LECTURE 1

The Classical Theory: Part I

The first two lectures will be largely elementary and expository. They will deal with the upper-half-plane $\mathcal{H}$ and Riemann sphere $\mathbb{P}^1$ from the points of view of Hodge theory, representation theory and complex geometry. The topics to be covered will be

(i) compact Riemann surfaces of genus one (= 1-dimensional complex tori) and polarized Hodge structures (PHS) of weight one;
(ii) the space $\mathcal{H}$ of PHS's of weight one and its compact dual $\mathbb{P}^1$ as homogeneous complex manifolds;
(iii) the geometry and representation theory associated to $\mathcal{H}$;
(iv) equivalence classes of PHS's of weight one, as parametrized by $\Gamma\backslash\mathcal{H}$, and automorphic forms;
(v) the geometric representation theory associated to $\mathbb{P}^1$, including the realization of higher cohomology by global, holomorphic data;
(vi) Penrose transforms in genus $g = 1$ and $g \geqq 2$.

Assumptions.

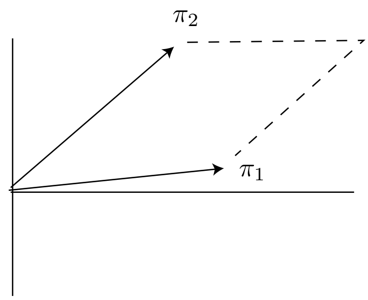
- basic knowledge of complex manifolds (in this lecture mainly Riemann surfaces);
- elementary topology and manifolds, including de Rham's theorem;
- some familiarity with classical modular forms will be helpful but not essential;[1]
- some familiarity with the basic theory of Lie groups and Lie algebras.

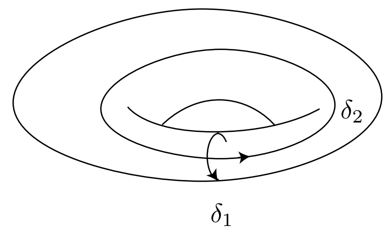
Complex tori of dimension one. We let X = compact, connected complex manifold of dimension one and genus one. Then X is a complex torus $\mathbb{C}/\Lambda$ where

$$\Lambda = \{n_1\pi_1 + n_2\pi_2\}_{n_1,n_2\in\mathbb{Z}} \subset \mathbb{C}$$

[1] The classical theory will be covered in the article [**Ke1**] by Matt Kerr in the Contemporary Mathematics volume, published by the AMS and that is associated to the CBMS conference.

is a lattice. The pictures are





Here $\delta_1 \leftrightarrow \pi_1$ and $\delta_2 \leftrightarrow \pi_2$ give a basis for $H_1(X, \mathbb{Z})$.

The complex plane $\mathbb{C} = \{z = x + iy\}$ is oriented by

$$dx \wedge dy = \left(\tfrac{i}{2}\right) dz \wedge d\bar{z} > 0.$$

We choose generators π_1, π_2 for Λ with $\pi_1 \wedge \pi_2 > 0$, and then the intersection number

$$\delta_1 \cdot \delta_2 = +1.$$

We set $V_{\mathbb{Z}} = H^1(X, \mathbb{Z})$, $V = V_{\mathbb{Z}} \otimes \mathbb{Q} = H^1(X, \mathbb{Q})$ and denote by

$$\begin{cases} Q : V \otimes V \to \mathbb{Q} \\ \mathbb{Q}(v, v') = -Q(v', v) \end{cases}$$

the cup-product, which via Poincaré duality $H_1(X, \mathbb{Q}) \cong H^1(X, \mathbb{Q})$ is the intersection form.

We have

$$\begin{array}{c} H^1(X, \mathbb{C}) \cong H^1_{\mathrm{DR}}(X) = \left\{ \begin{array}{c} \text{closed 1-forms } \psi \\ \text{modulo exact} \\ \text{1-forms } \psi = d\zeta \end{array} \right\} \\ \wr\| \\ H^1(X, \mathbb{Z})^* \otimes \mathbb{C} \end{array}$$

and it may be shown that

$$H^1_{\mathrm{DR}}(X) \cong \operatorname{span}_{\mathbb{C}} \{dz, d\bar{z}\}.$$

The pairing of cohomology and homology is given by *periods*

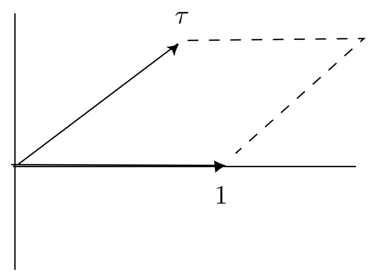
$$\pi_i = \int_{\delta_i} dz$$

and $\Pi = \binom{\pi_2}{\pi_1}$ is the *period matrix* (note the order of the π_i's).

Using the basis for $H^1(X, \mathbb{C})$ dual to the basis δ_1, δ_2 for $H_1(X, \mathbb{C})$, we have

$$\begin{array}{ccc} H^1(X, \mathbb{C}) & \cong \mathbb{C}^2 & = \text{column vectors} \\ \Cup & \Cup & \\ dz & = \Pi. & \end{array}$$

We may scale $\mathbb{C}$ by $z \to \lambda z$, and then $\Pi = \lambda \Pi$ so that the period matrix should be thought of as point in $\mathbb{P}^1$ with homogeneous coordinates $\left[\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right]$. By scaling, we may normalize to have $\pi_1 = 1$, so that setting $\tau = \pi_2$ the normalized period matrix is $\left[\begin{smallmatrix}\tau\\1\end{smallmatrix}\right]$ where $\operatorname{Im}\tau > 0$.



Differential forms on an n-dimensional complex manifold Y with local holomorphic coordinates $z_1, \dots, z_n$ are direct sums of those of *type* (p, q)

$$f\, \underbrace{dz_{i_1} \wedge \cdots \wedge dz_{i_p}}_{p} \wedge \underbrace{d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}}_{q}.$$

Thus the C^∞ forms of degree r on Y are

$$\begin{cases} A^r(Y) = \underset{p+q=r}{\oplus} A^{p,q}(Y) \\ A^{q,p}(Y) = \overline{A^{p,q}(Y)}. \end{cases}$$

Setting

$$H^{1,0}(X) = \operatorname{span}\{dz\}$$
$$H^{0,1}(X) = \operatorname{span}\{d\bar{z}\}$$

we have

$$\begin{cases} H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X) \\ H^{0,1}(X) = \overline{H^{1,0}(X)}. \end{cases}$$

This says that the above decomposition of the 1-forms on X induces a similar decomposition in cohomology. This is true in general for a compact Kähler manifold (Hodge's theorem) and is the basic starting point for Hodge theory. A recent source is [**Cat1**].

From $dz \wedge dz = 0$ and $\left(\frac{i}{2}\right) dz \wedge d\bar{z} > 0$, by using that cup-product is given in de Rham cohomology by wedge product and integration over X we have

$$\begin{cases} Q\big(H^{1,0}(X), H^{1,0}(X)\big) = 0 \\ iQ\big(H^{1,0}(X), \overline{H^{1,0}(X)}\big) > 0. \end{cases}$$

Using the above bases the matrix for Q is

$$Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and these relations are

$$\begin{cases} Q(\Pi, \Pi) = {}^t\Pi Q \Pi = 0 \\ iQ(\Pi, \overline{\Pi}) = i\,{}^t\overline{\Pi} Q \Pi > 0. \end{cases}$$

For $\Pi = \left[\begin{smallmatrix}\tau\\1\end{smallmatrix}\right]$ the second is just $\operatorname{Im}\tau > 0$.

DEFINITIONS. (i) A *Hodge structure* of weight one is given by a $\mathbb{Q}$-vector space V with a line $V^{1,0} \subset V_{\mathbb{C}}$ satisfying

$$\begin{cases} V_{\mathbb{C}} &= V^{1,0} \oplus V^{0,1} \\ V^{0,1} &= \overline{V}^{1,0}. \end{cases}$$

(ii) A *polarized Hodge structure* of weight one (PHS) is given by the above together with a non-degenerate form

$$Q : V \otimes V \to \mathbb{Q}, \qquad Q(v, v') = -Q(v', v)$$

satisfying the Hodge-Riemann bilinear relations

$$\begin{cases} Q(V^{1,0}, V^{1,0}) = 0 \\ iQ(V^{1,0}, \overline{V}^{1,0}) > 0. \end{cases}$$

In practice we will usually have $V = V_{\mathbb{Z}} \otimes \mathbb{Q}$. The reason for working with $\mathbb{Q}$ will be explained later.

When $\dim V = 2$, we may always choose a basis so that $V \cong \mathbb{Q}^2 =$ column vectors and Q is given by the matrix above. Then $V^{1,0} \cong \mathbb{C}$ is spanned by a point

$$\left[\begin{smallmatrix} \tau \\ 1 \end{smallmatrix}\right] \in \mathbb{P}V_{\mathbb{C}} \cong \mathbb{P}^1$$

Identification. The space of PHS's of weight one *(period domain)* is given by the upper-half-plane

$$\mathcal{H} = \{\tau : \operatorname{Im} \tau > 0\}.$$

The *compact dual* $\check{\mathcal{H}}$ given by subspaces $V^{1,0} \subset V_{\mathbb{C}}$ satisfying $Q(V^{1,0}, V^{1,0}) = 0$ (this is automatic in this case) is $\check{\mathcal{H}} = \mathbb{P}V_{\mathbb{C}} \cong \mathbb{P}^1$ where

$$\mathbb{P}^1 = \{\tau\text{-plane}\} \cup \infty = \text{lines through the origin in } \mathbb{C}^2.$$[2]

It is well known that $\mathcal{H}$ and $\mathbb{P}^1$ are *homogeneous complex manifolds*; i.e., they are acted on transitively by Lie groups. Here are the relevant groups. Writing

$$z = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}, \quad w = \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}$$

and using Q to identify $\Lambda^2 V$ with $\mathbb{Q}$ we have

$$Q(z, w) = {}^t w Q z = z \wedge w$$

and the relevant groups are

$$\begin{cases} \operatorname{Aut}(V_{\mathbb{R}}, Q) \cong \mathrm{SL}_2(\mathbb{R}) & \text{for } \mathcal{H} \\ \operatorname{Aut}(V_{\mathbb{C}}, Q) \cong \mathrm{SL}_2(\mathbb{C}) & \text{for } \mathbb{P}^1. \end{cases}$$

In terms of the coordinate τ the action is the familiar one:

$$\tau \to \frac{a\tau + c}{c\tau + d}$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2$. This is because $\tau = z_0/z_1$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{pmatrix} az_0 + bz_1 \\ cz_0 + dz_1 \end{pmatrix} = z_1 \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix}.$$

[2] [**CM-SP**] is a general reference for period domains and their differential geometric properties. A recent source is [**Ca**].

If we choose for our reference point $i \in \mathcal{H}$ $(= \left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right] \in \mathbb{P}^1)$, then we have the identifications

$$\begin{cases} \mathcal{H} \cong \mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}(2) \\ \mathbb{P}^1 \cong \mathrm{SL}_2(\mathbb{C})/B \end{cases}$$

where (this is a little exercise)

$$\mathrm{SO}(2) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a^2 + b^2 = 1 \right\} = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\}$$

$$B = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : i(a-d) = -b-c \right\}.$$

The Lie algebras are (here $\mathfrak{k} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$)

$$\mathrm{sl}_2(\mathfrak{k}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b \in \mathfrak{k} \right\}$$

$$\mathrm{so}(2) = \left\{ \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} : a \in \mathbb{R} \right\}$$

$$\mathfrak{b} = \left\{ \begin{pmatrix} a & -b \\ b & -a \end{pmatrix} : a, b \in \mathbb{C} \right\}.$$

REMARK. From a Hodge-theoretic perspective the above identifications of the period domain $\mathcal{H}$ and its compact dual $\check{\mathcal{H}}$ are the most convenient. From a group-theoretic perspective, it is frequently more convenient to set

$$\zeta = \frac{\tau - i}{\tau + i}, \qquad \operatorname{Im}\tau > 0 \Leftrightarrow |\zeta| < 1$$

and identify $\mathcal{H}$ with the unit disc $\Delta \subset \mathbb{C} \subset \mathbb{P}^1$. When this is done, $\mathrm{SL}_2(\mathbb{R})$ becomes the other real form

$$S\mathcal{U}(1,1)_{\mathbb{R}} = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) : {}^t\bar{g}\mathbb{H}g = \mathbb{H} \right\}$$

of $\mathrm{SL}_2(\mathbb{R})$, where here $\mathbb{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then

$$\mathcal{H} \ni i \leftrightarrow 0 \in \Delta$$

$$\mathrm{SO}(2) \leftrightarrow \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right\}$$

$$B \leftrightarrow \left\{ \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \right\}.$$

Thus, for the Δ model SO(2) becomes a "standard" maximal torus and B is a "standard" Borel subgroup.

We now think of $\mathcal{H}$ as the parameter space for the family of PHS's of weight one and with $\dim V = 2$. Over $\mathcal{H}$ there is the natural *Hodge bundle*

$$\mathbb{V}^{1,0} \to \mathcal{H}$$

with fibres

$$\mathbb{V}^{1,0}_\tau := V^{1,0}_\tau = \text{ line in } V_{\mathbb{C}}.$$

Under the inclusion $\mathcal{H} \hookrightarrow \mathbb{P}^1$, the Hodge bundle is the restriction of the tautological line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$. Both $\mathbb{V}^{1,0}$ and $\mathcal{O}_{\mathbb{P}^1}(-1)$ are examples of homogeneous vector bundles.

In general, given

- a homogeneous space
$$Y = A/B$$
where A is a Lie group and $B \subset A$ is a closed subgroup, and
- a linear representation $r : B \to \operatorname{Aut} E$ where E is a complex vector space,

there is an associated homogeneous vector bundle

$$\begin{array}{ccc} \mathbb{E} & := & A \times_B E \\ \downarrow & & \downarrow \\ Y & = & A/B \end{array}$$

where $A \times_B E$ is the trivial vector bundle $A \times E$ factored by the equivalence relation

$$(a, e) \sim (ab, r(b^{-1})e)$$

where $a \in A$, $e \in E$, $b \in B$. The group A acts on $\mathbb{E}$ by $a \cdot (a', e) = (aa', e)$ and there is an A-equivariant action on $\mathbb{E} \to Y$. There is an evident notion of a morphism of homogeneous vector bundles; then $\mathbb{E} \to Y$ is trivial *as a homogeneous vector bundle* if, and only if, $r : B \to \operatorname{Aut}(E)$ is the restriction to B of a representation of A.

EXAMPLE. Let $\tau_0 \in \mathcal{H} \subset \mathbb{P}^1$ be the reference point. For the standard linear representation of $\mathrm{SL}_2(\mathbb{C})$ on $V_{\mathbb{C}}$, the Borel subgroup B is the stability group of the flag

$$(0) \subset V^{1,0}_{\tau_0} \subset V_{\mathbb{C}}.$$

It follows that there is over $\mathbb{P}^1$ an exact sequence of $\mathrm{SL}_2(\mathbb{C})$-homogeneous vector bundles

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{V} \to \mathcal{O}_{\mathbb{P}^1}(1) \to 0$$

where $\mathbb{V} = \mathbb{P}^1 \times V_{\mathbb{C}}$ with $g \in \mathrm{SL}_2(\mathbb{C})$ acting on $\mathbb{V}$ by $g \cdot ([z], v) = ([gz], gv)$. The restriction to $\mathcal{H}$ of this sequence is an exact sequence of $\mathrm{SL}_2(\mathbb{R})$-homogeneous bundles

$$0 \to \mathbb{V}^{1,0} \to \mathbb{V} \to \mathbb{V}^{0,1} \to 0.$$

The bundle $\mathbb{V}^{1,0}$ is given by the representation

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \to e^{i\theta}$$

of $\mathrm{SO}(2)$. Using the form Q the quotient bundle $\mathbb{V}/\mathbb{V}^{1,0} := \mathbb{V}^{0,1}$ is identified with the dual bundle $\mathbb{V}^{1,0*}$.

The *canonical line bundle* is

$$\omega_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2).$$

Thus

$$\omega_{\mathcal{H}} \cong (\mathbb{V}^{1,0})^{\otimes 2}.$$

Proof. For the Grassmannian $Y = \mathrm{Gr}(n, E)$ of n-planes P in a vector space E there is the standard $\mathrm{GL}(E)$-equivariant isomorphism

$$T_P Y \cong \mathrm{Hom}(P, E/P).$$

In the case above where $E = \mathbb{C}^2$ and $z = \left[\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right] \in \mathbb{P}^1$ we have

$$T_z\mathbb{P}^1 \cong V_z^{1,0^*} \otimes V_{\mathbb{C}}/V_z^{1,0}$$

where $V_z^{1,0}$ is the line in $V_{\mathbb{C}}$ corresponding to z. If we use the group $\mathrm{SL}_2(\mathbb{C})$ that preserves Q in place of $\mathrm{GL}_2(\mathbb{C})$, then

$$V_{\mathbb{C}}/V_z^{1,0} \cong V_z^{1,0^*}.$$

Thus the cotangent space

$$T_z^*\mathbb{P}^1 \cong V_z^{2,0}$$

where in general we set $\mathbb{V}^{n,0} = (\mathbb{V}^{1,0})^{\otimes n}$. The above identification $\omega_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$ is an $\mathrm{SL}_2(\mathbb{C})$, but *not* $\mathrm{GL}_2(\mathbb{C})$, equivalence of homogenous bundles.

Convention. We set

$$\omega_{\mathcal{H}}^{1/2} = \mathbb{V}^{1,0}.$$

The Hodge bundle $\mathbb{V}^{1,0} \to \mathcal{H}$ has an $\mathrm{SL}_2(\mathbb{R})$-invariant metric, *the Hodge metric*, given fibrewise by the 2^{nd} Hodge-Riemann bilinear relation. The basic invariant of a metric is its *curvature*, and we have the following

General fact. Let $\mathbb{L} \to Y$ be an Hermitian line bundle over a complex manifold Y. Then the *Chern* (or curvature) *form* is

$$c_1(\mathbb{L}) = \left(\frac{1}{2\pi i}\right) \partial\overline{\partial} \log \|s\|^2$$

where $s \in \mathcal{O}(\mathbb{L})$ is any non-vanishing local holomorphic section and $\|s\|^2$ is its length squared.

Basic calculation.

$$c_1(\mathbb{V}^{1,0}) = \frac{1}{4\pi}\frac{dx \wedge dy}{y^2} = \frac{i}{2\pi}\frac{d\tau \wedge \overline{d\tau}}{(\operatorname{Im}\tau)^2}.$$

This has the following

Consequence. *The tangent bundle*

$$T\mathcal{H} \cong \mathbb{V}^{0,2}$$

has a metric

$$ds^2_{\mathcal{H}} = \frac{dx^2 + dy^2}{y^2} = \left(\frac{1}{(\operatorname{Im}\tau)^2}\right) \operatorname{Re}(dz\, d\bar{z})$$

of constant negative Gauss curvature.

Before giving the proof we shall make a couple of observations.

Any $\mathrm{SL}_2(\mathbb{R})$ invariant Hermitian metric on $\mathcal{H}$ is conformally equivalent to $dx^2 + dy^2$; hence it is of the form

$$h(x, y)\left(\frac{dx^2 + dy^2}{y^2}\right)$$

for a positive function $h(x, y)$. Invariance under translation $\tau \to \tau + b$, $b \in \mathbb{R}$, corresponding to the subgroup $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$, implies that $h(x, y) = h(y)$ depends only on y. Then invariance under $\tau \to a\tau$ corresponding to the subgroup $\left(\begin{smallmatrix} a^{1/2} & 0 \\ 0 & a^{-1/2} \end{smallmatrix}\right)$, $a > 0$, gives that $h(y) = \text{constant}$. A similar argument gives that $c_1(\mathbb{V}^{1,0})$ is a constant multiple of the form above.

The all important sign of the curvature K may be determined geometrically as follows: Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be a discrete group such that $Y = \Gamma\backslash\mathcal{H}$ is a compact

Riemann surface of genus $g \geqq 2$ with the metric induced from that on $\mathcal{H}$. By the Gauss-Bonnet theorem

$$0 > 2-2g = \chi(Y) = \frac{1}{4\pi}\int_Y K dA = K\left(\frac{\text{Area}(Y)}{4\pi}\right).$$

PROOF OF BASIC CALCULATION. We define a section $s \in \Gamma(\mathcal{H}, \mathbb{V}^{1,0})$ by

$$s(\tau) = \begin{pmatrix} \tau \\ 1 \end{pmatrix} \in \mathbb{V}^{1,0}_\tau.$$

The length squared is given by

$$\|s(\tau)\|^2 = i\,{}^t\overline{s(\tau)}Qs(\tau) = 2y.$$

Using for $\tau = x+iy$

$$\begin{cases} \partial_\tau = \frac{1}{2}(\partial_x - i\partial_y) \\ \partial_{\bar\tau} = \frac{1}{2}(\partial_x + i\partial_y) \end{cases}$$

we obtain

$$\frac{i}{2\pi}\overline{\partial}\partial = -\frac{1}{4\pi}(\partial_x^2+\partial_y^2)dx\wedge dy.$$

This gives

$$\frac{i}{2\pi}\overline{\partial}\partial \log \|s(\tau)\|^2 = \frac{1}{4\pi}\frac{dx\wedge dy}{y^2}.$$

REMARK. There is also a $S\mathcal{U}(2)$-invariant metric on $\mathcal{O}_{\mathbb{P}^1}(-1)$ induced from the standard metric on $\mathbb{C}^2$. For this metric

$$\|s(\tau)\|_c^2 = 1+|\tau|^2$$

(the subscript c on $\|\ \|_c^2$ stands for "compact"). Then we have

$$c_1(\mathbb{V}_c^{1,0}) = -\frac{1}{4\pi}\frac{dx\wedge dy}{(1+|\tau|^2)^2}.$$

Thus, $\mathbb{V}^{1,0} \to \mathcal{H}$ is a *positive* line bundle whereas $\mathbb{V}_c^{1,0} \to \mathbb{P}^1$ is a *negative* line bundle with

$$\deg \mathcal{O}_{\mathbb{P}^1}(-1) = \int_{\mathbb{P}^1} c_1(\mathbb{V}_c^{1,0}) = -1.$$

This *sign reversal* between the $\mathrm{SL}_2(\mathbb{R})$-invariant curvature on the open domain $\mathcal{H}$ and the $S\mathcal{U}(2)$ (= compact form of $\mathrm{SL}_2(\mathbb{C})$)-invariant metric on the compact dual $\check{\mathcal{H}} = \mathbb{P}^1$ will hold in general and is a fundamental phenomenon in Hodge theory.

Above we have holomorphically trivialized $\mathbb{V}^{1,0} \to \mathcal{H}$ using the section

$$s(\tau) = \begin{pmatrix} \tau \\ 1 \end{pmatrix}.$$

We have also noted that we have the isomorphism of $\mathrm{SL}_2(\mathbb{R})$-homogeneous line bundles

$$\omega_{\mathcal{H}} \cong \mathbb{V}^{2,0}.$$

Now $\omega_{\mathcal{H}}$ has a section $d\tau$ and a useful fact is that under this isomorphism

$$d\tau = s(\tau)^2.$$

The proof is by tracing through the isomorphism. To see why it should be true we make the following observation: Under the action of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, $s(\tau)$ transforms to

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix} = (c\tau + d) \begin{pmatrix} \frac{a\tau+b}{c\tau+d} \\ 1 \end{pmatrix};$$

i.e., $s(\tau)$ transforms by $(c\tau + d)^{-1}$. On the other hand, using $ad - bc = 1$ we find that

$$d\left(\frac{a\tau + b}{c\tau + d}\right) = \frac{d\tau}{(c\tau + d)^2}.$$

Thus $s(\tau)^2$ and $d\tau$ transform the same way under $\mathrm{SL}_2(\mathbb{R})$, and consequently their ratio is a constant function on $\mathcal{H}$.

Beginnings of representation theory[3]

In these lectures we shall be primarily concerned with infinite dimensional representations of real, semi-simple Lie groups and with finite dimensional representations of reductive $\mathbb{Q}$-algebraic groups. Leaving aside some matters of terminology and definitions for the moment we shall briefly describe the basic examples of the former in the present framework.

Denote by $\Gamma(\mathcal{H}, \mathbb{V}^{n,0})$ the space of global holomorphic sections over $\mathcal{H}$ of the n^{th} tensor power of the Hodge bundle, and by $d\mu(\tau)$ the $\mathrm{SL}_2(\mathbb{R})$ invariant area form $dx \wedge dy/y^2$ on $\mathcal{H}$.

Definition. For $n \geqq 2$ we set

$$\mathcal{D}_n^+ = \left\{\psi \in \Gamma(\mathcal{H}, \mathbb{V}^{n,0}) : \int_{\mathcal{H}} \|\psi(\tau)\|^2 d\mu(\tau) < \infty\right\}.$$

There is an obvious natural action of $\mathrm{SL}_2(\mathbb{R})$ on $\Gamma(\mathcal{H}, \mathbb{V}^{n,0})$ that preserves the pointwise norms, and it is a basic result [**Kn2**] that the map

$$\mathrm{SL}_2(\mathbb{R}) \to \mathrm{Aut}(\mathcal{D}_n^+)$$

gives an irreducible, unitary representation of $\mathrm{SL}_2(\mathbb{R})$.

As noted above there is a holomorphic trivialization of $\mathbb{V}^{1,0} \to \mathcal{H}$ given by the non-zero section

$$\sigma(\tau) = \begin{pmatrix} \tau \\ 1 \end{pmatrix}.$$

Then using the definition of the Hodge norm and ignoring the factor of 2,

$$\|\sigma(\tau)\|^2 = y.$$

Writing

$$\psi(\tau) = f_\psi(\tau)\sigma(\tau)$$

we have

$$\int_{\mathcal{H}} \|\psi(\tau)\|^2 d\mu(\tau) = \left(\frac{i}{2}\right) \iint |f_\psi(\tau)|^2 (\operatorname{Im}\tau)^{n-2} d\tau \wedge d\bar{\tau}.$$

Thus we may describe $\mathcal{D}_n^+$ as

$$\left\{f \in \Gamma(\mathcal{H}, \mathcal{O}_{\mathcal{H}}) : \iint |f_\psi(x + iy)|^2 y^{n-2} dx \wedge dy < \infty\right\}.$$

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[4]A general reference for this is [**Ke1**].

For $n = 1$ we define the norm by

$$\sup_{y>0} \int_{-\infty}^{\infty} |f_\psi(x+iy)|^2 dx.$$

The spaces $\mathcal{D}_n^-$ are described analogously using the lower half plane.

FACT ([**Kn2**]). *The* $\mathcal{D}_n^\pm$ *for* $n \geqq 2$ *are the discrete series representations of* $\mathrm{SL}_2(\mathbb{R})$. *For* $n = 1$, $\mathcal{D}_1^\pm$ *are the limits of discrete series.*

The terminology arises from the fact that in the spectral decomposition of $L^2(\mathrm{SL}_2(\mathbb{R}))$ the $\mathcal{D}_n^\pm$ for $n \geqq 2$ occur discretely.

There is an important duality between the orbits of $\mathrm{SL}_2(\mathbb{R})$ and of $\mathrm{SO}(2,\mathbb{C})$ acting on $\mathbb{P}^1$. Anticipating terminology to be used later in these lectures we set

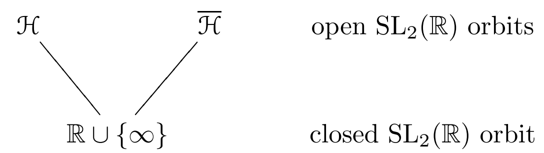
- $\mathbb{P}^1$ = *flag variety* $\mathrm{SL}_2(\mathbb{C})/B$ where B is the Borel subgroup fixing $i = \left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right]$;
- $\mathrm{SL}_2(\mathbb{R})$ = *real form* of $\mathrm{SL}_2(\mathbb{C})$ relative to the conjugation $A \to \overline{A}$;
- $\mathrm{SO}(2)$ = *maximal compact subgroup* of $\mathrm{SL}_2(\mathbb{R})$ (in this case it is $\mathrm{SL}_2(\mathbb{R}) \cap B$);
- $\mathcal{H}$ = *flag domain* $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$;
- $\mathrm{SO}(2,\mathbb{C})$ = *complexification* of $\mathrm{SO}(2)$.

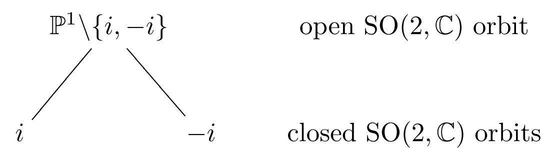
We note that $\mathrm{SO}(2,\mathbb{C}) \cong \mathbb{C}^*$.

Matsuki duality is a one-to-one correspondence of the sets

$$\{\mathrm{SL}_2(\mathbb{R})\text{-orbits in } \mathbb{P}^1\} \leftrightarrow \{\mathrm{SO}(2,\mathbb{C})\text{-orbits in } \mathbb{P}^1\}$$

that reverses the relation "in the closure of." The orbit structures in this case are





The lines mean "in the closure of."[5] The correspondence in Matsuki duality is

$$\begin{cases} \mathcal{H} \leftrightarrow i \\ \overline{\mathcal{H}} \leftrightarrow -i \end{cases}$$

$$\mathbb{R} \cup \{0\} \leftrightarrow \mathbb{P}^1 \backslash \{i, -i\}.$$

Matsuki duality arises in the context of representation theory as follows: A *Harish-Chandra module* is a representation space W for $\mathrm{sl}_2(\mathbb{C})$ and for $\mathrm{SO}(2,\mathbb{C})$ that satisfies certain conditions (to be explained in Lecture 5). A *Zuckerman module* is,

[5] Matsuki duality for flag varieties is discussed in [**FHW**] and in [**Sch3**] where its connection to representation is taken up.

for these lectures, a module obtained by taking finite parts of completed unitary $\mathrm{SL}_2(\mathbb{R})$-modules. For the $\mathcal{D}_n^+$ the modules are formal power series

$$\psi = \sum_{k \geqq 0} a_k (\tau - i)^k d\tau^{\otimes n/2}.$$

We think of these as associated to $G_\mathbb{R}$-modules arising from the open orbit $\mathcal{H}$. The Lie algebra $\mathrm{sl}_2(\mathbb{C})$, thought of as vector fields on $\mathbb{P}^1$, operates on ψ above by the Lie derivative, and $\mathrm{SO}(2, \mathbb{C})$ operates by linear fractional transformations.

Associated to the closed $\mathrm{SO}(2, \mathbb{C})$ orbit i are formal Laurent series

$$\gamma = \sum_{l \geqq 1} \frac{b_l}{(\tau - i)^l} \left(\frac{\partial}{\partial \tau} \right)^{\otimes n/2} dz.$$

This is also a $(\mathrm{sl}_2(\mathbb{C}), \mathrm{SO}(2, \mathbb{C}))$-module. The pairing between $\mathrm{SO}(2, \mathbb{C})$-finite vectors, i.e., finite power and Laurent series, is

$$\langle \psi, \gamma \rangle = \mathrm{Res}_i(\psi, \gamma).$$

There are also representations associated to the closed $\mathrm{SL}_2(\mathbb{R})$ orbit and open $\mathrm{SO}(2, \mathbb{C})$ orbit that are in duality (cf. [**Sch3**]).

There is a similar picture if one takes the other real form $S\mathcal{U}(1,1)_\mathbb{R}$ of $\mathrm{SL}_2(\mathbb{C})$. It is a nice exercise to work out the orbit structure and duality in this case.

We shall revisit Matsuki duality in this case, but set in a general context, in Lecture 2.

Why we work over $\mathbb{Q}$. Setting $X_\Lambda = C/\Lambda$ we say that X_Λ and $X_{\Lambda'}$ are *isomorphic* if there is a linear mapping

$$\alpha : \mathbb{C} \xrightarrow{\sim} \mathbb{C}$$

with $\alpha(\Lambda) = \Lambda'$. This is equivalent to X_Λ and $X_{\Lambda'}$ being biholomorphic as compact Riemann surfaces. Normalizing the lattices as above the condition is

$$\tau' = \frac{a\tau + b}{c\tau + c}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Thus the equivalence classes of compact Riemann surfaces of genus one is identified with the quotient space $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{H}$.

For many purposes a weaker notion of equivalence is more useful. We say that X_Λ and $X_{\Lambda'}$ are *isogeneous* if the condition $\alpha(\Lambda) = \Lambda'$ is replaced by $\alpha(\Lambda) \subseteq \Lambda'$. Then $\Lambda'/\alpha(\Lambda)$ is a finite group and there is an unramified covering map

$$X_\Lambda \to X_{\Lambda'}.$$

More generally, we may say that $X_\Lambda \sim X_{\Lambda'}$ if there is a diagram of isogenies

$$\begin{array}{ccc} & X_{\Lambda''} & \\ \swarrow & & \searrow \\ X_\Lambda & & X_{\Lambda'}. \end{array}$$

Identifying each of the universal covers with the same $\mathbb{C}$, we have $\Lambda \subset \Lambda''$, $\Lambda' \subset \Lambda''$ and then

$$\Lambda \otimes \mathbb{Q} = \Lambda'' \otimes \mathbb{Q} = \Lambda' \otimes \mathbb{Q}.$$

The converse is true, which suggests one reason for working over $\mathbb{Q}$.

REMARK. Among the important subgroups of $\mathrm{SL}_2(\mathbb{Z})$ are the *congruence subgroups*

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\mathrm{mod}\, N) \right\}.$$

Then $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$. Geometrically the quotient spaces $M_{\Gamma(N)} := \Gamma(N)\backslash\mathcal{H}$ arise as parameter spaces for complex tori X_τ plus additional "rigidifying" data. In this case the additional data is "marking" the N-torsion points

$$X_\tau(N) := (1/N)\Lambda/\Lambda \cong (\mathbb{Z}/N\mathbb{Z})^2.$$

When we require that an ismorphism $X_\Lambda(N) \cong X_\Lambda(N)$ take marked points to marked points the the equivalence classes of $X_\Lambda(N)$'s are $\Gamma(N)\backslash\mathcal{H}$.

Later in these talks we will encounter arithmetic groups Γ which have compact quotients.

LECTURE 2

The Classical Theory: Part II

This lecture is a continuation of the first one. In it we will introduce and illustrate a number of the basic concepts and terms that will appear in the later lectures, where also the formal definitions will be given.

Holomorphic automorphic forms. We have seen above that the equivalence classes of PHS's of weight one with $\dim V = 2$ may be identified with $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$. More generally, for geometric reasons discussed earlier one wishes to consider congruence subgroups $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ and the quotient spaces

$$M_\Gamma := \Gamma\backslash\mathcal{H}.$$

We make two important remarks concerning these spaces:

(i) The fixed points of $\gamma \in \Gamma$ acting on $\mathcal{H}$ occur when we have a PHS

$$V_{\mathbb{C}} = V_\tau^{1,0} \oplus V_\tau^{0,1}$$

left invariant by $\gamma \in \mathrm{Aut}(V_{\mathbb{Z}}, Q)$. Thus γ is an integral matrix that lies in the compact subgroup of $\mathrm{SL}_2(\mathbb{R})$ which preserves the positive Hermitian form $iQ(V_\tau^{1,0}, \overline{V}_\tau^{1,0})$. It follows that γ is of finite order, so that locally there is a disc Δ around τ with a coordinate t on Δ such that

$$\gamma(t) = \zeta \cdot t, \qquad \zeta^m = 1$$

for some integer m (in fact, $m = 2$ or 3). The map

$$s = t^m$$

then gives a local biholomorphism between Δ modulo the action of the group $\{\gamma^m\}$ and the s-disc. In this way M_Γ is a Riemann surface. We define sections of the bundle $\mathbb{V}^{n,0}$ over the quotient space $\{\gamma^k, k \in \mathbb{Z}\}\backslash\Delta$ of the disc modulo the action of γ to be given by γ-invariant sections of $\mathbb{V}^{n,0} \to \Delta$.

Remark. It will be a general fact, with essentially the same argument as above, that isotropy group of a general polarized Hodge structure that lies in an arithmetic group is finite.

(ii) M_Γ will not be compact but will have *cusps*, which are biholomorphic to the punctured disc Δ^*. The model here is the quotient of the region

$$\mathcal{H}_c = \{\mathrm{Im}\,\tau > c\}, \qquad c > 0$$

by the subgroup $\Gamma_0 = \{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) : n \in \mathbb{Z}\}$ of translations. Setting

$$q = e^{2\pi i \tau}$$

we obtain a biholomorphism

$$\Gamma_0\backslash\mathcal{H}_c \xrightarrow{\sim} \{0 < |q| < e^{-2\pi c}\}$$

of the quotient space with a punctured disc.

DEFINITION. A *holomorphic automorphic form of weight* n is given by a holomorphic section $\psi \in \Gamma(M_\Gamma, \mathbb{V}^{n,0})$ that is finite at the cusps.

These will be referred to simply as *modular forms.*[1]

We recall that $\omega_{\mathcal{H}} \cong \mathbb{V}^{2,0}$, so that $\omega_{\mathcal{H}}^{\otimes n/2} \cong \mathbb{V}^{n,0}$ and the sections of $\omega_{M_\Gamma}^{\otimes n/2}$ around the fixed points of Γ are defined as above. Thus automorphic forms of weight n are given by

$$\psi(\tau) = f_\psi(\tau) d\tau^{n/2}$$

where $f_\psi(\tau)$ is holomorphic on $\mathcal{H}$ and satisfies

$$f_\psi\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^n f_\psi(\tau).$$

Around a cusp as above one sets $q = e^{2\pi i \tau}$ and expands the resulting well-defined function $F_\psi(q) = f_\psi(\tau)$ in a Laurent series

$$F_\psi(q) = \sum_n a_n q^n.$$

By definition, the finiteness condition at the cusp is $a_n = 0$ for $n < 0$.

From a Hodge-theoretic perspective there is a *canonical extension* $\mathbb{V}_e^{1,0} \to \Delta$ of the Hodge bundle $\mathbb{V}^{1,0} \to \Delta^*$ given by the condition that the Hodge length of a section have at most *logarithmic growth* in the Hodge norm as one approaches the puncture (cf. [**Cat2**]). Modular forms are then the holomorphic sections of $\mathbb{V}^{n,0} \to \Gamma\backslash\mathcal{H}$ that extend to holomorphic sections of $\mathbb{V}_e^{n,0} \to \overline{\Gamma\backslash\mathcal{H}}$. In this way they are defined purely Hodge-theoretically.

Among the modular forms are the special class of *cusp forms* ψ, defined by the equivalent conditions

- $\int_{\Gamma\backslash\mathcal{H}} \|\psi\|^2 d\mu < \infty$;[2]
- $a_0 = 0$;
- ψ vanishes at the origin in the canonical extensions at the cusps.

Representation theory associated to $\mathbb{P}^1$. It is convenient to represent $\mathbb{P}^1$ as the compact dual of $\Delta = S\mathcal{U}(1,1)_\mathbb{R}/T$. Thus

$$\mathrm{SL}_2(\mathbb{C}) \cong S\mathcal{U}(1,1)_\mathbb{C}.$$

At the Lie algebra level we then have

$$\mathrm{su}(1,1)_\mathbb{R} = \left\{ \begin{pmatrix} i\alpha & \beta \\ \overline{\beta} & -i\alpha \end{pmatrix},\ \alpha, \beta \in \mathbb{R} \right\}$$

$$\mathrm{sl}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix},\ a, b, c \in \mathbb{C} \right\}$$

where $\mathrm{sl}_2(\mathbb{C}) = \mathrm{su}(1,1)_\mathbb{R} + i\,\mathrm{su}(1,1)_\mathbb{R}$ via

$$\begin{cases} a = \alpha + i\alpha' \\ b = \beta + i\beta' \\ c = \overline{\beta} + i\overline{\beta}'. \end{cases}$$

[1] We refer to [**Ke1**] for a general discusssion of classical modular forms, and to [**Ke2**] for a treatment of modular forms as they arise in the theory of Shimura varieties.

[2] This is not the usual condition, which involves the integral of f_ψ over a horizontal path in $\mathcal{H}$. We have used it in order to have a purely Hodge-theoretic formulation.

As basis for $\mathrm{sl}_2(\mathbb{C})$ we take the standard generators

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then setting

$$\mathfrak{h} = \mathbb{C}H, \quad \mathfrak{n}^+ = \mathbb{C}X, \quad \mathfrak{n}^- = \mathbb{C}Y$$

$\mathfrak{h}$ is a *Cartan sub-algebra* and the structure equations are

$$\begin{cases} [H, X] = 2X \\ [H, Y] = -2Y \\ [X, Y] = H. \end{cases}$$

The *weight lattice* P are the integral linear forms on $\mathbb{Z}H \subset \mathfrak{h}$. Thus $P \cong \mathbb{Z}$ with $\langle 1, H\rangle = 1$. The *root vectors* are the eigenvectors X, Y of $\mathfrak{h}$ acting on $\mathrm{sl}_2(\mathbb{C})$, and the *roots* are the corresponding eigenvalues $+2, -2$ viewed in the evident way as weights. They generate the *root lattice* $R \subset P$ with $P/R \cong \mathbb{Z}/2\mathbb{Z}$. The *positive root* is $+2$ and

$$\begin{cases} \mathfrak{n}^+ = \text{span of positive root vector } X \\ \mathfrak{n}^- = \text{span of negative root vector } Y. \end{cases}$$

For the *Borel subgroup* $B = \left\{ \left(\begin{smallmatrix} a & 0 \\ c & a^{-1} \end{smallmatrix} \right) \right\}$, which is the stability group of $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right] \in \mathbb{P}^1$ corresponding to the origin $0 \in \Delta$, the Lie algebra

$$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^-.$$

We note that the roots are purely imaginary on the Lie algebra

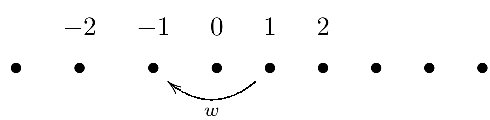
$$\mathfrak{t} = \left\{ \begin{pmatrix} i\theta & 0 \\ 0 & -i\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}$$

of the maximal torus $T \subset S\mathfrak{U}(1,1)_{\mathbb{R}}$.

As is customary notation in representation theory we set

$$\rho = \frac{1}{2}(\Sigma \text{ positive roots}) = 1.$$

The *Weyl group* W acting on $\mathfrak{h}$ is generated by the reflections in the hyperplanes defined by roots; in this case it is just $\pm\,\mathrm{id}$. One usually draws the picture of $i\mathfrak{t} \subset \mathfrak{h}$ with the roots and weights identified. In this case it is $2\pi i\mathfrak{t} = \mathbb{R}$, $P = \mathbb{Z}$, $R = 2\mathbb{Z}$.



where "2" is the positive root and W is generated by the identity and w where $w(x) = -x$.

Given a representation

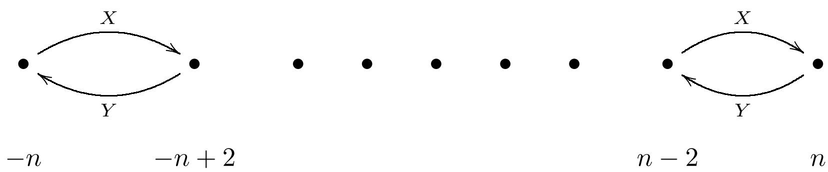
$$r : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Aut}\, E$$

where E is a complex vector space, the *weights* are the simultaneous eigenvalues of $r(\mathfrak{h})$. In this case they are the eigenvalues of $r(H)$. The *standard representation* is given by $E = \mathbb{C}^2$. The *weight vectors* are the eigenvectors for $r(\mathfrak{h})$. For the standard representation they are

$$e_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad e_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

with weights ± 1.

Any irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ is isomorphic to $S^n := \mathrm{Sym}^n E$ for $n = 0, 1, 2, \dots$. The picture of S^n is



where the dots represent the 1-dimensional weight spaces with weights $-n, -n+2, \dots, n-2, n$. The actions on X and Y are as indicated. If we make the identifications

$$\begin{cases} z_0 \leftrightarrow e_+ \\ z_1 \leftrightarrow e_- \end{cases}$$

then

- S^n = homogeneous polynomials $F(z_0, z_1)$ of degree n;
- $X = \partial_{z_1}, Y = \partial_{z_0}$;
- z_0^n is the *highest weight vector.*

As $\mathrm{SL}_2(\mathbb{C})$-modules we have

$$H^0(\mathcal{O}_{\mathbb{P}^1}(n)) \cong S^n.$$

Geometrically, since $\mathcal{O}_{\mathbb{P}^1}(n) = \mathcal{O}_{\mathbb{P}^1}(-n)^*$ we see that on each line L in $\mathbb{C}^2$, $F(z_0, z_0)$ restricts to a form that is homogeneous of degree n. Thus

$$F\big|_L \in \mathrm{Sym}^n L^* = \text{fibre of } \mathcal{O}_{\mathbb{P}^1}(n) \text{ at } L.$$

As a homogeneous line bundle

$$\mathcal{O}_{\mathbb{P}^1}(n) = \mathrm{SL}_2(\mathbb{C}) \times_B \mathbb{C}$$

where $\left(\begin{smallmatrix} a & 0 \\ b & a^{-1} \end{smallmatrix}\right) \in B$ acts on $\mathbb{C}$ by the character a^n. With our convention above, the differential of this character, viewed as a linear form on $\mathfrak{h}$, is the weight n.

With the notation to be used later we have

$$\mathcal{O}_{\mathbb{P}^1}(n) = L_n$$

where the subscript on L denotes the weight, which is the differential of the character that defines the homogeneous line bundle.

By Kodaira-Serre duality

$$H^1\big(\mathcal{O}_{\mathbb{P}^1}(-k-2)\big)^* \cong H^0(\omega_{\mathbb{P}^1}(k)),$$

and using the isomorphism of $\mathrm{SL}_2(\mathbb{C})$-homogeneous line bundles

$$\omega_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$$

$$H^1\big(\mathcal{O}_{\mathbb{P}^1}(-k-2)\big)^* \cong H^0\big(\mathcal{O}_{\mathbb{P}^1}(k)\big) \cong S^k.$$

Penrose transform for $\mathbb{P}^1$. One of the main aspects of these lectures will be to use the method of Eastwood-Gindikin-Wong [EGW] to represent higher degree sheaf cohomology by *global, holomorphic* data. We will now illustrate this for $H^1(\mathcal{O}_{\mathbb{P}^1}(-k-2))$.

For this we set

$$\check{\mathcal{W}} = \mathbb{P}^1 \times \mathbb{P}^1 \backslash (\text{diagonal}).$$

Using homogeneous coordinates $z = \left[\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right]$ we have

$$\check{\mathcal{W}} = \{(z, w) \in \mathbb{P}^1 \times \mathbb{P}^1 : z_0 w_1 - z_1 w_0 \neq 0\}.$$

For simplicity of notation we identify $\Lambda^2\mathbb{C}^2 = \mathbb{C}$ and then have $z \wedge w = z_0 w_1 - z_1 w_0$.[3] For calculations it is, as usual, convenient to work upstairs in the open set U in $\mathbb{C}^2 \times \mathbb{C}^2$ lying over $\check{\mathcal{W}}$ and keep track of the bi-homogeneity of a function defined in U.

The *correspondence space* $\check{\mathcal{W}}$ has the properties

(A) $\check{\mathcal{W}}$ is a Stein manifold (it is an affine algebraic variety);

(B) the fibres of the projection $\check{\mathcal{W}} \xrightarrow{\pi} \mathbb{P}^1$ on the first factor are contractible (they are just copies of $\mathbb{C}$).

Under these conditions [EGW] showed that there is a natural isomorphism

$$(*) \qquad H^q(\mathcal{O}_{\mathbb{P}^1}(m)) \cong H^q_{\mathrm{DR}}\big(\Gamma(\check{\mathcal{W}}, \Omega^\bullet_\pi(m)); d_\pi\big).$$

As we will now explain, the RHS of $(*)$ is a global, holomorphic object. A further detailed explanation will be given in Lecture 7. We will explain “in coordinates” what the various terms mean.

- Ω^q_π = sheaf of relative differentials on $\check{\mathcal{W}}$;
- $(\Omega^\bullet_\pi, d_\pi)$ is the complex $\cdots \to \Omega^q_\pi \xrightarrow{d_\pi} \Omega^{q+1}_\pi \to \dots$;
- $\Omega^\bullet_\pi(m) = \Omega^\bullet_\pi \otimes_{\mathcal{O}_{\check{\mathcal{W}}}} \pi^*\mathcal{O}_{\mathbb{P}^1}(m)$ where $\pi^*\mathcal{O}_{\mathbb{P}^1}(m)$ is the pullback bundle;
- $H^q_{\mathrm{DR}}\big(\Gamma(\check{\mathcal{W}}, \Omega^\bullet_\pi(m)); d_\pi\big)$ is the de Rham cohomology arising from the global sections of the above complex.

The relative forms are defined by

$$\Omega^q_\pi = \Omega^q_{\check{\mathcal{W}}}/\mathrm{image}\left\{\pi^*\Omega^1_{\mathbb{P}^1} \otimes \Omega^{q-1}_{\check{\mathcal{W}}} \to \Omega^q_{\check{\mathcal{W}}}\right\},$$

and d_π is induced by the usual exterior differential d. We think of $\pi^*\mathcal{O}_{\mathbb{P}^1}(m) \to \check{\mathcal{W}}$ as a vector bundle whose transition functions are constant on the fibres of π, and then d_π is well defined on sections of $\pi^*\mathcal{O}_{\mathbb{P}^1}(m)$.

The pullback sheaf $\pi^{-1}\mathcal{O}_{\mathbb{P}^1}(m)$ is the sheaf over $\check{\mathcal{W}}$ whose sections over an open set $Z \subset \check{\mathcal{W}}$ are the sections of $\mathcal{O}_{\mathbb{P}^1}(m)$ over $\pi(Z)$. We have an inclusion

$$\pi^{-1}\mathcal{O}_{\mathbb{P}^1}(m) \hookrightarrow \pi^*\mathcal{O}_{\mathbb{P}^1}(m)$$

where the subsheaf $\pi^{-1}\mathcal{O}_{\mathbb{P}^1}(m)$ is given by the sections of the bundle $\pi^*\mathcal{O}_{\mathbb{P}^1}(m)$ that are constant on the fibres of $\check{\mathcal{W}} \to \mathbb{P}^1$.

In coordinates $(z,w) = (z_0, z_1; w_0, w_1)$ on U, $\Omega^\bullet_\pi$ means that we mod out by dz_0 and dz_1. Setting

$$\Psi = w_1 dw_0 - w_0 dw_1$$

we have

- $\Gamma\big(\check{\mathcal{W}}, \pi^{-1}\mathcal{O}_{\mathbb{P}^1}(m)\big) = \left\{\begin{array}{l} F(z,w) \text{ holomorphic in } U \text{ and homogeneous} \\ \text{of degree } m \text{ in } z \text{ and of degree zero in } w \end{array}\right\};$
- $d_\pi F(z,w) = F_{w_0}(z,w)dw_0 + F_{w_1}(z,w)dw_1$.[4]

Using Euler's relation

$$w_0 F_{w_0} + w_1 F_{w_1} = 0$$

when $F(z,w)$ is homogenous of degree zero in w we obtain

$$d_\pi F(z,w) = \left(\frac{F_{w_0}}{w_1}\right)\Psi = -\left(\frac{F_{w_1}}{w_0}\right)\Psi.$$

---

[3]Thus our symmetry group is $\mathrm{SL}_2(\mathbb{C})$ and not $\mathrm{GL}_2(\mathbb{C})$.

[4]This equation is true for an $F(z,w)$ with any bi-homogeneity in z, w.

For the reasons to be seen below, it is now convenient to set $m = -k - 2$. Then

- $\Gamma(\check{\mathcal{W}}, \Omega^1_\pi(-k-2)) = \left\{ \begin{array}{l} \frac{G(z,w)\Psi}{(z\wedge w)^{k+2}} \text{ where } G(z,w) \text{ is homogeneous of} \\ \text{degree zero in } z \text{ and of degree } k \text{ in } w \end{array} \right\}.$

THEOREM. *Every class in* $H^1_{\mathrm{DR}}\big(\Gamma(\check{\mathcal{W}}, \Omega^\bullet_\pi(-k-2))\big)$ *has a unique representative of the form*

$$\frac{H(w)\Psi}{(z\wedge w)^{k+2}}$$

where $H(w)$ *is a homogeneous polynomial of degree* k.

Discussion. Given $\frac{G(z,w)\Psi}{(z\wedge w)^{k+2}}$ as above, we have to show that the equation

$$\frac{G(z,w)\Psi}{(z\wedge w)^{k+2}} = d_\pi\left(\frac{F(z,w)}{(z\wedge w)^{k+2}}\right) + \frac{H(w)\Psi}{(z\wedge w)^{k+2}},$$

has a unique solution where F has degree zero in z and degree $k+2$ in w and $H(w)$ is as above. Using Euler's relation $w_0F_{w_0} + w_1F_{w_1} = (k+2)F$ gives

$$d_\pi\left(\frac{F(z,w)}{(z\wedge w)^{k+2}}\right) = \frac{z_0F_{w_0}(z,w) + z_1F_{w_1}(z,w)\Psi}{(z\wedge w)^{k+3}}.$$

Then the equation to be solved is, after a calculation,

$$z_0F_{w_0}(z,w) + z_1F_{w_1}(z,w) = (z_0w_1 - z_1w_0)G(z,w) + (z_0w_1 - z_1w_0)H(w).$$

We shall first show that a solution is unique; i.e.,

$$z_0F_{w_0} + z_1F_{w_1} = (z_0w_1 - z_1w_0)H(w) \Rightarrow H(w) = 0.$$

Taking the forms that are homogeneous of degree one in z_0, z_1 gives

$$\begin{cases} F_{w_0} = w_1H \\ F_{w_1} = -w_0H. \end{cases}$$

Applying ∂_{w_1} to the first and ∂_{w_0} to the second leads to

$$H + w_1H_{w_1} = -H - w_0H_{w_0}.$$

Euler's relation then gives that $H(w)$ is homogeneous of degree -2, which is a contradiction.[5]

It is an interesting exercise to directly show by a calculation the existence of a solution to be above equation. On general grounds we know that this must be so because the map

$$(**) \qquad H(w) \longrightarrow \frac{H(w)\Psi}{(z\wedge w)^{k+2}}$$

has been shown to be injective and $\dim H^1\big(\mathcal{O}_{\mathbb{P}^1}(-k-2)\big) = k+1 = \dim S^k$.

The map $(**)$ has the following interpretation: Let $\mathbb{P}^1_z$ and $\mathbb{P}^1_w$ be $\mathbb{P}^1$ with coordinates z and w respectively. Then we have a *correspondence diagram*

$$\begin{array}{ccc} & \check{\mathcal{W}} & \\ {}^{\pi_w}\swarrow & & \searrow{}^{\pi_z} \\ \mathbb{P}^1_w & & \mathbb{P}^1_z. \end{array}$$

[5]One may wonder why the degree -2 appears, when all that is needed is degree -1. The philosophical reason is that $H^1(\mathcal{O}_{\mathbb{P}^1}(-1)) = (0)$.

Setting $\mathcal{O}_{\check{\mathcal{W}}}(a,b) = \pi_z^*\mathcal{O}_{\mathbb{P}_z^1}(a) \boxtimes \pi_w^*\mathcal{O}_{\mathbb{P}_w^1}(b)$ and using the theorem of EGW we obtain a diagram

$$\begin{array}{ccc} H^0\big(\mathcal{O}_{\mathbb{P}_w^1}(k)\big) & \overset{\mathcal{P}}{\dashrightarrow} & H^1\big(\mathcal{O}_{\mathbb{P}_z^1}(k-2)\big) \\ \wr\Vert & & \wr\Vert \\ H^0_{\mathrm{DR}}\big(\Gamma(\check{\mathcal{W}}, \Omega^\bullet_{\pi_w}(0,k)); d_{\pi_w}\big) & \xrightarrow{\frac{\Psi}{(z\wedge w)^{k+2}}} & H^1_{\mathrm{DR}}\big(\Gamma(\check{\mathcal{W}}, \Omega^\bullet_{\pi_z}(-k-2,0)); d_{\pi_z}\big) \end{array}$$

where the isomorphism

$$H^0\big(\mathcal{O}_{\mathbb{P}_w^1}(k)\big) \xrightarrow{\mathcal{P}} H^1\big(\mathcal{O}_{\mathbb{P}_z^1}(-k-2)\big)$$

is termed a *Penrose transform.* Letting $\mathrm{SL}_2(\mathbb{C})$ act on $\check{\mathcal{W}} \subset \mathbb{P}_w^1 \times \mathbb{P}_z^1$ diagonally in the above correspondence diagram we see that $\mathcal{P}$ is an isomorphism of $\mathrm{SL}_2(\mathbb{C})$-modules.

In fact, it is a geometric way of realizing in this special case the isomorphism in the *Borel-Weil-Bott* (BWB) *theorem.* The line bundle L_{-k-2} has weight $-k-2$, and for $k \geq 0$

$$\underbrace{-k-2}+\rho = -k-1$$

is *regular* in the sense that its value on every root vector is non-zero. Moreover

$$\#\{\text{positive root vectors } X \text{ with } \langle -k-1, X\rangle < 0\} = 1.$$

For $w \in W$ as above

$$w(-k-1) - \rho = k+1-1 = k.$$

The BWB states that for $k \geqq 0$, $H^q(\mathcal{O}_{\mathbb{P}^1}(-k-2)) \neq 0$ only for $q=1$, and that this group is the irreducible $\mathrm{SL}_2(\mathbb{C})$ module with highest weight $w(-k-2+\rho)-\rho = k$. The Penrose transform $\mathcal{P}$ realizes this identification.

The general discussion of the BWB will be given in the appendices to Lectures 5 and 7, where the special role of the weight ρ and transformation $w(\mu+\rho)-\rho$, where μ is a weight, will be explained.

Penrose transform for elliptic curves. The mechanism of the EGW theorem and resulting Penrose transform will be a basic tool in these lectures. We now illustrate it for compact Riemann surfaces of genus $g = 1$ and then shall do the same for genus $g > 1$.

For reasons deriving from the work of Carayol that will be discussed in the last lecture, it is convenient to take our complex torus

$$E' = \mathbb{C}/\mathcal{O}_{\mathbb{F}}$$

where $\mathbb{F}$ is a quadratic imaginary number field and $\mathcal{O}_{\mathbb{F}}$ is the ring of integers in $\mathbb{F}$; e.g., $\mathbb{F} = \mathbb{Q}(\sqrt{-d})$. We set

$$\mathcal{W} = \mathbb{C}\times\mathbb{C} \text{ with coordinates } (z', z'')$$

and consider the diagram

$$\begin{array}{ccc} & \mathcal{O}_{\mathbb{F}}\backslash\mathcal{W} & \\ {}^{\pi'}\swarrow & & \searrow{}^{\pi''} \\ E' & & E'' \end{array}$$

where $\alpha \in \mathcal{O}_{\mathbb{F}}$ acts by $\overline{\alpha}$ in the first factor and by $-\alpha$ in the second. It may be easily checked that $\mathcal{O}_{\mathbb{F}}\backslash\mathcal{W}$ is Stein and the fibres of π', π'' are contractible (they are just $\mathbb{C}$'s). Thus the EGW theorem applies to the above diagram.

We will describe line bundles $L'_r \to E'$ and $L''_r \to E''$, where r is a positive integer, and then shall define the Penrose transform to give an isomorphism

$$H^0(E', L'_r) \xrightarrow{\sim} H^1(E'', L''_{-r}).$$

For this we let β be a complex number with

$$\begin{cases} \beta + \overline{\beta} = |\alpha|^2 \\ \operatorname{Im}\beta = \beta_0 > 0. \end{cases}$$

Sections of $L'_r \to E'$ are given by entire holomorphic functions $\theta'_r(z')$ where

$$\theta'_r(z' + \overline{\alpha}) = \theta'_r(z') \exp\left(\frac{2\pi i r}{\beta_0}\left(\alpha z' + \frac{|\alpha|^2}{2}\right)\right).$$

These are *theta functions* viewed as sections of $L'_r \to E'$ where

$$L'_r = \mathbb{C} \times_{\mathcal{O}_{\mathbb{F}}} \mathbb{C}$$

with the equivalence relation

$$(z', \xi) \sim \left(z' + \overline{\alpha}, \exp\left(\frac{2\pi i r}{\beta_0}\left(\alpha z' + \frac{|\alpha|^2}{2}\right)\xi\right)\right).$$

Then

$$p(\theta')(z', z'') := \theta'(z') \exp\left(\frac{2\pi i r}{\beta_0} z' z''\right) dz'$$

gives a relative differential for $\pi'' : \mathcal{O}_{\mathbb{F}}\backslash\mathcal{W} \to E''$, and the functional equation

$$p(\theta')(z' + \overline{\alpha}, z'' - \alpha) = p(\theta')(z', z'') \exp\left(\frac{2\pi i r}{\beta_0}(\alpha z'' + \beta)\right)$$

shows that $p(\theta')$ has values in $\pi''^*(L''_{-r})$. Thus

$$p(\theta') \in H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{O}_{\mathbb{F}}\backslash\mathcal{W}, \Omega^\bullet_{\pi''}(L''_{-r})); d\pi''\big) \cong H^1(E'', L''_{-r})$$

and defines the Penrose transform alluded to above.

The relative 1-form $\exp\left(\frac{2\pi i r}{\beta_0} z' z''\right) dz'$ plays the role of the form ω in the $\mathbb{P}^1$-case. As suggested above the notation has been chosen to align with Carayol's work which will be discussed in the last lecture and in the appendix to that lecture.

Penrose transforms for curves of higher genus. We let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be a co-compact, discrete group and set

$$X' = \Gamma\backslash\mathcal{H}, \quad X = \Gamma\backslash\overline{\mathcal{H}}.$$

Here we take τ' as coordinate in $\mathcal{H}$ and τ as coordinate in $\overline{\mathcal{H}}$. The perhaps mysterious appearance of $\mathcal{H}$ and the complex conjugate $\overline{\mathcal{H}}$ will be "explained" when in Lecture 6 we discuss cycle spaces associated to flag domains $G_{\mathbb{R}}/T$ where G is of Hermitian type. We set $\mathcal{W} = \mathcal{H} \times \overline{\mathcal{H}}$ and consider the diagram

$$\begin{array}{ccc} & \Gamma\backslash\mathcal{W} & \\ {\scriptstyle\pi'}\swarrow & & \searrow{\scriptstyle\pi} \\ X' & & X. \end{array}$$

It is again the case that $\Gamma\backslash\mathcal{W}$ is Stein and the fibres of π, π' are contractible. The Penrose transform will be an isomorphism

$$H^0(X', L'_k) \to H^1(X, L_{k-2}).$$

In order to have $L'_k \to X'$ be a positive line bundle we must have $k = -1, -2, \dots$. Then

$$L_{k-2} = L_k \otimes \omega_X$$

where $L_k \to X$ is *negative* since $X = \Gamma\backslash\overline{\mathcal{H}}$.

We let $f(\tau') \in H^0(X', L'_k)$ be a modular form of weight $-k$, given by a holomorphic function on $\mathcal{H}$ satisfying the usual functional equation under the action of Γ. We then set

$$p(f)(\tau', \tau) = f(\tau')\left(\frac{\tau - \tau'}{2i}\right)^{k-2} d\tau'.$$

This is a relative differential for $\Gamma\backslash\mathcal{W} \to X$, and the transformation formula under $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ given by

$$\gamma^*\left(\left(\frac{\tau - \tau'}{2i}\right)^{k-2} d\tau'\right) = (c\tau + d)^{2-k}(c\tau' + d)^{-k}\left(\frac{\tau - \tau'}{2i}\right)^{k-2} d\tau'$$

shows that we obtain a class

$$p(f) \in H^1_{\mathrm{DR}}\Big(\Gamma\big(\Gamma\backslash\mathcal{W}, \Omega^\bullet_\pi(\pi^* L_{k-2})\big); d\pi\Big) \cong H^1(X, L_{k-2})$$

(apologies for the double appearance of Γ). It is a nice exercise to show that $p(f) \neq 0$, and since

$$\dim H^0(X', L'_k) = \dim H^1(X, L_{k-2})$$

we see that the resulting map $H^0(X', L'_k) \to H^1(X, L_{k-2})$ is an isomorphism.

Orbit structure for $\mathbb{P}^1$. The main groups we shall consider acting on $\mathbb{P}^1$ are

- $G_\mathbb{C} = \mathrm{SL}_2(\mathbb{C})$;
- $K = \mathrm{SO}(2)$ and its complexification $K_\mathbb{C}$;
- $G_\mathbb{R} = \mathrm{SL}_2(\mathbb{R})$ = real form of $G_\mathbb{C}$.

The compact real form $G_c = S\mathcal{U}(2)$ also acts on $\mathbb{P}^1$, but in these lectures we shall only make occasional use of it. The complex group $G_\mathbb{C}$ acts transitively on $\mathbb{P}^1$, but $K_\mathbb{C}$ and $G_\mathbb{R}$ do not act transitively and their orbit structure will be of interest. The central point is *Matsuki duality*, which is

the orbits of $K_\mathbb{C}$ and $G_\mathbb{R}$ are in a 1-1 correspondence.

We have already mentioned this in Lecture 1; here we formulate it in a manner that suggests the general statement. The correspondence is defined as follows: Let $z \in \mathbb{P}^1$ and $G_\mathbb{R} \cdot z, K_\mathbb{C} \cdot z$ the corresponding orbits. Then

$G_\mathbb{R} \cdot z$ and $K_\mathbb{C} \cdot z$ are dual exactly when their intersection consists of one closed K orbit.

The following table illustrates this duality.

	$G_{\mathbb{R}}$-orbits	$K_{\mathbb{C}}$-orbits	
open $G_{\mathbb{R}}$ orbits	$\left\{ \begin{array}{l} \mathcal{H} \\ \overline{\mathcal{H}} \end{array} \right.$	$\left. \begin{array}{r} i \\ -i \end{array} \right\}$	closed $K_{\mathbb{C}}$ orbits
closed $G_{\mathbb{R}}$ orbit	$\left\{ \mathbb{R} \cup \{0\} \right.$	$\left. \mathbb{P}^1 \backslash \{i, -i\} \right\}$	open $K_{\mathbb{C}}$ orbit

Description of the material in the later lectures and the appendices. We will now informally describe the content of the remaining lectures in this series. The overall objective is to present aspects of the relationship between Hodge theory and representation theory, especially those that may be described using complex geometry. One specific objective is to discuss and prove special cases of recent results of Carayol, and some extensions of his work, that open up new perspectives on this relationship and may have the possiblity to introduce new aspects into arithmetic automorphic representation theory, aspects that are thus far inaccessible by the traditional approaches through Shimura varieties. Whether or not this turns out to be successful, Carayol's work is a beautiful story in complex geometry.

Lecture 3 will introduce and illustrate the basic terms and concepts in Hodge theory. We emphasize that we will *not* take up the extensive and central topic of the *Hodge theory of algebraic varieties*.[6] Rather our emphasis is on the Hodge structures as objects of interest in their own right, especially as they relate to representation theory and complex geometry.

The basic symmetry groups of Hodge theory are the *Mumford-Tate groups*, and associated to them are basic objects of the related complex geometry, the *Mumford-Tate domains*, consisting of the set of polarized Hodge structures whose generic member has a given Mumford-Tate group G. In Lecture 4 we will describe which G's can occur as a Mumford-Tate group, and in how many ways this can happen. The fundamental concept here is a *Hodge representation*, consisting roughly of a character and a co-character. As homogeneous complex manifolds the corresponding Mumford-Tate domains depend only on the co-character. This lecture will explain and illustrate this.[7]

Lecture 5 is concerned with *discrete series* (DS) and $\mathfrak{n}$-*cohomology.* The central point is the realization of the DS's via complex geometry, specifically the L^2-cohomology of holomorphic line bundles over flag domains.[8] The latter may be realized, in multiple ways, as Mumford-Tate domains and this will be seen to be an important aspect in Carayol's work. The realization described above is largely the work of Schmid.[9] An important ingredient in this analysis is the description of the L^2-cohomology groups via Lie algebra cohomology, in this case what is termed $\mathfrak{n}$-*cohomology.* We will discuss these latter groups in some detail as they will play an important role in the material of the later lectures and the work of Carayol.

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[6] There are many excellent references for this subject. Three such ones are [**Vo1**], [**Vo2**] and [**PS**]. More recent sources are [**Cat1**], [**Cat2**], [**Ke2**], [**Ca**], and [**ET**].

[7] [**GGK1**] and [**Ro1**] are references for this.

[8] Another realization due to Atiyah and Schmid [**AS**], is via L^2 solutions to the Dirac equation on the associated Riemannian symmetric spaces. This realization has advantageous aspects, but since in these lectures our primary interest is in the complex geometric aspects of Hodge theory and representation theory we will not discuss it here.

[9] cf. [**Sch1**] and [**Sch2**] and the references cited therein.

The flag domains fall into two classes, the *classical* ones that fibre holomorphically or anti-holomorphically over an Hermitian symmetric domain, and the *non-classical* ones for which this not the case. For this work it is the non-classical ones that are of the primary interest.

Lectures 6 and 7 will take up the basic constructions and results in the geometry of homogeneous complex manifolds that will play a central role in the remaining lectures, as well as being a very interesting topic in their own right. The main point is that associated to a flag domain there are complex manifolds — including the dual flag variety $\check{D}$ — in which the group $G_{\mathbb{R}}$ acts, albeit non-transitively, and these capture aspects of the complex geometry that provide the basic tools for understanding the cohomology of homogeneous line bundles over flag domains. One of these, the *cycle spaces*, are classical and originated from Hodge theory and which have been the subject of extensive study over the years, culminating in the recent monograph [**FHW**]. The other tool, the *correspondence spaces*, are of more recent vintage [**GG1**] and in several ways may be the object that for this work best interpolates between flag domains and the various associated spaces. Their basic property of *universality* will be introduced and illustrated in these two lectures. We emphasize that even though the cycle and correspondence spaces may be defined for any flag domain D, it is the case when D is non-classical that the geometry is particularly rich.

Lectures 8 and 9 will introduce and study the *Penrose transforms*, which among other things allow one to relate cohomologies on different flag domains and on their quotients by arithmetic groups. The main specific results here are the analysis of Penrose transforms in the case when $G = \mathcal{U}(2,1)$ studied by Carayol in [**C1**], [**C2**], [**C3**] and when $G = \mathrm{Sp}(4)$, which is a new case that is discussed in [**GGK2**] and in [**Ke3**]. Using the Penrose transform to relate classical automorphic forms to non-classical automorphic cohomology, we discuss how the cup-products of the images of Penrose transform reach the automorphic cohomology groups associated to totally degenerate limits of discrete series (TDLDS), which are the central representation-theoretic objects of interest in these lectures. This result for $\mathcal{U}(2,1)$ is due to Carayol and for $\mathrm{Sp}(4)$ will appear in [**Ke3**].

In the last Lecture 10 we discuss some topics that were not covered earlier and some open issues that arise from the material in the lectures, together with some new results that have appeared since the lectures were given, and which are related to questions posed in the lectures. Among the topics covered is the study by Carayol of cuspidal automorphic cohomology expanded about boundary components in the Kato-Usui completion, or partial compactifications, of quotients of of Mumford-Tate domains by arithmetic groups in the case of $S\mathcal{U}(2,1)$. This seems to be a very interesting area for further work (cf. [**KP1**]).

Turning to the appendices, the appendix to Lecture 5 discusses the Borel-Weil-Bott (BWB) theorem, which is the basic result relating complex geometry and the finite dimensional representation theory of complex semi-simple Lie groups. We recall Kostant's $\mathfrak{n}$-cohomology interpretation of the BWB theorem, which through the use of the Hochschild-Serre spectral sequence and the decomposition of a general Harish-Chandra module into its K-types plays a central role in the analysis of the $\mathfrak{n}$-cohomology of those modules.

The rather lengthy appendix to Lecture 6 contains descriptions, with illustrative examples, of the $G_{\mathbb{R}}$-orbit structures of $\check{D}$ and $\mathcal{U}$, and of the $K_{\mathbb{C}}$-orbit structure of $\check{D}$ that is dual to the $G_{\mathbb{R}}$ one (cf. [**FHW**]). Included are computations of intrinsic Levi forms for both $\check{D}$ and $\mathcal{U}$. These are interesting in the case of $G_{\mathbb{R}}$-orbits in ∂D where D is a Mumford-Tate domain, since as discussed in the appendix to Lecture 10 these will relate to boundary components given by limiting mixed Hodge structures. This provides a further connection between Hodge theory and representation theory "at the boundary," a topic that we suspect may have significant further developments.

The main objective of the discussion of the $G_{\mathbb{R}}$-orbit structure of $\mathcal{U}$ is to give a proof of the fundamental result in [**BHH**] that there exist strongly plurisubharmonic exhaustion functions modulo $G_{\mathbb{R}}$ on $\mathcal{U}$. This result implies that for $\Gamma \subset G_{\mathbb{R}}$ discrete and co-compact the quotient $\Gamma\backslash\mathcal{U}$ is Stein, a result that is basic to the use of Penrose transforms to study automorphic cohomology relating those groups between the classical and non-classical cases. Along the way we identify the tangent, normal and CR-tangent spaces to $G_{\mathbb{R}}$-orbits in $\mathcal{U}$. This is done in [**FHW**], but for the computational purposes in the present work we have proceeded in a somewhat different way.

In the appendix to Lecture 7 we revisit the Borel-Weil-Bott theorem in the context of Penrose transforms. Specifically, the BWB theorem gives the various geometric realizations, indexed by the Weyl group W, of the same irreducible $G_{\mathbb{C}}$-module as cohomology groups over the flag variety $\check{D} = G_{\mathbb{C}}/B$. In this appendix we show how these abstract isomorphisms between cohomology groups may be realized geometrically by Penrose transforms. The analogue of this for flag domains, where now the Penrose transform is between Harish-Chandra modules with the same infinitesimal character realized as cohomology groups over flag domains given by open $G_{\mathbb{R}}$-orbits D and D' in $\check{D}$ and where the complex structures of D and D' may be inequivalent, is fundamental to Lectures 8 and 9. The point here is that the infinitesimal character is an invariant of W whereas the inequivalent homogeneous complex structures on $G_{\mathbb{R}}/T$ are indexed by W/W_K, so that that Penrose transform enables one to relate geometrically classical and non-classical objects.

There are three appendices to Lecture 9. In the first we give the K-types for the totally degenerate limits of discrete series in our two running examples $S\mathcal{U}(2,1)$ and $\mathrm{Sp}(4)$; this was used in the lecture where among other things the mechanism underlying the degeneration of the Hochschild-Serre spectral sequence was presented. It will also be used in the paper [**Ke3**] where the results of the lecture and the appendix will be used in the proof of the analogue for $\mathrm{Sp}(4)$ of Carayol's cup-product theorem. For $\mathrm{Sp}(4)$ this result is particularly subtle because it involves the interplay between the two inequivalent TDLDS's.

In the second appendix to Lecture 9 we have given an exposition of Schmid's proof of the degeneration of the Hochschild-Serre spectral sequences for the TDLDS's in the $S\mathcal{U}(2,1)$ and $\mathrm{Sp}(4)$ cases. This is a particular illustration of the use of Zuckerman tensoring and the Casselman-Osborne theorem in the computation of $\mathfrak{n}$-cohomology. The third appendix applies Zuckerman tensoring and the work of Schmid to obtain a general construction of TDLDS via Dolbeault cohomology of line bundles on nonclassical Mumford-Tate domains.

The lengthy appendix to Lecture 10 has two purposes. One is to combine the root-theoretic analysis of the $G_{\mathbb{R}}$-orbits in ∂D with the theory of limiting mixed Hodge structures (LMHS) to give an analogue of the realization of certain open $G_{\mathbb{R}}$-orbits in flag varieties as Mumford-Tate domains for polarized Hodge structures on

$(\mathfrak{g}, \mathbb{B})$. The main point here is the analysis of the period-type map

$$\left\{\begin{matrix}\text{Kato-Usui boundary} \\ \text{components}\end{matrix}\right\} \xrightarrow{\Phi_\infty} \{G_{\mathbb{R}}\text{-orbits in } \partial D\}\,.$$

In doing this we introduce and discuss mixed Hodge structures and limiting mixed Hodge structures, Kato-Usui boundary components (nilpotent orbits), and Kato-Usui extensions, all of which will be used in the proof of Carayol's result mentioned above. The underlying point is that much of the theory discussed in these lectures relating Hodge theory and representation theory should be extended "to the boundary." This would be an analogue of the well-known principle in algebraic geometry that Hodge theory is frequently simpler and more tractable when an algebraic variety degenerates to a singular one.

Of particular note here is the quite different behavior of the differential $\Phi_{\infty,*}$ between the classical and non-classical cases. In the former it only detects the associated graded to the limiting mixed Hodge structures, analogous to the Satake-Borel-Baily compactifications, whereas in the non-classical case some — but not all — of the extension data is captured by $\Phi_{\infty,*}$.

We would like to call attention to the papers [**KP1**] and [**KP2**], where among other things the Mumford-Tate groups associated to nilpotent orbits are defined and criteria given for when a Kato-Usui boundary component is "classical," thereby preparing the way to extend Carayol's result to other situations, including a case where the arithmetically interesting automorphic cohomology group is associated to an automorphic representation of G_2. Some of the discussion in the first part of this appendix (and part of the appendix to Lecture 6) overlaps with part of what is discussed in the above works.

A second purpose of the appendix is to discuss the proof of the result of Carayol [**C3**] where for $S\mathcal{U}(2,1)$ automorphic cohomology is expanded about a Kato-Usui boundary component. This gives an analogue of the expansion of classical automorphic forms about a cusp and suggests a possible definition of arithmeticity for automorphic cohomology in this case.