

Contents

Acknowledgement	xi
Note to the reader	xiii
Chapter 1. Overview	1
1.1. Applications	4
1.2. A Scheme for Understanding Groups	5
Chapter 2. Nonpositively Curved Cube Complexes	7
2.1. Definitions	7
2.2. Some Favorite 2-Dimensional Examples	8
2.3. Right-Angled Artin Groups	12
2.4. Hyperplanes	13
Chapter 3. Cubical disk diagrams, hyperplanes, and convexity	15
3.1. Disk Diagrams	15
3.2. Properties of Hyperplanes	18
3.3. Local Isometries and Convexity	22
3.4. Background on Quasiconvexity	25
3.5. Cores, Hulls, and Superconvexity	27
Chapter 4. Special Cube Complexes	31
4.1. Hyperplane Definition of Special Cube Complex	31
4.2. Separability Criteria for Virtual Specialness	33
4.3. Canonical Completion and Retraction	35
4.4. Separability in the Hyperbolic Case	36
4.5. Wall-Injectivity and a Fundamental Commutative Diagram	39
4.6. Wall Projection Controls Retraction	40
Chapter 5. Virtual Specialness of Malnormal Amalgams	43
5.1. Specializing Malnormal Amalgams	43
5.2. Proof of the Isomorphic Elevation Lemma	48
Chapter 6. Wallspaces and their Dual Cube Complexes	53
6.1. Wallspaces	53
6.2. The Dual $CAT(0)$ Cube Complex	53
6.3. C is $CAT(0)$	55
6.4. Some Examples	56
6.5. Wallspaces from Codimension-1 Subgroups	58

Chapter 7. Finiteness properties of the dual cube complex	61
7.1. The Cubes of C :	61
7.2. The Bounded Packing Property and Finite Dimensionality:	62
7.3. Cocompactness in the Hyperbolic Case	63
7.4. Relative Cocompactness in the Relatively Hyperbolic Case	63
7.5. Properness of the G Action on $C(\tilde{X})$	65
7.6. The Cut-Wall Criterion for Properness	67
Chapter 8. Cubulating Malnormal Graphs of Cubulated Groups	69
8.1. A Wallspace for an Easy Non-Hyperbolic Group	69
8.2. Extending Walls	71
8.3. Constructing Turns	72
8.4. Cubulating Malnormal Amalgams	73
Chapter 9. Cubical Small Cancellation Theory	77
9.1. Cubical Presentations	78
9.2. The Fundamental Theorem of Small-Cancellation Theory	79
9.3. Combinatorial Gauss-Bonnet Theorem	81
9.4. Greendlinger's Lemma and the Ladder Theorem	82
9.5. Reduced Diagrams	84
9.6. Producing Examples	87
9.7. Rectified Diagrams	88
Chapter 10. Walls in Cubical Small-Cancellation Theory	95
10.1. Walls in Classical $C'(\frac{1}{6})$ Small-Cancellation Complexes	95
10.2. Wallspace Cones	95
10.3. Producing Wallspace Cones	96
10.4. Walls in \tilde{X}^*	97
10.5. Quasiconvexity of Walls in \tilde{X}^*	98
Chapter 11. Annular Diagrams	101
11.1. Classification of Flat Annuli	101
11.2. The Doubly Collared Annulus Theorem	103
11.3. Almost Malnormality	104
Chapter 12. Virtually Special Quotients	107
12.1. The Malnormal Special Quotient Theorem	107
12.2. Case Study: $F_2/\langle\langle W_1^{n_1}, \dots, W_r^{n_r} \rangle\rangle$	109
12.3. The Special Quotient Theorem	113
Chapter 13. Hyperbolicity and Quasiconvexity Detection	115
13.1. Cubical Version of Filling Theorem	115
13.2. Persistence of Quasiconvexity	117
13.3. No Missing Shells and Quasiconvexity	117
Chapter 14. Hyperbolic groups with a quasiconvex hierarchy	121
Chapter 15. The relatively hyperbolic setting	125

Chapter 16. Applications	129
16.1. Baumslag's Conjecture	129
16.2. 3-Manifolds	131
16.3. Limit Groups	132
Bibliography	135
Index of notation and defined terms	139