Introduction

In the past 25 years or so, there has been considerable interest in the study of nonlinear partial differential equations, modeling phenomena of wave propagation, coming from physics and engineering. The areas that gave rise to these equations are water waves, optics, lasers, ferromagnetism, general relativity, sigma models and many others. These equations also have connections to geometric flows and to Kähler and Minkowski geometries. Examples of such equations are the generalized KdV equations:

$$\begin{cases} \partial_t u + \partial_x^3 u + u^k \partial_x u = 0, x \in \mathbb{R}, t \in \mathbb{R} \\ u|_{t=0} = u_0, \end{cases}$$

the nonlinear Schrödinger equations:

$$\begin{cases} i\partial_t u + \Delta u \pm |u|^p u = 0, x \in \mathbb{R}^N, t \in \mathbb{R} \\ u|_{t=0} = u_0, \end{cases}$$

and the nonlinear wave equation:

$$\begin{cases} \partial_t^2 u - \Delta u = \pm|u|^p u, x \in \mathbb{R}^N, t \in \mathbb{R} \\ u|_{t=0} = u_0 \\ \partial_t u|_{t=0} = u_1. \end{cases}$$

Inspired by the theory of ODE one defines a notion of well-posedness for these initial value problems (IVP), with data u_0 $((u_0, u_1))$ in a given function space B. Since these equations are time reversible, the intervals of time to be considered are symmetric around the origin. Well-posedness entails existence, uniqueness of a solution which describes a continuous curve in the space B, for $t \in I$, the interval of existence, and continuous dependence of the curve on the initial data. If I is finite we call this local well-posedness (lwp); if I is the whole line, we call this global well-posedness (gwp). The first stage of development of the theory concentrated on the "local theory of the Cauchy problem", which established local well-posedness results on Sobolev spaces B, or global well-posedness for small data in B. Pioneering works were due to Segal, Strichartz, Kato, Ginibre-Velo, Pecher and many others. In the late 80's, in collaboration with Ponce and Vega we introduced the systematic use of the machinery of modern harmonic analysis to study the "local theory of the Cauchy problem". Further contributions came from work of Bourgain, Klainerman-Machedon, Tataru, Tao and many others.

In recent years, there has been a lot of interest in the study, for nonlinear dispersive equations, of the long-time behaviour of solutions, for large data. Issues like blow-up, global existence and scattering have come to the forefront, especially in critical problems. These problems are natural extensions of nonlinear elliptic problems, which were studied earlier. To explain this connection, recall that in the late 1970's and early 1980's, there was a great deal of interest in the study of semilinear elliptic equations, to a great degree motivated by geometric applications.

For instance, recall the Yamabe problem: Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Is there a conformal metric $\widetilde{g} = cg$, so that the scalar curvature of $(M, \widetilde{g})$ is constant?

In this context, the following equation was studied extensively: $\Delta u + |u|^{\frac{4}{n-2}} u = 0, x \in \mathbb{R}^n$ where $u \in \dot{H}^1(\mathbb{R}^n) = \left\{u : \nabla u \in L^2(\mathbb{R}^n)\right\}$. Using this information, Trudinger, Aubin and Schoen solved the Yamabe problem in the affirmative (see [**89**] and references therein). We will concentrate in the case $n = 3$, so that the equation becomes $\Delta u + u^5 = 0, x \in \mathbb{R}^3, u \in \dot{H}^1(\mathbb{R}^3)$. This equation is "critical" because the linear part (Δ) and the nonlinear part (u^5) have the same "strength", since if u is a solution, so is $\frac{1}{\lambda^{\frac{1}{2}}} u\left(\frac{x}{\lambda}\right)$ and both the linear part and the nonlinear part transform "in the same way" under this change. The equation is "focusing" because the linear part (Δ) and the nonlinear part (u^5) have opposite signs and hence they "fight each other". Note that for the much easier "defocusing" problem $\Delta u - u^5 = 0, u \in \dot{H}^1(\mathbb{R}^3)$, it is easy to see that there are no non-zero solutions. The difficulty in the study of $\Delta u + u^5 = 0$ in $\mathbb{R}^3$ comes from the "lack of compactness" in the Sobolev embedding $\|u\|_{L^6(\mathbb{R}^3)} \leq C_3 \|\nabla u\|_{L^2(\mathbb{R}^3)}$, where C_3 is the best constant. $\left(C_3 = \pi^{-\frac{1}{2}} 3^{-\frac{1}{2}} \left[\frac{\Gamma(3)}{\Gamma(\frac{3}{2})}\right]^{\frac{1}{3}}\right)$ (See [**100**]). Modulo translation and scaling, the only non-negative solution to $\Delta u + u^5 = 0, u \in \dot{H}^1(\mathbb{R}^3)$ is $W(x) = \left(1 + \frac{|x|^2}{3}\right)^{-\frac{1}{2}}$ (Gidas-Ni-Nirenberg [**42**], Kwong [**72**]). Also W is the unique minimizer to the Sobolev inequality above (Talenti [**100**]). W is called the "ground state". W is also the unique radial solution in $\dot{H}^1\left(\mathbb{R}^3\right)$, (without imposing a sign condition). (Pohozaev [**88**], Kwong [**72**]). On the other hand, Ding [**22**] constructed infinitely many variable sign solutions, which are non-radial. Pohozaev [**88**] also showed that the only solution to the boundary value problem

$$\begin{cases} \Delta u + u^5 = 0 \text{ in } B_1 ß \mathbb{R}^3 \\ \quad u|_{\partial B_1} \equiv 0 \end{cases}$$

is $u \equiv 0$. If instead, we consider the problem

$$\begin{cases} \quad \Delta u_\varepsilon + u_\varepsilon^5 = 0 \text{ in } B_1 \backslash B_\varepsilon \\ u_\varepsilon|_{\partial B_1 \cup \partial B_\varepsilon} = 0 \end{cases}$$

then there are non-zero solutions. If we normalize them and let $\varepsilon \to 0$, we have $u_\varepsilon \sim \sum_{j=1}^J (-1)^j \frac{W\left(\frac{x}{\lambda_j}\right)}{\lambda_j^{\frac{1}{2}}}$, where $0 \leq \lambda_1(\varepsilon) \ll \lambda_2(\varepsilon) \ll \cdots \ll \lambda_J(\varepsilon)$ (Musso-Pistoia [**84**], "towers of bubbles").

Through the study of these and related problems, in works of Talenti, Trudinger, Aubin, Schoen, Taubes, Schoen-Uhlenbeck, Sachs-Unlenbeck, Bahri-Coron, Struwe, Brézis-Coron, etc., many important techniques were developed. In particular, through these and others works, the study of the "defect of compactness" and the "bubble decomposition" were systematized through the work of P-L. Lions on concentration-compactness.

For nonlinear dispersive equations there are also critical problems, which are related to $\Delta u + u^5 = 0$.

In this monograph we will concentrate on the energy critical nonlinear wave equation in $3d$,

$$(\mathrm{NLW}_\pm) \qquad \begin{cases} \partial_t^2 u - \Delta u = \mp u^5, x \in \mathbb{R}^3, t \in \mathbb{R} \\ u|_{t=0} = u_0 \in \dot{H}^1(\mathbb{R}^3) \\ \partial_t u|_{t=0} = u_1 \in L^2\left(\mathbb{R}^3\right). \end{cases}$$

The hope is that the results obtained for $(\mathrm{NLW}_\pm)$ will be a model for what to strive for in other critical dispersive problems. In (NLW_+) we have the "defocusing case", while in (NLW_-) we have the "focusing case". $(\mathrm{NLW}_\pm)$ are "energy critical" because if u is a solution, so is $\frac{1}{\lambda^{\frac{1}{2}}} u\left(\frac{x}{\lambda}, \frac{t}{\lambda}\right)$, and the scaling leaves invariant the norm of the Cauchy data in $\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. Both problems have energies that are constant in time

$$E_\pm\left(u(t), \partial_t u(t)\right) = \frac{1}{2}\int |\nabla u(t)|^2 + (\partial_t u(t))^2 \pm \frac{1}{6}\int u^6(t).$$

where $+$ on the right hand side, corresponds to the defocusing case and $-$ on the right hand side corresponds to the focusing case.

We now summarize the "local theory of the Cauchy problem", for equations $(\mathrm{NLW}_\pm)$. This is described in detail in Chapter 1.

If $\|(u_0, u_1)\|_{\dot{H}^1 \times L^2}$ is small, $\exists!$ solution u, defined for all time, such that $u \in C\left((-\infty, +\infty); \dot{H}^1 \times L^2\right) \cap L^8_{xt}$, which scatters, i.e.,

$$\left\|(u(t), \partial_t u(t)) - S(t)\left(u_0^\pm, u_1^\pm\right)\right\|_{\dot{H}^1 \times L^2} \overset{t \to \pm\infty}{\longrightarrow} 0,$$

for some $\left(u_0^\pm, u_1^\pm\right) \in \dot{H}^1 \times L^2$.

Moreover, for any data $(u_0, u_1) \in \dot{H}^1 \times L^2$, we have short time existence and hence there exists a maximal interval of existence $I = (T_-(u), T_+(u))$.

Here, $S(t)(u_0, u_1)$ is the solution of the linear wave equation $\partial_t^2 - \Delta$, with initial Cauchy data (u_0, u_1). Also, the meaning of, say, $T_+(u) < \infty$, is that if $\{t_n\}$ is a sequence of times converging to $T_+(u), (u(t_n), \partial_t u(t_n))$ has no convergent subsequence in $\dot{H}^1 \times L^2$.

Question: What about large data?

We first turn to the defocusing case, which was studied in works of Struwe [**97**], Grillakis [**46**], [**47**], Shatah-Struwe [**92**],[**93**], Kapitanski [**43**], Bahouri-Shatah [**5**] (80's and 90's). They established:

(+) <u>Global regularity and well-posedness conjecture</u>

(For critical defocusing problems): There is global in time well-posedness and scattering for arbitrary data in $\dot{H}^1 \times L^2$. Moreover more regular data keep this regularity for all time. This closes the study of the dynamics in (NLW_+).

For the focusing problem, (+) fails. In fact, H. Levine (1974) [**73**] showed that if $(u_0, u_1) \in H^1 \times L^2$, $E(u_0, u_1) \leq 0, (u_0, u_1) \neq (0, 0)$, ($\dot{H}^1 \times L^2$ in the radial case), then $|T_\pm(u_0, u_1)| < \infty$.

Levine's proof is of the "obstruction" type. He shows that there is an obstruction for the global existence, but does not give information on the nature of the "blow-up".

Moreover, $u(x,t) = \left(\frac{3}{4}\right)^{\frac{1}{4}} (1-t)^{-\frac{1}{2}}$ is a solution. It is not in $\dot{H}^1 \times L^2$, but we can truncate it and use finite speed of propagation to find data in $\dot{H}^1 \times L^2$

such that $\lim_{t\uparrow 1}\|(u(t),\partial_t u(t))\|_{\dot H^1\times L^2}=\infty$. (Type I blow-up, or ODE blow-up). Also, W, which solves $\Delta W+W^5=0$ and is independent of time, is a global in time solution, which does not scatter. (If a solution u scatters, $\int_{|x|\leq 1}|\nabla u(x,t)|^2dx \overset{t\to\infty}{\longrightarrow} 0$. This clearly fails for W). Moreover, Krieger-Schlag-Tataru [**71**], Krieger-Schlag [**69**] have constructed type II blow-up solutions, i.e., solutions with $T_+<\infty$, and $\sup_{0<t<T_+}\|(u(t),\partial_t u(t))\|_{\dot H^1\times L^2}<\infty$, which are radial. Also, Donninger-Krieger [**29**] have constructed radial, global in time solutions, bounded in $\dot H^1\times L^2$, which do not scatter to either a linear solution or to W.

In the rest of this monograph, we will study the focusing case and call it (NLW) and its energy E. Here, the analog of (+) is

(++) <u>Ground State Conjecture</u>

(For critical focusing problems): There exists a "ground state", whose energy is a threshold for global existence and scattering.

In 2006-09, Frank Merle and I developed a program to attack critical dispersive problems and establish (+) and, for the first time (++) in focusing problems. We call this the "concentration-compactness/rigidity theorem method", which was partly inspired by the earlier elliptic problems. The method gives a "road map" to attack both (+) and for the first time (++). The "road map" has already found an enormous range of applicability, to previously intractable problems, in work of many researchers. See for instance [**64**], [**65**], [**66**], [**67**], [**23**], [**24**], [**25**],[**26**],[**27**],[**28**],[**6**], [**70**] and many others.

The result of Kenig-Merle [**62**], establishing the ground state conjecture (++) for (NLW), using the "concentration-compactness/rigidity theorem method" is:

THEOREM 2.6. *If* $E(u_0,u_1)<E(W,0)$ *then*

i) *If* $\|\nabla u_0\|<\|\nabla W\|$, *global existence, scattering.*
ii) *If* $\|\nabla u_0\|>\|\nabla W\|, T_+, |T_-|<\infty$.
iii) *The case* $\|\nabla u_0\|=\|\nabla W\|$ *is impossible.*

The "concentration-compactness/rigidity theorem" method, as well as the proof of Theorem 2.6 above are discussed in detail in Chapters 2 and 3 of this monograph. In Chapters 4 and 5 of this monograph, we study solutions of (NLW) with the "compactness property", an important class of "non-dispersive" solutions. In proving Theorem 2.6, a rigidity theorem for solutions with the compactness property and further size restrictions is crucial. In Chapter 4 we study solutions with the compactness property and no further size restriction. The main results are collected in Theorem 4.77. The main conjecture here is the "rigidity conjecture" for solutions with the compactness property, namely that they are all solitary waves, i.e., Lorentz transforms of stationary solutions (stationary solutions solve the elliptic equation $\Delta Q+Q^5=0$). This conjecture was established in [**32**] in the radial case, and in [**35**] under a non-degeneracy assumption. These results are dealt with in Chapter 5. They comprise Theorem 4.17, Theorem 4.18 and Theorem 5.6. In Chapter 4 we also give an extension (Theorem 4.4) of i) in Theorem 2.6, which was proved in [**31**] and uses the rigidity results of Chapter 5.

In Chapter 6, we begin the systematic study of type II blow-up solutions and more generally, of extended type II solutions, namely non-zero solutions u for which $sup_{0<t<T_+(u)}\|(u(t),\partial_t u(t))\|_{\dot H^1\times L^2}<\infty$.

As was mentioned earlier there are examples of type II solutions, constructed in [**69**] and [**71**] and of generalized type II solutions, such as W, or more generally the solitary waves, i.e., Lorentz transforms $Q_{\vec{l}}$ of non-zero solutions to $\Delta Q + Q^5 = 0$ (this class of Q is denoted by Σ), as well as constructions in [**29**] with complicated asymptotics. In [**36**], Duyckaerts-Kenig-Merle developed a very general compactness argument, which connects extended type II solutions and solutions with the compactness property (see Theorem 6.12). In the case of NLW, a more detailed result is possible, Theorem 6.13, obtained in [**36**]. A simple version of this is:

THEOREM 6.14. *Let u be an extended type (II) solution of* (NLW). *Then there exists $Q \in \Sigma, \vec{l} \in \mathbb{R}^3, \left|\vec{l}\right| < 1$ and sequences $\{t_n\} \in [0, T_+(u)), \{x_n\} \in \mathbb{R}^3, \{\lambda_n\} \in \mathbb{R}^+, t_n \to T_+(u)$ and such that*

$$\left(\lambda_n^{\frac{1}{2}} u\left(x_n + \lambda_n x, t_n\right), \lambda_n^{\frac{3}{2}} \partial_t u\left(x_n + \lambda_n x, t_n\right)\right) \overset{n}{\rightharpoonup} \left(Q_{\vec{l}}(0), \partial_t Q_{\vec{l}}(0)\right),$$

where the convergence is weak in $\dot{H}^1 \times L^2$.

$$\left(Q_{\vec{l}}(x,t) = Q\left(x + \left[\frac{1}{l^2}\left(\frac{1}{\sqrt{1-l^2}} - 1\right)\vec{l} \cdot x - \frac{t}{\sqrt{1-t^2}}\right]\vec{l}\right), l = \left|\vec{l}\right|\right).$$

Theorem 6.14 is proved in Chapter 6. We regard the result in Theorem 6.13 of Chapter 6 as a first step towards the proof of a full decomposition, for u an extended type II solution,

$$\begin{aligned} u(x,t_n) &= \sum_{j=1}^{J} \frac{1}{\lambda_{j,n}^{\frac{1}{2}}} Q_{\vec{l}_j}^j\left(\frac{x - x_{j,n}}{\lambda_{j,n}}, 0\right) + v\left(x, t_n\right) + o_n(1) \\ \partial_t u(x,t_n) &= \sum_{j=1}^{J} \frac{1}{\lambda_{j,n}^{\frac{3}{2}}} \partial_t Q_{\vec{l}_j}^j\left(\frac{x - x_{j,n}}{\lambda_{j,n}}, 0\right) + \partial_t v\left(x, t_n\right) + o_n(1), \end{aligned}$$

where v is a radiation term (a solution of the linear equation), $o_n(1)$ goes to 0 in $\dot{H}^1 \times L^2, \left|\vec{l_j}\right| < 1, \vec{l_j} \in \mathbb{R}^3, Q^j \in \Sigma, \{x_{j,n}\} \in \mathbb{R}^3, \{\lambda_{j,n}\} > 0$, and $t_n \to T_+(u)$, which is a special case of the soliton resolution conjecture for (NLW), which says that any extended type II solution u of (NLW) can be written, as $t \to T_+(u)$ as a sum of decoupled solitary waves and a radiation term, plus a term that goes to 0 in $\dot{H}^1 \times L^2$. The proof of this conjecture is our final goal in this direction. In Chapter 7, we discuss "channels of energy" and "outer energy lower bounds" for the linear wave equation, see Corollary 7.6 and Proposition 7.9, which were discovered in [**31**], [**32**], and which are our main tools in passing from weak convergence results, of the type in Theorem 6.13 and Theorem 6.14, to strong convergence results. The first such result is discussed in Chapter 8. It shows the universality of the construction in [**69**], [**71**], as well as establishing a case of the soliton resolution conjecture discussed earlier, when $T_+(u) < \infty$, under an additional smallness hypothesis. This result is from [**31**] and [**32**].

THEOREM 8.1. *Assume that u is a type II solution, $T_+ = 1$.*

i) *If u is radial and*

$$\sup_{0<t<1} \|\nabla u(t)\| \leq \|\nabla W\| + \eta_0,$$

$\eta_0 > 0$, small. Then $\exists (v_0, v_1) \in \dot{H}^1 \times L^2, \lambda(t) > 0, t \in (0,1), i_0 \in \{\pm 1\}$ such that $(u(t), \partial_t u(t)) = \left(\frac{i_0}{\lambda(t)^{\frac{1}{2}}} W\left(\frac{x}{\lambda(t)}\right), 0\right) + (v_0, v_1) + o(1)$ in $\dot{H}^1 \times L^2$, where $\lambda(t) = o(1-t)$.

ii) *Non-radial case. Assume that*

$$\sup_{0<t<1} \|\nabla u(t)\|^2 + \frac{1}{2}\|\partial_t u(t)\|^2 \leq \|\nabla W\|^2 + \eta_0,$$

η_0 small. Then, after rotation, translation of $\mathbb{R}^3$, $\exists (v_0, v_1) \in \dot{H}^1 \times L^2$, $i_0 \in \{\pm 1\}$, l small, $x(t) \in \mathbb{R}^3, \lambda(t) > 0, t \in (0,1)$ such that $(u(t), \partial_t v(t)) = \left(\frac{i_0}{\lambda(t)^{\frac{1}{2}}} W_l\left(\frac{x - x(t)}{\lambda(t)}, 0\right), \frac{i_0}{\lambda(t)^{\frac{3}{2}}} \partial_t W_l\left(\frac{x - x(t)}{\lambda(t)}, 0\right)\right) + (v_0, v_1) + o(1)$ in $\dot{H}^1 \times L^2$, where $\lambda(t) = o(1-t), \lim_{t\uparrow 1} \frac{x(t)}{1-t} = l\vec{e_1}, \vec{e_1} = (1,0,0), |l| \leq C\eta_0^{\frac{1}{4}}$, and $W_l(x,t) = W\left(\frac{x_1 - tl}{\sqrt{1-l^2}}, x_2, x_3\right)$ is the Lorentz transform of W.

The final three chapters of the monograph are devoted to the proof of the soliton resolution conjecture for (NLW) in the radial case, by Duyckaerts-Kenig-Merle. This was obtained, for a particular sequence of times, in [**30**] and in [**33**] for general times. The result is:

THEOREM 9.1. *Let u be a radial solution of* (NLW)*. Then, one of the following holds:*

a) *Type I blow-up: $T_+ < \infty$ and*

$$\lim_{t\uparrow T_+} \|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} = \infty.$$

b) *Type II blow-up: $T_+ < \infty$ and $\exists (v_0, v_1) \in \dot{H}^1 \times L^2, J \in \mathbb{N}\setminus\{0\}$ and $\forall j \in \{1, \dots, J\}, i_j \in \{\pm 1\}$ and $\lambda_j(t) > 0$ such that $0 < \lambda_1(t) \ll \lambda_2(t) \ll \dots \ll \lambda_J(t) \ll T_+ - t$, and $(u(t), \partial_t u(t)) = \left(\sum_{j=1}^J \frac{i_j}{\lambda_j(t)^{\frac{1}{2}}} W\left(\frac{x}{\lambda_j(t)}\right), 0\right) + (v_0, v_1) + o(1)$ in $\dot{H}^1 \times L^2$.*

c) *$T_+ = \infty$ and $\exists$ a solution v_L of* (LW)*, $J \in \mathbb{N}$ and for all $j \in \{1, \dots, J\}$, $i_J \in \{\pm 1\}, \lambda_j(t) > 0$ such that $0 < \lambda_1(t) \ll \lambda_2(t) \ll \dots \ll \lambda_J(t) \ll t$, and $(u(t), \partial_t u(t)) = \left(\sum_{j=1}^J \frac{i_j}{\lambda_j(t)^{\frac{1}{2}}} W\left(\frac{x}{\lambda_j(t)}\right), 0\right) + (v_L(t), \partial_t v_L(t)) + o(1)$ in $\dot{H}^1 \times L^2$.*

Here $a(t) \ll b(t)$ means that $\frac{a(t)}{b(t)} \to 0$, and (LW) *denotes the linear wave equation.*

A fundamental new ingredient of the proof of Theorem 9.1 is the following dispersive property that all global in time radial solutions to (NLW) (other than $0, \pm W$ up to scaling) must have:

$$\int_{|x|>R+|t|} |\nabla_{x,t} u(x,t)|^2 dx \geq \eta, \text{ for some } R > 0, \eta > 0 \text{ and all } t \geq 0 \text{ or all } t \leq 0.$$

This is in Proposition 9.17. The proof is a consequence of the "channel of energy" property in Chapter 7.

As far as exposition goes, most of the results mentioned above are proved in full, for others the proofs are merely sketched or ommitted completely. In the last two cases, appropriate references are given.

Warning: We generally use $\| \;\|$ and $\| \;\|_{L^2}$ interchangeably, and the same goes for expressions like "orthogonality of parameters" and "pseudo-orthogonality of parameters".

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