangle} \cong R\langle V\rangle$$

of R-modules.

Proof. For V finite dimensional and R complete, the proposition follows from [**AM69**, Proposition 10.13]. If R is Artin, $\mathfrak{m}^i = 0$ for i sufficiently large, so the inverse limit in (1.3) stabilizes. The topology of $\widehat{R\langle V\rangle}$ is therefore discrete and $\widehat{R\langle V\rangle} \cong R\langle V\rangle$ as (discrete) topological R-modules. $\square$

Lemma 1.9. *Let U and V be (discrete) vector spaces, and $R = (R, \mathfrak{m})$ a local complete Noetherian ring with residue field $\Bbbk$. Then there is a natural one-to-one correspondence*

$$\Phi : Lin\big(U, \widehat{R\langle V\rangle}\big) \stackrel{\cong}{\longleftrightarrow} Lin^c_R\big(\widehat{R\langle U\rangle}, \widehat{R\langle V\rangle}\big) : \Psi, \tag{1.7}$$

where $Lin(-,-)$ denotes, as usual, the space of $\Bbbk$-linear maps and $Lin^c_R(-,-)$ the space of continuous R-linear maps. Moreover, (1.7) restricts, for each $k \geq 0$, to the isomorphism

$$\Phi_k : Lin\big(U, \mathfrak{m}^k \mathbin{\widehat{\otimes}} V\big) \stackrel{\cong}{\longleftrightarrow} Lin^c_R\big(\widehat{R\langle U\rangle}, \mathfrak{m}^k \mathbin{\widehat{\otimes}} V\big) : \Psi_k.$$

Proof. The lemma, of course, follows from the universal property of $\widehat{R\langle U\rangle}$ in the category of complete topological R-modules, but we include a direct proof here. Let us define first the correspondences Φ and Ψ.

For a $\Bbbk$-linear map $\phi : U \to \widehat{R\langle V\rangle}$ denote by $\tilde{\phi} : R \otimes U \to R \mathbin{\widehat{\otimes}} V$ its R-linear extension given by $\tilde{\phi}(r \otimes u) := r\phi(u)$, for $u \in U$ and $r \in R$. Clearly,

$$\tilde{\phi}(\mathfrak{m}^i \otimes U) = \mathfrak{m}^i \phi(U) \subset \mathfrak{m}^i (R \mathbin{\widehat{\otimes}} V),$$

so, by (1.5), $\tilde{\phi}(\mathfrak{m}^i \otimes U) \subset \mathfrak{m}^i \mathbin{\widehat{\otimes}} V$. The map $\tilde{\phi}$ therefore induces a map

$$\frac{R \otimes U}{\mathfrak{m}^i \otimes U} \longrightarrow \frac{R \mathbin{\widehat{\otimes}} V}{\mathfrak{m}^i \mathbin{\widehat{\otimes}} V}$$

of the quotients which, combined with the isomorphisms

$$\frac{R \otimes U}{\mathfrak{m}^i \otimes U} \cong R/\mathfrak{m}^i \otimes U \quad \text{and} \quad \frac{R \mathbin{\widehat{\otimes}} V}{\mathfrak{m}^i \mathbin{\widehat{\otimes}} V} \cong R/\mathfrak{m}^i \otimes V$$

gives a map $\tilde{\phi}_i : R/\mathfrak{m}^i \otimes U \to R/\mathfrak{m}^i \otimes V$. Define finally, in the notation (1.4) for elements of the inverse limits,

$$\Phi(\phi)(x_1, x_2, x_3, \ldots) := (\tilde{\phi}_1(x_1), \tilde{\phi}_2(x_2), \tilde{\phi}_3(x_3), \ldots).$$

It is easy to verify that $\Phi(\phi) : \widehat{R\langle U\rangle} \to \widehat{R\langle V\rangle}$ is a well-defined continuous map.

The definition of the inverse correspondence Ψ is even simpler. Given an R-linear map $f : \widehat{R\langle U\rangle} \to \widehat{R\langle V\rangle}$, $\Psi(f) : U \to \widehat{R\langle V\rangle}$ is the composition

$$\Psi(f) : U \stackrel{\iota}{\hookrightarrow} \widehat{R\langle U\rangle} \stackrel{f}{\longrightarrow} \widehat{R\langle V\rangle}.$$

It is clear that $\Psi \circ \Phi = \mathbb{1}$. It therefore remains to prove that Ψ is a monomorphism.

So assume that $\Psi(f) = 0$ for a continuous $f : \widehat{R\langle U\rangle} \to \widehat{R\langle V\rangle}$ and prove that then $f = 0$. Since $R \otimes V$ is dense in $\widehat{R\langle V\rangle}$, it is enough to show that $f(r \otimes v) = 0$ for $r \in R$ and $v \in V$. But this is obvious, since

$$f(r \otimes v) = rf(1 \otimes v) = rf(\iota(v)) = r\Psi(f)(v) = 0.$$

We leave the proof that the isomorphisms Φ resp. Ψ restrict to Φ_k resp. Ψ_k as an exercise. $\square$

EXAMPLE 1.10. Let $\Bbbk[[t]]$ be the formal power series ring and U, V (discrete) $\Bbbk$-vector spaces. Let us show that *each* $\Bbbk[[t]]$-linear map $f : \Bbbk[[t]] \widehat{\otimes} U \to \Bbbk[[t]] \widehat{\otimes} V$ is automatically *continuous*.

We verify this fact by proving that, for each sequence $\{a_n\}_1^\infty$ converging to $a \in \Bbbk[[t]] \widehat{\otimes} U$, the sequence $\big\{f(a_n)\big\}_1^\infty$ converges to $f(a)$ in $\Bbbk[[t]] \widehat{\otimes} V$. The convergence $\{a_n\}_0^\infty \to a$ means that, for each $n \geq 0$ there exists $k \geq 1$ such that $a - a_k \in (t^n) \widehat{\otimes} U$. It is obvious from the description of $\Bbbk[[t]] \widehat{\otimes} U$ in terms of power series with coefficients in U given in Example 1.7 that the last condition in fact says that $a - a_k$ is divisible by t^n:

$$a - a_k = t^n \cdot u_k^n, \ \text{ for some } \ u_k^n \in \Bbbk[[t]] \widehat{\otimes} U.$$

We conclude that $f(a) - f(a_k) \in (t)^n \widehat{\otimes} V$, which shows that $\big\{f(a_n)\big\}_1^\infty$ converges to $f(a)$ in the topology of $\Bbbk[[t]] \widehat{\otimes} V$.

We leave as an exercise based on Theorem 11.22 of [**AM69**] to prove the following generalization of Example 1.10.

PROPOSITION 1.11. *Let R be a regular*[3] *local complete Noetherian ring and U, V discrete vector spaces. Then each R-linear map $f : \widehat{R\langle U\rangle} \to \widehat{R\langle V\rangle}$ is continuous.*

Topologized tensor products. Suppose we are given a ring S and topological S-modules M and N. One may topologize the tensor product $M \otimes_S N$ by requiring that the subspaces

$$M \otimes_S \mathcal{V} + \mathcal{U} \otimes_S N \subset M \otimes_S N, \tag{1.8}$$

where $\mathcal{U}$ (resp. $\mathcal{V}$) are open S-submodules of M (resp. N), form a basis of open neighborhoods of zero in $M \otimes_S N$. This topology has a certain universal property with respect to uniformly continuous maps which we formulate below. Let us recall some necessary definitions.

A *uniformity* on a set X is a system $\mathbb{U}$ of neighborhoods of the diagonal

$$\Delta(X) := \big\{(x, x) \mid x \in X\big\} \subset X \times X$$

satisfying suitable axioms [**Kel55**, Chapter 6]. A set with a uniformity is called a *uniform space*. Each uniformity induces a topology on X, with a basis of open neighborhoods of $x \in X$ given by the sets

$$\mathcal{U}_x := \{x' \in X \mid (x', x) \in \mathcal{U}\},$$

where $\mathcal{U} \in \mathbb{U}$.

A map $f : (X, \mathbb{U}) \to (Y, \mathbb{V})$ is *uniformly continuous* if, for each $\mathcal{V} \in \mathbb{V}$, there exists $\mathcal{U} \in \mathbb{U}$ such that

$$(x_1, x_2) \in \mathcal{U} \Longrightarrow \big(f(x_1), f(x_2)\big) \in \mathcal{V}.$$

Each uniformly continuous map is continuous with respect to the induced topologies.

The cartesian product $X_1 \times X_2$ of uniform spaces $(X_i, \mathbb{U}_i)$, $i = 1, 2$, has a uniformity $\mathbb{U}_1 \times \mathbb{U}_2$ given by the subsets

$$\mathcal{U}_1 \times \mathcal{U}_2 \subset (X_1 \times X_1) \times (X_2 \times X_2) \cong (X_1 \times X_2) \times (X_1 \times X_2) = \Delta(X_1 \times X_2),$$

where $\mathcal{U}_i \in \mathbb{U}_i$. The induced topology on $X_1 \times X_2$ is the product of the induced topologies.

[3] See [**AM69**, Theorem 11.22] for a definition of regular local rings.

Important examples of uniform spaces are provided by topological linear spaces. Such a space X is uniform, with the uniformity $\mathbb{U}$ given by the sets

$$\big\{(x_1, x_2) \in X \times X \mid x_1 - x_2 \in \mathcal{U}\big\},$$

where $\mathcal{U}$ runs over a basis of open neighborhoods of 0 in X. It is clear that the topology induced by $\mathbb{U}$ is the original one. A linear map between linear topological spaces is continuous if and only if it is uniformly continuous.

Let us finally formulate the universal property of the topology (1.8).

PROPOSITION 1.12. *Suppose that M, N and Z are topological S-modules. The map $\widetilde{F} : M \otimes_S N \to Z$ induced by an S-bilinear map $F : M \times N \to Z$ is continuous if and only if F is* uniformly *continuous.*[4]

PROOF. The continuity of $\widetilde{F}$ means by definition that, for each open neighborhood $\mathcal{W}$ of 0 in Z, there exist open S-linear subspaces $\mathcal{U} \subset M$ and $\mathcal{V} \subset N$ such that

$$\widetilde{F}(M \otimes_S \mathcal{V} + \mathcal{U} \otimes_S N) \subset \mathcal{W}. \tag{1.9}$$

By the definition of $\widetilde{F}$ and its linearity one has

$$\begin{aligned}\widetilde{F}(M \otimes_S \mathcal{V} + \mathcal{U} \otimes_S N) &= \widetilde{F}(M \otimes_S \mathcal{V}) + \widetilde{F}(\mathcal{U} \otimes_S N)\\ &= \mathrm{Span}_S\big(F(M, \mathcal{V})\big) + \mathrm{Span}_S\big(F(\mathcal{U}, N)\big),\end{aligned}$$

where $\mathrm{Span}_S(-)$ denotes the S-linear envelope. We see that (1.9) is equivalent to

$$F(M, \mathcal{V}) \subset \mathcal{W} \ \&\ F(\mathcal{U}, N) \subset \mathcal{W}. \tag{1.10}$$

On the other hand, the uniform continuity of F means that, for each open neighborhood $\mathcal{W}$ of 0 in Z, there exist neighborhoods $\mathcal{U} \subset M$ and $\mathcal{V} \subset N$ such that

$$x' - x'' \in \mathcal{U} \ \&\ y' - y'' \in \mathcal{V} \Longrightarrow F(x', y') - F(x'', y'') \in \mathcal{W}. \tag{1.11}$$

One has, by bilinearity,

$$F(x', y') - F(x'', y'') = F(x' - x'', y') + F(x'', y' - y'').$$

Assume the inclusions (1.10). Then $F(x' - x'', y') \in \mathcal{W}$ because $F(\mathcal{U}, N) \subset \mathcal{W}$, and $F(x'', y' - y'') \in \mathcal{W}$ because $F(M, \mathcal{V}) \subset \mathcal{W}$. So (1.10) implies (1.11).

Putting $y' = y'' = y$ in (1.11) we get that

$$F(x', y) - F(x'', y) = F(x' - x'', y) \in \mathcal{W},$$

whenever $x' - x'' \in \mathcal{U}$, therefore $F(\mathcal{U}, N) \subset \mathcal{W}$. The same argument establishes the inclusion $F(M, \mathcal{V}) \subset \mathcal{W}$, thus (1.11) implies (1.10). The lemma is proved. □

The tensor product $- \otimes_S -$ with the topology (1.8) is suited for studying multilinear maps and adequate to our purposes. It makes the category of topological S-modules a symmetric monoidal category with the unit object S. This, by definition, means the existence of natural isomorphisms of topological S-modules

$$\begin{aligned}S \otimes_S M \cong M \otimes_S S \cong M &\qquad \text{(unitality)},\\ M \otimes_S N \cong N \otimes_S M &\qquad \text{(commutativity)},\\ M \otimes_S (N \otimes_S O) \cong (M \otimes_S N) \otimes_S O &\qquad \text{(associativity)},\end{aligned}$$

[4]Bilinear maps are *not* linear, so their continuity does not imply their uniform continuity.

satisfying appropriate coherence relations, see [**ML63b**]. Also the quotients by open submodules are easy to describe; by standard linear algebra,

$$\frac{M \otimes_S N}{M \otimes_S \mathcal{V} + \mathcal{U} \otimes_S N} \cong M/\mathcal{U} \otimes_S N/\mathcal{V}.$$

Therefore the completion of the tensor product $M \otimes_S N$ with respect to topology (1.8) equals

$$M \mathbin{\widehat{\otimes}}_S N := \lim_{\longleftarrow \mathcal{U},\mathcal{V}} M/\mathcal{U} \otimes_S N/\mathcal{V}. \tag{1.12}$$

We say that an algebraic structure is *topological* if all its structure operations are *uniformly* continuous.

There exits another topology on $M \otimes_S N$ having the universal property of Proposition 1.12 with respect to *all* continuous S-bilinear maps. This topology, however, does not have nice properties and it does not seem to play any rôle in multilinear algebra, see the discussion in [**BH96**, §24]. There are yet some important cases where R-bilinear continuous maps are automatically uniformly continuous. Let us prove the following

Lemma 1.13. *Let R be a local complete Noetherian ring with residue field $\Bbbk$ and V_1, V_2, W discrete $\Bbbk$-vector spaces. Any continuous R-bilinear map*

$$F : (R \mathbin{\widehat{\otimes}} V_1) \times (R \mathbin{\widehat{\otimes}} V_2) \to R \mathbin{\widehat{\otimes}} W$$

is uniformly continuous. There is thus an one-to-one correspondence between continuous R-bilinear maps

$$(R \mathbin{\widehat{\otimes}} V_1) \times (R \mathbin{\widehat{\otimes}} V_2) \to R \mathbin{\widehat{\otimes}} W$$

and R-linear continuous maps

$$(R \mathbin{\widehat{\otimes}} V_1) \otimes_R (R \mathbin{\widehat{\otimes}} V_2) \to R \mathbin{\widehat{\otimes}} W.$$

Proof. The R-submodules $\mathfrak{m}^k \mathbin{\widehat{\otimes}} V_1$ (resp. $\mathfrak{m}^k \mathbin{\widehat{\otimes}} V_2$, resp. $\mathfrak{m}^k \mathbin{\widehat{\otimes}} W$), $k \geq 0$, form a basis of open neighborhoods of 0 in $R \mathbin{\widehat{\otimes}} V_1$, (resp. $R \mathbin{\widehat{\otimes}} V_2$, resp. $R \mathbin{\widehat{\otimes}} W$). By (1.10), the uniform continuity of F will therefore be established if we prove that, for each $k \geq 0$, there exist $k_1, k_2 \geq 0$ such that

$$F(R \mathbin{\widehat{\otimes}} V_1, \mathfrak{m}^{k_1} \mathbin{\widehat{\otimes}} V_2) \subset \mathfrak{m}^k \mathbin{\widehat{\otimes}} W \quad\text{and}\quad F(\mathfrak{m}^{k_2} \mathbin{\widehat{\otimes}} V_1, R \mathbin{\widehat{\otimes}} V_2) \subset \mathfrak{m}^k \mathbin{\widehat{\otimes}} W. \tag{1.13}$$

Let us prove that the first inclusion is satisfied with $k_1 := k$. Since F is continuous separately in each variable and $\mathfrak{m}^k \mathbin{\widehat{\otimes}} W$ is complete, it is enough to verify the inclusion

$$F(R \mathbin{\widehat{\otimes}} V_1, \mathfrak{m}^k \otimes V_2) \subset \mathfrak{m}^k \mathbin{\widehat{\otimes}} W. \tag{1.14}$$

The R-bilinearity of F implies that

$$F(R \mathbin{\widehat{\otimes}} V_1, \mathfrak{m}^k \otimes V_2) = \mathfrak{m}^k F(R \mathbin{\widehat{\otimes}} V_1, V_2). \tag{1.15}$$

Since $F(R \mathbin{\widehat{\otimes}} V_1, V_2) \subset R \mathbin{\widehat{\otimes}} W$ and $\mathfrak{m}^k(R \mathbin{\widehat{\otimes}} W) \subset \mathfrak{m}^k \mathbin{\widehat{\otimes}} W$, (1.15) implies (1.14) and thus also the first inclusion of (1.13) with $k_1 = k$. The second inclusion can be treated analogously. □

Examples 1.14. Let R be a complete Noetherian local ring and V a discrete k vector space. The completed $\widehat{R\langle V\rangle} = R \mathbin{\widehat{\otimes}} V$ of (1.3) is a particular instance of the completed tensor product (1.12), with $S = \Bbbk$, $M = R$, and discrete $N = V$.

Consider the map $\epsilon \otimes \mathbb{1}_V : R \otimes V \to \Bbbk \otimes V \cong V$, where $\epsilon : R \to \Bbbk$ is the augmentation. There are two open subspaces of $\Bbbk$, $\{0\}$ and the whole $\Bbbk$. It is clear that both subspaces

$$\mathfrak{m} \otimes V = (\epsilon \otimes \mathbb{1}_V)^{-1}(0) \;\text{ and }\; R \otimes V = (\epsilon \otimes \mathbb{1}_V)^{-1}(\Bbbk)$$

are open in the topology (1.8), so $(\epsilon \otimes \mathbb{1}_V)$ is continuous. The space V is, as each discrete space, complete, so $\epsilon \otimes \mathbb{1}_V$ uniquely extends into a continuous map

$$\epsilon \mathbin{\widehat{\otimes}} \mathbb{1}_V : R \mathbin{\widehat{\otimes}} V \to V. \tag{1.16}$$

Another important particular case is $S = R$, $M = \widehat{R\langle V_1\rangle}$ and $N = \widehat{R\langle V_2\rangle}$ for some discrete $\Bbbk$-vector spaces V_1, V_2. The completed tensor product then equals

$$\widehat{R\langle V_1\rangle} \mathbin{\widehat{\otimes}}_R \widehat{R\langle V_2\rangle} = \varprojlim_{a,\, b} \, (R/\mathfrak{m}^a \otimes V_1) \otimes_R (R/\mathfrak{m}^b \otimes V_2).$$

From the obvious isomorphism

$$(R/\mathfrak{m}^a \otimes V_1) \otimes_R (R/\mathfrak{m}^b \otimes V_2) \cong R/\mathfrak{m}^{\max\{a,b\}} \otimes V_1 \otimes V_2$$

we obtain

$$\widehat{R\langle V_1\rangle} \mathbin{\widehat{\otimes}}_R \widehat{R\langle V_2\rangle} \cong \varprojlim_i \, R/\mathfrak{m}^i \otimes V_1 \otimes V_2 \cong \widehat{R\langle V_1 \otimes V_2\rangle}. \tag{1.17}$$

Iterating (1.17) gives a natural isomorphisms

$$\textstyle\bigotimes_R^k \widehat{R\langle V\rangle} \cong \widehat{R\langle \bigotimes^k V\rangle}, \text{ for each } k \geq 0. \tag{1.18}$$

As the last example of the completed tensor product, consider the situation $S = R$, $M = \Bbbk$ and $N = R \mathbin{\widehat{\otimes}} V$. Since $\Bbbk \otimes_R (R/\mathfrak{m}^n \otimes V) \cong V$ for each $n \geq 1$, one has

$$\Bbbk \mathbin{\widehat{\otimes}}_R (R \mathbin{\widehat{\otimes}} V) = \varprojlim_n \, \Bbbk \otimes_R (R/\mathfrak{m}^n \otimes V) \cong \varprojlim_n \, V \cong V. \tag{1.19}$$

This isomorphism induces, for each R-linear continuous $\phi : \widehat{R\langle V\rangle} \to \widehat{R\langle V\rangle}$, the $\Bbbk$-linear map $\overline{\phi} : V \to V$ via the commutativity of the diagram

$$\begin{array}{ccc} \Bbbk \mathbin{\widehat{\otimes}}_R (R \mathbin{\widehat{\otimes}} V) & \xrightarrow{\Bbbk \widehat{\otimes}_R \phi} & \Bbbk \mathbin{\widehat{\otimes}}_R (R \mathbin{\widehat{\otimes}} V) \\ \big\updownarrow \cong & & \big\updownarrow \cong \\ V & \xrightarrow{\quad \overline{\phi} \quad} & V. \end{array} \tag{1.20}$$

The map (1.16) then fits into the diagram

$$\begin{array}{ccc} R \mathbin{\widehat{\otimes}} V & \xrightarrow{\quad \phi \quad} & R \mathbin{\widehat{\otimes}} V \\ \big\downarrow \epsilon \widehat{\otimes} V & & \big\downarrow \epsilon \widehat{\otimes} V \\ V & \xrightarrow{\quad \overline{\phi} \quad} & V. \end{array} \tag{1.21}$$

Deformations of associative algebras. We will illustrate basic notions of deformation theory on the particular example of associative algebras. We will see in chapters 9 and 10 that most of the material extends to a broad class of equationally given structures as Lie, commutative associative, Poisson, Leibniz algebras, various bialgebras, and their diagrams. Recall

DEFINITION 1.15. An *associative R-algebra*[5] is a couple $B = (M, \mu)$ consisting of an R-module M with an R-bilinear multiplication $\mu : M \times M \to M$ satisfying

$$a(bc) = (ab)c, \quad \text{for all } a, b, c \in M,$$

where we abbreviate, as usual, $\mu(a, b) := ab$, &c. The algebra $B = (M, \mu)$ is *commutative*, if

$$ab = ba, \quad \text{for all } a, b \in M.$$

It is *unital*, if there exists a *unit* $e \in M$ satisfying

$$ae = ea, \quad \text{for each } a \in M.$$

If M is a topological R-module, we assume that μ is *uniformly*[6] continuous. If R is the basic field $\Bbbk$, we sometimes call an R-algebra simply an *algebra*.

The R-module M is the *underlying module* of the R-algebra B. A *morphism* $f : B' \to B''$ from the associative algebra $B' = (M', \mu')$ to the associative algebra $B' = (M'', \mu'')$ is a morphism $f : M' \to M''$ of the underlying modules commuting with the multiplications, that is, satisfying $f\mu' = \mu''(f \times f)$.

We remind the reader that the R-bilinearity of the multiplication of B means that, for each $a', a'', b', b'' \in M$ and $r', r'', s', s'' \in R$,

$$(r'a' + r''a'')(s'b' + s''b'') = (r's')(a'b') + (r's'')(a'b'') + (r''s')(a''b') + (r''s'')(a''b'').$$

The universal property of the tensor product implies that each R-bilinear map $\mu : M \times M \to M$ gives rise to an R-module morphism (denoted by the same symbol) $\mu : M \otimes_R M \to M$. We will usually use this tensor-product notation for structure operations of algebraic systems. The associativity of μ can then be expressed as the equality

$$\mu(\mathbb{1}_M \otimes_R \mu) = \mu(\mu \otimes_R \mathbb{1}_M)$$

of R-linear maps $M \otimes_R M \otimes_R M \to M$. If M is topological and the multiplication uniformly continuous, then $\mu : M \otimes_R M \to M$ is continuous in the topology (1.8). If M is, moreover, complete, μ uniquely extends into a continuous map (denoted again by the same symbol) $\mu : M \widehat{\otimes}_R M \to M$ from the completed tensor product. The central definition of this chapter reads:

DEFINITION 1.16. Let A be an associative $\Bbbk$-algebra with the underlying vector space V, and R a local complete Noetherian ring with residue field $\Bbbk$. An *R-deformation* of A is an associative continuous[7] R-algebra structure on the topological R-module $R\langle V \rangle = R \widehat{\otimes} V$ such that the map

$$\epsilon \widehat{\otimes} \mathbb{1}_V : R \widehat{\otimes} V \to V \tag{1.22}$$

induced by the augmentation $\epsilon : R \to \Bbbk$ is a morphism of associative $\Bbbk$-algebras. The *trivial R-deformation* of A is the one given by the R-linear extension of the original multiplication of A to $R \widehat{\otimes} V$. Deformations in the above sense will sometimes be called deformations with the *base* R or deformations *over* R.

[5] We sometimes say also an *associative algebra over R*.

[6] See the discussion on page 8.

[7] By Lemma 1.13, such a structure is automatically *uniformly* continuous.

REMARK 1.17. We need different symbols for algebras and for its underlying modules. This will make our notation somehow heavy but, since we will sometimes consider *several* algebras with the *same* underlying module, such a distinction is necessary. We will, however, try to simplify the notation if no confusion may occur. For instance, the trivial deformation of an associative algebra $A = (V, \mu)$ will be denoted by $R \widehat{\otimes} A$.

In the following definition, $\overline{\phi} : V \to V$ denotes the $\mathbb{k}$-linear map induced by a continuous R-linear endomorphism $\phi : R \widehat{\otimes} V \to R \widehat{\otimes} V$ as in diagram (1.20).

DEFINITION 1.18. Two R-deformations $(R \widehat{\otimes} V, \mu')$ and $(R \widehat{\otimes} V, \mu'')$ of an associative algebra A with the underlying vector space V are *equivalent* if there exists a continuous R-algebra isomorphism

$$\phi : (R \widehat{\otimes} V, \mu') \xrightarrow{\cong} (R \widehat{\otimes} V, \mu'')$$

such that $\overline{\phi} : V \to V$ is the identity automorphism $\mathbb{1}_V$ of V. We denote by $\mathfrak{Def}_A(R)$ the set of equivalence classes of deformations of A with the base R.

In the important particular case when R is Artin,[8] all topologies involved in Definitions 1.16 and 1.18 are discrete, so we can omit the completions $\widehat{\ }$. If R is regular local complete Noetherian, then the continuity of all maps follows from their R-linearity, see Proposition 1.11. We can therefore avoid the topologies also in this case and work in the realm of 'standard' algebra. As a matter of fact, all base rings in this monograph will be of one of the above two types.

Let us show that the set $\mathfrak{Def}_A(R)$ behaves functorially in R. Assume that $f : R' \to R''$ is a morphism of augmented rings and $(R' \widehat{\otimes} V, \mu')$ an R'-deformation of A. One can easily check that μ' induces an associative multiplication $f_!(\mu)$ on

$$R'' \widehat{\otimes} A \cong R'' \widehat{\otimes}_{R'} (R' \widehat{\otimes} V)$$

which is an R''-deformation of A. We will call $f_!(\mu')$ the *push-forward* of the deformation μ'. This construction induces a natural map (denoted by the same symbol)

$$f_! : \mathfrak{Def}_A(R') \to \mathfrak{Def}_A(R'').$$

The sets $\mathfrak{Def}_A(R)$ therefore assemble into a covariant functor $\mathfrak{Def}_A(-)$ from the category of complete local Noetherian rings with a given residue field, into the category of sets. This point of view was, in the Artin case, pioneered by M. Artin and M. Schlessinger [**Sch68**]. We include a brief subsection devoted to this approach at the end of Chapter 4.

We denote by $\mathrm{Def}_A(R)$ the set of R-deformations of an associative algebra A as in Definition 1.16. Denote also by $\mathrm{G}_A(R)$ the group of R-module automorphisms $\phi : R \widehat{\otimes} V \to R \widehat{\otimes} V$ such that $\overline{\phi} = \mathbb{1}_V$, with the group structure given by the composition. We will call $\mathrm{G}_A(R)$ the *gauge group*. An automorphism $\phi \in \mathrm{G}_A(R)$ acts on $\mu' \in \mathrm{Def}_A(R)$ by $\phi \cdot \mu' = \mu''$, where

$$\mu''(a, b) := \phi \circ \mu'\big(\phi^{-1}(a), \phi^{-1}(b)\big),\ a, b \in R \otimes V. \tag{1.23}$$

The next proposition follows immediately from definitions.

[8]Recall [**AM69**, Theorem 8.5] that Artin local rings are precisely complete Noetherian rings of global dimension zero.

PROPOSITION 1.19. *The set of equivalence classes* $\mathfrak{Def}_A(R)$ *of* R*-deformations of an associative algebra* A *is the quotient*

$$\mathfrak{Def}_A(R) \cong \mathrm{Def}_A(R)/\mathrm{G}_A(R). \tag{1.24}$$

Variants. One can modify Definition 1.16 in several ways. For instance, one can take as R an arbitrary ring augmented over $\Bbbk$ and consider deformations with the (uncompleted) $R \otimes V$ as the underlying space. In [**Fia08**], these deformations are called *global*. For deformations in this sense one however loses the cohomology as a tool and several other statements, as the invertibility of Proposition 1.21, cease to hold.

Another modification is to define an R-deformation of an associative $\Bbbk$-algebra A as an associative R-algebra B with a $\Bbbk$-algebra isomorphism $\Bbbk \mathbin{\widehat{\otimes}_R} B \stackrel{\cong}{\to} A$. There is, however, not much to be said about R-deformations without some additional assumptions on the underlying R-module M of B. In our Definition 1.16 we assumed that it was a free complete R-module. Another assumption frequently used in algebraic geometry [**Har77**, Section III.§9] is that M is flat which, by definition, means that the functor $M \otimes_R -$ is left exact. One then speaks about *flat deformations*.

If R is a local Noetherian ring, a finitely generated R-module is flat if and only if it is free (see Exercise 7.15, Corollary 10.16 and Corollary 10.27 of [**AM69**]). Therefore, for A with a finite-dimensional underlying vector space, free deformations are the same as the flat ones. Our restriction to free deformations includes most of instances of algebraic deformation theory, including the (mini)versal deformations recalled on page 16 and, of course, also the classical setup of [**Ger64**].

The R-linearity built in Definitions 1.16 and 1.18 implies the following lemma.

LEMMA 1.20. *Let* $(R \mathbin{\widehat{\otimes}} V, \mu)$ *be an* R*-deformation of* A *as in Definition 1.16. Then the multiplication* μ *is determined by its restriction to*

$$V \otimes V \subset (R \mathbin{\widehat{\otimes}} V) \mathbin{\widehat{\otimes}_R} (R \mathbin{\widehat{\otimes}} V).$$

Likewise, every equivalence of deformations $\phi : (R \mathbin{\widehat{\otimes}} V, \mu') \to (R \mathbin{\widehat{\otimes}} V, \mu'')$ *is determined by its restriction to* $V \subset R \mathbin{\widehat{\otimes}} V$.

PROOF. To prove the first part of the lemma, we invoke the isomorphism

$$(R \mathbin{\widehat{\otimes}} V) \mathbin{\widehat{\otimes}_R} (R \mathbin{\widehat{\otimes}} V) \cong R\langle V \otimes V\rangle$$

of (1.17) and apply Lemma 1.9 to the case $U = V \otimes V$. The second part of the proposition follows from the same lemma with $U = V$. □

The following proposition will also be useful.

PROPOSITION 1.21. *Assume that* V *is a* $\Bbbk$*-vector space and* R *a local complete Noetherian ring with residue field* $\Bbbk$.[9] *Then every* R*-linear continuous morphism*

$$\phi : (R \mathbin{\widehat{\otimes}} V) \to (R \mathbin{\widehat{\otimes}} V)$$

of R*-modules such that* $\overline{\phi} = \mathbb{1}_V$ *is invertible.*

The proposition will follow from:

[9]This of course includes also the Artin case.

LEMMA 1.22. *Let* $(U, \cdot, e)$ *be an associative unital* $\Bbbk$*-algebra with a complete linear topology given by a descending sequence* $\{U_i\}_{i\geq 1}$ *of ideals such that*

$$U_a \cdot U_b \subset U_{a+b}, \quad \text{for each } a, b \geq 1. \tag{1.25}$$

Then each element $v \in U$ *such that* $v - e \in U_1$ *is invertible. In other words, the subset*

$$(U_1 + e) := \{v \in V \mid v - e \in U_1\}$$

of U *is a group.*

PROOF. Let $\alpha := v - e$ so that $v = e + \alpha$. For any $k \geq 1$ consider the element

$$v_k^{-1} := e - \alpha + \alpha^2 - \alpha^3 + \cdots + (-1)^k \alpha^k.$$

By (1.25), $\alpha^i \in U_i$ for each $i \geq 1$ and, since U is complete, the sequence $\{v_k^{-1}\}_{k\geq 1}$ converges to an element $v^{-1} \in U$. One clearly has, for each $k \geq 1$,

$$e - v_k^{-1} v = e - v v_k^{-1} \in U_{k+1},$$

which implies that $e - v^{-1}v = e - vv^{-1} \in \bigcap_{i\geq 1} U_i$. Since complete linear spaces are Hausdorff, $\bigcap_{i\geq 1} U_i = \{0\}$, therefore $v^{-1}v = vv^{-1} = e$. □

By (1.10), condition (1.25) implies the uniform continuity of the multiplication $\cdot : U \times U \to U$. Notice also that all ideals U_i, $i \geq 1$, must be proper. Indeed, if $U_i = U$ for some i, then certainly $U = U_1$ for the filtration is descending. Thus, by (1.25),

$$U_a \cdot U_1 = U_a \cdot U = U_a \subset U_{a+1}.$$

This implies that $U_a = U$ for each $a \geq 1$, so $\bigcap_{i\geq 1} U_i = U \neq \{0\}$, which contradicts the completeness of U.

PROOF OF PROPOSITION 1.21. We apply Lemma 1.22 to the unital associative algebra $Lin_R^c(R \mathbin{\widehat{\otimes}} V, R \mathbin{\widehat{\otimes}} V)$ of continuous R-linear maps, with the multiplication given by the (point-wise) composition, and the identity map $\mathbb{1} : R \mathbin{\widehat{\otimes}} V \to R \mathbin{\widehat{\otimes}} V$ as the unit. It is easy to see that the descending filtration

$$Lin_R^c(R \mathbin{\widehat{\otimes}} V, R \mathbin{\widehat{\otimes}} V)_i := Lin_R^c(R \mathbin{\widehat{\otimes}} V, \mathfrak{m}^i \mathbin{\widehat{\otimes}} V), \ i \geq 1, \tag{1.26}$$

satisfies (1.25) and that, for $\phi \in Lin_R^c(R \mathbin{\widehat{\otimes}} V, R \mathbin{\widehat{\otimes}} V)$, the induced map $\overline{\phi} : V \to V$ is the identity if and only if

$$\phi - \mathbb{1} \in Lin_R^c(R \mathbin{\widehat{\otimes}} V, R \mathbin{\widehat{\otimes}} V)_1.$$

Lemma 1.22 now produces an inverse map ϕ^{-1}. □

Another useful consequence of Lemma 1.22 is the following standard

LEMMA 1.23. *Let* $R = (R, \mathfrak{m})$ *be a local, not necessary complete, Noetherian ring with residue field* $\Bbbk$*. For each* $n \geq 0$*, the quotient* $\mathfrak{a}_n := R/\mathfrak{m}^{n+1}$ *is a local Artin ring with the maximal ideal* $\mathfrak{m}/\mathfrak{m}^{n+1}$*.*

PROOF. Since the completion of a local Noetherian ring is again local Noetherian [**AM69**, Proposition 10.16] and since

$$R/\mathfrak{m} \cong \widehat{R}/\widehat{\mathfrak{m}},$$

we may assume that R is complete. It is clear that $\mathfrak{m}/\mathfrak{m}^{n+1}$ is an ideal in $R/\mathfrak{m}^{n+1}$. Since

$$\frac{R/\mathfrak{m}^{n+1}}{\mathfrak{m}/\mathfrak{m}^{n+1}} \cong \frac{R}{\mathfrak{m}} \cong \Bbbk,$$

$\mathfrak{m}/\mathfrak{m}^{n+1}$ is maximal. Let us show that any *proper* ideal $I \subset \mathfrak{a}$ is contained in $\mathfrak{m}/\mathfrak{m}^{n+1}$. If $I \not\subset \mathfrak{m}/\mathfrak{m}^{n+1}$, then there exists $\alpha \in I$ represented by some $x \in R$ satisfying $\epsilon(x) \neq 0$, where $\epsilon : R \to R/\mathfrak{m} \cong \Bbbk$ is the augmentation map. By multiplying with a scalar if necessary, we may clearly achieve that $\epsilon(x) = 1$ that is, $x - 1 \in \mathfrak{m}$. By Lemma 1.22, there exists $y \in R$ such that $xy = 1$. Then, for the equivalence classes in $\mathfrak{a} = R/\mathfrak{m}^{n+1}$ we have $1 = [x][y] = \alpha[y]$, therefore $1 \in I$, so $I = \mathfrak{a}$.

Since $\mathfrak{m}/\mathfrak{m}^{n+1}$ contains all proper ideals, it is the *unique* maximal proper ideal, which proves that $\mathfrak{a}$ is local. The Artin property of $\mathfrak{a}$ is also clear: any proper ideal $I \subset \mathfrak{a}$ is contained in $\mathfrak{m}/\mathfrak{m}^{n+1}$, so $I^{m+1} \subset (\mathfrak{m}/\mathfrak{m}^{n+1})^{m+1} = 0$. □

Let us review three most important types of deformations. As usual, for elements a, b of an associative algebra A we denote by ab their product. Also the notation $\mathbb{N} := \{1, 2, 3, \ldots\}$ of the set of natural numbers is standard.

Definition 1.24. A *formal deformation* is a deformation over the complete local augmented ring $\Bbbk[[t]]$.

Proposition 1.25. *A formal deformation of $A = (V, \cdot)$ is given by a family*

$$\{\mu_i : V \otimes V \to V \mid i \in \mathbb{N}\} \tag{1.27}$$

satisfying, for each $k \geq 1$ and $a, b, c \in V$,

$$\begin{aligned} \mu_k(a,b)c + \mu_k(ab,c) + \sum_{i+j=k} \mu_i(\mu_j(a,b),c) = \\ a\mu_k(b,c) + \mu_k(a,bc) + \sum_{i+j=k} \mu_i(a,\mu_j(b,c)). \end{aligned} \tag{D_k}$$

Proof. By Lemma 1.20, the deformed multiplication μ is determined by its restriction to $V \otimes V \subset (\Bbbk[[t]] \mathbin{\widehat{\otimes}} V) \mathbin{\widehat{\otimes}}_{\Bbbk[[t]]} (\Bbbk[[t]] \mathbin{\widehat{\otimes}} V)$. Now expand $\mu(a,b)$, for $a, b \in V$, into a power series

$$\mu(a,b) = \mu_0(a,b) + t\mu_1(a,b) + t^2\mu_2(a,b) + t^3\mu_3(a,b) + \cdots$$

with some $\Bbbk$-bilinear functions $\mu_i : V \otimes V \to V$, $i \geq 0$. Clearly, μ_0 must be the original multiplication in A so, in fact,

$$\mu(a,b) = ab + t\mu_1(a,b) + t^2\mu_2(a,b) + t^3\mu_3(a,b) + \cdots . \tag{1.28}$$

It is easy to see that μ is associative if and only if (D_k) is satisfied for each $k \geq 1$. □

Proposition 1.25 shows that the set $\mathrm{Def}_A(\Bbbk[[t]])$ of formal deformations of A consists of families (1.27) satisfying the infinite system (D_k), $k \geq 1$, of quadratic equations. The trivial deformation is the one with $\mu_i = 0$ for each $i \geq 1$. It is also clear that

$$\mathrm{G}_A\big(\Bbbk[[t]]\big) \cong \big\{u = \mathbb{1}_V + \phi_1 t + \phi_2 t^2 + \phi_3 t^3 + \cdots \mid \phi_i \in \mathit{Lin}(V,V)\big\}, \tag{1.29}$$

with the group multiplication

$$\begin{aligned} (\mathbb{1}_V + &\phi_1' t + \phi_2' t^2 + \phi_3'' t^3 + \cdots)(\mathbb{1}_V + \phi_1'' t + \phi_2'' t^2 + \phi_3' t^3 + \cdots) := \\ &\mathbb{1}_V + (\phi_1' + \phi_1'')t + (\phi_2' + \phi_1'\phi_1'' + \phi_2'')t^2 + (\phi_3' + \phi_2'\phi_1'' + \phi_1'\phi_2'' + \phi_3'')t^3 + \cdots , \end{aligned}$$

where $\phi_i'\phi_j''$ denotes the standard composition of linear maps. Observe that, by Proposition 1.21, each u as in (1.29) indeed induces an *invertible*

$$\phi : \Bbbk[[t]] \mathbin{\widehat{\otimes}} V \to \Bbbk[[t]] \mathbin{\widehat{\otimes}} V.$$

By (1.24),

$$\mathfrak{Def}_A\big(\Bbbk[[t]]\big) \cong \mathrm{Def}_A\big(\Bbbk[[t]]\big)/\mathrm{G}_A\big(\Bbbk[[t]]\big).$$

It is the quotient of an infinite dimensional affine quadratic algebraic variety, modulo an action of a pro-unipotent group. From the point of view of singularity theory, this is the worst situation.

Expansion (1.28) exhibits μ as a one-dimensional family, depending on the parameter t, of associative products whose value at $t = 0$ is the original undeformed multiplication. Its 'formality' means that no kind of convergence is required, so the series (1.28) has only a 'formal' meaning. In [**Fia08**], all deformations with a complete local base are called formal.

Let $\Bbbk[t]$ be the polynomial ring as in Example 1.3, with the augmentation $\epsilon_0 : \Bbbk[t] \to \Bbbk$ defined by $\epsilon_0(f) := f(0) \in \Bbbk$, for $f \in \Bbbk[t]$. Associative $\Bbbk[t]$-algebra structures on the (uncompleted) $\Bbbk[t] \otimes V$ such that $\epsilon \otimes \mathbb{1}_V : \Bbbk[t] \otimes V \to V$ is a morphism of associative algebras are examples of *global deformations* of $A = (V, \cdot\,)$ in the sense of [**FP02**]. It is easy to verify that these deformations are precisely *finite* expressions (1.28).

Definition 1.26. An *infinitesimal deformation*, sometimes also called a *first order deformation*, of an algebra A is a deformation over the local Antin ring $D := \Bbbk[t]/(t^2)$ of dual numbers.

Notice that in [**Fia08**], all deformations over a local base $(R, \mathfrak{m})$ with $\mathfrak{m}^2 = 0$ are called infinitesimal. We leave the proof of the following version of Proposition 1.25 as an exercise.

Proposition 1.27. *An infinitesimal deformation of* $A = (V, \cdot\,)$ *is given by a linear map* $\mu_1 : V \otimes V \to V$ *fulfilling*

$$a\mu_1(b,c) - \mu_1(ab,c) + \mu_1(a,bc) - \mu_1(a,b)c = 0 \tag{1.30}$$

for each $a, b, c \in V$.

Therefore $\mathrm{Def}_A(D)$ consists of linear maps $\mu_1 : V \otimes V \to V$ satisfying (1.30). It is easy to see that

$$\mathrm{G}_A(D) \cong \big\{u = \mathbb{1}_V + \phi_1 t \mid \phi_1 \in \mathit{Lin}(V,V)\big\} \cong \mathit{Lin}(V,V),$$

with the abelian group structure of point-wise addition of linear maps, and the action on $\mu_1 \in \mathrm{Def}_A(D)$ given by

$$\phi_1(\mu_1)(a,b) := \mu_1(a,b) + \phi_1(ab) - \phi_1(a)b - a\phi_1(b). \tag{1.31}$$

Not very surprisingly, the set

$$\mathfrak{Def}_A(D) = \mathrm{Def}_A(D)/\mathrm{G}_A(D) \tag{1.32}$$

of isomorphism classes of infinitesimal deformations of A is a vector space that equals the second Hochschild cohomology group $HH^2(A,A)$ recalled in the next chapter, see Theorem 2.3.

Definition 1.28. Let $n \geq 1$. An *n-deformation* of an algebra A is a deformation over the local Artin ring $\Bbbk[t]/(t^{n+1})$.

1-deformations are infinitesimal (= first-order) deformations of Definition 1.26. We have the following version of Proposition 1.25 which generalizes Proposition 1.27.

PROPOSITION 1.29. *An n-deformation of $A = (V, \cdot\,)$ is given by a family*

$$\{\mu_i : V \otimes V \to V \mid 1 \leq i \leq n\}$$

of $\Bbbk$-linear maps satisfying condition (D_k) of Theorem 1.25 for $1 \leq k \leq n$.

The proof is obvious, as well as the description

$$\mathrm{G}_A\big(\Bbbk[t]/(t^{n+1})\big) \cong \big\{u = \mathbb{1}_V + \phi_1 t + \phi_2 t^2 + \cdots + \phi_n t^n \mid \phi_1, \ldots, \phi_n \in Lin(V, V)\big\}$$

of the gauge group. We leave both as an exercise.

(Mini)versal deformations. A deformation ω of an associative algebra A with a base S would be *universal*, if for any other deformation μ with the base R there exists a *unique* ring morphism $f : S \to R$ such that the push-forward $f_!(\omega)$ of ω along f is equivalent to μ. Unfortunately, as a consequence of the fact that the category of algebraic varieties is not closed under quotients, universal deformations seldom exist, the uniqueness of f is too much to ask. Under some mild assumptions, there however exist *miniversal deformations*. Recall that a deformation ω of an algebra A with the base S is miniversal, if

(i) for any deformation μ of A with the base R there exists a, not necessarily unique, ring morphism $f : S \to R$ such that $f_!(\omega)$ is equivalent to μ, and
(ii) if the maximal ideal of R satisfies $\mathfrak{m}^2 = 0$, then f is unique.

Deformations satisfying (i) only are called *versal*. The existence of miniversal deformations for a large class of algebras was proved in [**FP02**] and the citations therein; see also [**SS84**].

Deformations in algebraic geometry. Let us mention briefly how deformations are treated in algebraic geometry. Since we are not going to follow this direction in the sequel, we will be very telegraphic here, referring to [**Har10, Ser06**] for terminology and details.

For a point α of a scheme Y, let $(\mathcal{O}_\alpha, \mathfrak{m}_\alpha)$ denote the local ring of α and $k(\alpha) := \mathcal{O}_\alpha/\mathfrak{m}_\alpha$ its residue field. The inclusion $\{\alpha\} \hookrightarrow Y$ induces a morphism Spec $k(\alpha) \to Y$. Let E be another scheme and $p : E \to B$ a proper flat morphism. The *fiber* of p over $\alpha \in Y$ is the pull-back

(1.33)
$$\begin{array}{ccc} F_\alpha & \longrightarrow & E \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ \mathrm{Spec}\ k(\alpha) & \longrightarrow & Y. \end{array}$$

It turns out that F_α is a scheme over $k(\alpha)$ whose underlying topological space equals $p^{-1}(b) \subset E$.

Let X be a scheme over $k(b)$ isomorphic to F_b, via a fixed isomorphism which is considered to be a part of the structure. The morphism $p : E \to Y$ can be viewed as a family of deformations of the scheme X parametrized by the points of Y. The flatness and properness of the morphism $p : E \to Y$ guarantee that the fibers vary in a 'controlled' way, and that the above concept is invariant under the base change.

An R-deformation B of an associative $\Bbbk$-algebra A as in Definition 1.16 should then 'ideologically' be the same as a morphism Spec $B \to$ Spec R of spectra whose fiber F_0 is isomorphic to Spec A. Such an interpretation is, however, very superficial. Besides the non-commutativity of A and B, we do not assume B to be unital, so there is no natural algebra morphism $R \to B$ that would induce the above map of spectra.