CHAPTER 2

The "Road Map": The Concentration Compactness/Rigidity Theorem Method for Critical Problems I

In this chapter and the next, we will describe the concentration/compactness rigidity theorem method introduced by Kenig-Merle [**61**],[**62**] in order to study global well-posedness and scattering in critical problems. We will do so in the context of the focusing, energy-critical non-linear wave equation. This method is designed to address the large data/large time situation left out from the "local theory of the Cauchy problem" discussed in Chapter 1. The proofs presented in this chapter are from [**61**], [**62**], [**59**], [**60**]. See also the surveys [**50**], [**51**], [**52**], [**54**]. We first discuss briefly the defocusing case.

$$
(\mathrm{NLW}_+) \qquad \begin{cases} \partial_t^2 u - \Delta u = -u^5, x \in \mathbb{R}^3, t \in \mathbb{R} \\ u|_{t=0} = u_0 \in \dot{H}^1(\mathbb{R}^3) \\ \partial_t u|_{t=0} = u_1 \in L^2\left(\mathbb{R}^3\right). \end{cases}
$$

In the focusing case of (NLW), the energy is

$$
E(u_0, u_1) = \frac{1}{2}\int |\nabla u_0|^2 + (u_1)^2 - \frac{1}{6}\int u_0^6. \tag{2.1}
$$

From the identity

$$
\partial_t e(u)(x,t) = \sum_{j=1}^{3} \partial_{x_j}\left(\partial_{x_j} u(x,t)\cdot \partial_t u(x,t)\right), \tag{2.2}
$$

with $e(u)(x,t) = \frac{1}{2}\left(\partial_t u\right)^2(x,t) + \frac{1}{2}\left|\nabla u\right|^2(x,t) - \frac{1}{6}u^6(x,t)$, for smooth solutions of (NLW) and Remark 1.15, we see that, if u is a solution of (NLW), $t \in I_{\max}(u)$,

$$
E\left(u(t), \partial_t u(t)\right) = E(u_0, u_1). \tag{2.3}
$$

For the defocusing (NLW_+), similar considerations, with

$$
E_+(u_0, u_1) = \frac{1}{2}\int |\nabla u_0|^2 + u_1^2 + \frac{1}{6}\int u_0^6,
$$

lead to the "a priori" bound

$$
\sup_{t \in I_{\max}(u)} \frac{1}{2}\int |\nabla u(t)|^2 + \left(\partial_t u(t)\right)^2 \le C\left\|(u_0, u_1)\right\|^2_{\dot{H}^1 \times L^2}.
$$

The defocusing case was studied in the 80's and early 90's, in a series of works by Struwe, Grillakis, Shatah-Struwe, Kapitanski, Bahouri-Shatah ([**97**],[**46**],[**47**],[**92**], [**93**],[**48**],[**5**]) who established:

(2.4) <u>Global Regularity and Well-Posedness Conjecture</u>

(For critical defocusing problems): There is global in time well-posedness and scattering for arbitrary data in $\dot{H}^1 \times L^2$. Moreover more regular data keep this regularity for all time. (This closes the study of the dynamics for defocusing (NLW_+)).

For the focusing problem (2.4) fails. In fact, H. Levine ([**73**]) showed that if $(u_0, u_1) \in H^1 \times L^2$, $E(u_0, u_1) \leq 0, (u_0, u_1) \neq (0, 0)$. ($\dot{H}^1 \times L^2$ in the radial case), then $|T_\pm(u_0, u_1)| < \infty$. This is done by an indirect argument ("an obstruction argument") that does not explicitly analyze the singularity formation.

Moreover, $u(x,t) = \left(\frac{3}{4}\right)^{\frac{1}{4}} (1-t)^{-\frac{1}{2}}$ is a solution. It is not in $\dot{H}^1 \times L^2$, but we can truncate it and use finite speed of propagation to find data in $\dot{H}^1 \times L^2$ such that $\lim_{t\uparrow 1} \|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} = \infty$. (Type I blow-up, or ODE blow-up). Also, W, which solves $\Delta W + W^5 = 0$ and is independent of time, is a global in time solution, which does not scatter. (If a solution u scatters, $\int_{|x|\leq 1} |\nabla u(x,t)|^2 dx \overset{t\to\infty}{\longrightarrow} 0$. This clearly fails for W). Moreover, Krieger-Schlag-Tataru ([**71**]), Krieger-Schlag ([**70**]) have constructed type II blow-up solutions, i.e., solutions with $T_+ < \infty$, and $\sup_{0<t<T_+} \|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} < \infty$, which are radial. More on these solutions later on. Also, Donninger-Krieger ([**29**]) have constructed radial, global in time solutions, bounded in $\dot{H}^1 \times L^2$, which do not scatter to either a linear solution or to W.

In the rest of this monograph we will try to understand the focusing case. Here the analog of (2.4) is

(2.5) <u>Ground State Conjecture</u>

(For critical focusing problems): There exists a "ground state", whose energy is a threshold for global existence and scattering.

In 2006-09, Frank Merle and I developed a program to attack critical dispersive problems and establish (2.4) and, for the first time (2.5) in focusing problems. We call this the "concentration-compactness/rigidity theorem method", which was partly inspired by the earlier elliptic problems. The method gives a "road map" to attack both (2.4) and for the first time (2.5). The "road map" has already found an enormous range of applicability, to previously intractable problems, in work of many researchers. I will now describe the results on (NLW) in the last few years, which we are going to be discussing in these two chapters, starting with the proof of (2.5) for (NLW), via the "road map".

THEOREM 2.6 ([**62**]). *If* $E(u_0, u_1) < E(W, 0)$ *then*

i) *If* $\|\nabla u_0\| < \|\nabla W\|$, *global existence, scattering.*
ii) *If* $\|\nabla u_0\| > \|\nabla W\|, T_+, |T_-| < \infty$.
iii) *The case* $\|\nabla u_0\| = \|\nabla W\|$ *is impossible.*

(2.7) <u>The road map: A quick summary</u>

We next describe, in a schematic way, the "road map" for the concentration-compactness/rigidity theorem method.

a) *Variational arguments* (Only needed in focusing problems). These are "static" arguments, which exploit the variational characterization of the ground state W. In our case, it is the extremal in the Sobolev embedding $\|u\|_{L^6(\mathbb{R}^3)} \leq C_3 \|\nabla u\|_{L^2(\mathbb{R}^3)}$. Combining these variational arguments with preservation of the energy and continuity of the flow, yields: if $E(u_0, u_1) < E(W, 0)$, $\|\nabla u_0\| < \|\nabla W\|$, then, for

$t \in I = (T_-, T_+)$, $E(u(t), \partial_t u(t)) \simeq \|(u(t), \partial_t u(t))\|^2_{\dot{H}^1 \times L^2} \simeq \|(u_0, u_1)\|^2_{\dot{H}^1 \times L^2}$, so that $\sup_{t \in I} \|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} < \infty$. Because of the Krieger-Schlag-Tataru [**71**] example, this does not suffice.

b) *Concentration-compactness procedure.* Since in our situation, by a) $E(u(t), \partial_t u(t)) \simeq \|(u_0, u_1)\|^2_{\dot{H}^1 \times L^2}$, if $E(u_0, u_1)$ is small, by the "local theory of Cauchy Problem", we have global existence and scattering. Hence, there is a critical level of energy E_c, with

$$0 \leq \delta_1 \leq E_c \leq E(W, 0)$$

such that if $E(u_0, u_1) < E_c, \|\nabla u_0\| < \|\nabla W\|$, we have global existence and scattering, and E_c is optimal with this property. i) in our Theorem is the statement $E_c = E(W, 0)$. If $E_c < E(W, 0)$ we will reach a contradiction by proving:

Proposition A (Existence of critical elements). $\exists (u_{0,c}, u_{1,c})$ *with* $E(u_{0,c}, u_{1,c}) = E_c, \|\nabla u_{0,c}\| < \|\nabla W\|$, *such that either I is finite or if I is infinite, u_c does not scatter. u_c is called a "critical element".*

Proposition B (Compactness of critical elements). *There exists* $\lambda(t) \in \mathbb{R}^+$, $x(t) \in \mathbb{R}^3, t \in I_+ = I$ß$[0, \infty)$ *such that*

$$K = \left\{ \left(\frac{1}{\lambda(t)^{\frac{1}{2}}} u_c \left(\frac{x - x(t)}{\lambda(t)}, t \right), \frac{1}{\lambda(t)^{\frac{3}{2}}} \partial_t u_c \left(\frac{x - x(t)}{\lambda(t)}, t \right) \right) : t \in I_+ \right\}$$

has compact closure in $\dot{H}^1 \times L^2$. (non-dispersive property of u_c, "minimality"). (Or corresponding proposition for I_-).

c) *Rigidity Theorem.* If $\overline{K}$, corresponding to a solution u is compact, and $E(u_0, u_1) < E(W, 0)$, $\|\nabla u_0\| < \|\nabla W\|$, then $(u_0, u_1) = (0, 0)$.

This gives a contradiction, since $E(u_{0,c}, u_{1,c})) = E_c \geq \delta_1 > 0$.

We now proceed to the proof of Theorem 2.6, using the "road map". The first part of the proof is

a) *Variational Estimates*: Recall that $W(x) = \left(1 + \frac{|x|^2}{3}\right)^{-\frac{1}{2}}$ is a stationary solution which solves the elliptic equation $\Delta W + W^5 = 0, W \geq 0$ and is radially decreasing. By the obvious invariances of the elliptic equation $W_{\lambda_0, x_0}(x) = \lambda_0^{\frac{1}{2}} W(\lambda_0 (x - x_0))$ is still a solution. Aubin and Talenti ([**3**],[**100**]) gave the following variational characterization of W: let C_3 be the best constant in the Sobolev embedding $\|u\|_{L^6} \leq C_3 \|\nabla u\|_{L^2}$, $C_3 = (3\pi)^{-\frac{1}{2}} \left(\frac{\Gamma(3)}{\Gamma(\frac{3}{2})}\right)^{\frac{1}{2}}$. Then, if u is real valued, $\|u\|_{L^6} = C_3 \|\nabla u\|_{L^2}, u \not\equiv 0$, we have $u = W_{\lambda_0, x_0}$. Note that by the elliptic equation $\int |\nabla W|^2 = \int W^6$. Also, $C_3 \|\nabla W\|_{L^2} = \|W\|_{L^6}$, so that $C_3^2 \|\nabla W\|^2_{L^2} = \left(\int |\nabla W|^2\right)^{\frac{1}{3}}$. Hence, $\int |\nabla W|^2 = \frac{1}{C_3^3}$. Moreover, $E((W, 0)) = \left(\frac{1}{2} - \frac{1}{6}\right) \int |\nabla W|^2 = \frac{1}{3C_3^3}$.

Lemma 2.8. *Assume that* $\|\nabla v\| < \|\nabla W\|, E(v, 0) \leq (1 - \delta_0) E(W, 0), \delta_0 > 0$. *Then,* $\exists \overline{\delta} = \overline{\delta}(\delta_0)$ *such that*

i) $\|\nabla v\|^2 \leq (1 - \overline{\delta}) \|\nabla W\|^2$

ii) $\int |\nabla v|^2 - |v|^6 \geq \overline{\delta} \int |\nabla v|^2$

Proof. Let $f(y) = \frac{1}{2} y - \frac{C_3^6}{6} y^3$. Note that if $\overline{y} = \|v\|^2, f(\overline{y}) \leq E(v, 0)$. Note that $f(y) = 0 \Leftrightarrow y = 0$ or $y = y^* = \frac{\sqrt{3}}{C_3^3} = \sqrt{3} \int |\nabla W|^2$ (for $y \geq 0$), so that

$f(y) > 0, 0 < y < y^*$. Also, $f'(y) = 0, y > 0 \Leftrightarrow y = y_c = \frac{1}{C_3^3} = \|\nabla W\|^2$. Also, $f(y_c) = \frac{1}{3C_3^3} = E(W,0)$, and $f''(y_c) \neq 0$. Since $0 \le \overline{y} < y_c, f(\overline{y}) \le (1-\delta_0)f(y_c)$ and f is non-negative, strictly increasing in $0 \le y < y_c$, we obtain $\overline{y} \le (1-\overline{\delta})y_c = (1-\overline{\delta})\|\nabla W\|^2$, that is i).

For ii), note that

$$\int |\nabla v|^2 - v^6 \ge \int |\nabla v|^2 - C_3^6 \left(\int |\nabla v|^2\right)^3 = \int |\nabla v|^2 \left[1 - C_3^6 \left(\int |\nabla v|^2\right)^2\right]$$

$$\ge \int |\nabla v|^2 \left[1 - C_3^6 (1-\overline{\delta})^2 \left(\int |\nabla W|^2\right)^2\right] = \int |\nabla v|^2 \left[1 - (1-\overline{\delta})^2\right],$$

which gives ii). □

Corollary 2.9. *If* $\|\nabla v\|^2 \le \sqrt{3}\|\nabla W\|^2, E(v,0) \ge 0$.

(Follows from the proof above).

Lemma 2.10. *If* $\|\nabla v\| \le \|\nabla W\|, E(v,0) \le E(W,0) \Rightarrow \|\nabla v\|^2 \le \frac{\|\nabla W\|^2}{E(W,0)} E(v,0)$ $= 3E(v,0)$.

Proof. Let f be as in previous lemma. Note that f is concave on $\mathbb{R}^+$, $f(0) = 0, f(\|\nabla W\|^2) = E(W,0), f(\|\nabla v\|^2) \le E(v,0)$. For $s \in (0,1)$, $f\left(s\|\nabla W\|^2\right) \ge sf\left(\|\nabla W\|^2\right) = sE(W,0)$. Choose $s = \frac{\|\nabla v\|^2}{\|\nabla W\|^2}$. □

Corollary 2.11. $E(v,0) < E(W,0), \|\nabla v\| = \|\nabla W\|^2$ *is impossible.*

Corollary 2.12 (Energy trapping). $(u_0,u_1) \in \dot{H}^1 \times L^2, E(u_0,u_1) < (1-\delta_0)E(W,0)$,
$\|\nabla u_0\| < \|\nabla W\|$. *Then, if* u *is the solution with maximal interval* I, $\exists \overline{\delta} = \overline{\delta}(\delta_0)$ *such that* $\forall t \in I, \|\nabla u(t)\| \le (1-\overline{\delta})\|\nabla W\|, \int |\nabla u(t)|^2 - u^6(t) \ge \overline{\delta} \int |\nabla u(t)|^2, E(u(t),0) \ge 0, E(u(t), \partial_t u(t)) \simeq \|(u(t), \partial_t u(t))\|^2_{\dot{H}^1 \times L^2} \simeq \|(u_0,u_1)\|^2_{\dot{H}^1 \times L^2}$, *with comparability constants depending only on* δ_0.

Remark 2.13. If $E(u_0,u_1) \le (1-\delta_0)E(W,0), \|\nabla u_0\|^2 > \|\nabla W\|^2$, then, for $t \in I, \|\nabla u(t)\|^2 \ge (1+\overline{\delta})\|\nabla W\|^2$. This follows as in Lemma 2.8 i).

Let us now turn to the proof of Theorem 2.6, ii), having already dealt with iii). We will do it in the case $u_0 \in L^2$. This additional assumption can be eliminated easily using finite speed of propagation. (See [**62**]). The argument comes from [**73**].

Thus, assume $u_0 \in L^2, E(u_0,u_1) < (1-\delta_0)E(W,0), \|\nabla u_0\| > \|\nabla W\|$. We want to show $T_+ < \infty$. Assume not. By Remark 2.13, $\int |\nabla u(t)|^2 \ge (1+\overline{\delta}) \int |\nabla W|^2$, $t \in I, E(W,0) \ge E(u(t), \partial_t u(t)) + \widetilde{\delta}, t \in I$, so that $\frac{1}{6}\int u(t)^6 \ge \frac{1}{2}\int (\partial_t u(t))^2 + \frac{1}{2}\int |\nabla u(t)|^2 - E(W,0) + \widetilde{\delta}$ and so $\int u(t)^6 \ge 3\int (\partial_t u(t))^2 + 3\int |\nabla u(t)|^2 - 6E(W,0) +$

$6\widetilde{\delta}$. Let $y(t) = \int u^2(t), y'(t) = 2\int u(t)\partial_t u(t)$. A simple calculation using the equation, integration by parts, gives $y''(t) = 2\int \left[\partial_t u(t)^2 + u(t)^6 - |\nabla u(t)|^2\right]$. Thus,

$$\begin{aligned} y''(t) &\geq 2\int (\partial_t u(t))^2 + 6\int (\partial_t u(t))^2 + 4\int |\nabla u(t)|^2 - 12E(W,0) + \widetilde{\widetilde{\delta}} \\ &= 8\int (\partial_t u(t))^2 + 4\int |\nabla u(t)|^2 - 4\int |\nabla W|^2 + \widetilde{\widetilde{\delta}} \\ &\geq 8\int (\partial_t u(t))^2 + \widetilde{\widetilde{\delta}}. \end{aligned}$$

Since $I \cap [0,\infty) = [0,\infty), \exists t_0 > 0$ such that $y'(t_0) > 0, y'(t) > 0, t > t_0$. For $t > t_0$, $y(t)y''(t) \geq 8\left(\int \partial_t u(t)^2\right)\left(\int u^2(t)\right) \geq 2y'(t)^2$, so that $\frac{y''(t)}{y'(t)} \geq 2\frac{y'(t)}{y(t)}$ or $y'(t) \geq C_0 y(t)^2$ for $t > t_0$, which leads to finite time blow-up for $y(t)$, a contradiction.

We next turn to b) in the road map, namely the “concentration-compactness” procedure, in order to establish i) in Theorem 2.6. Note that in the defocusing case, the variational estimates are not needed. Note also that because of Corollary 2.12, we already know that $\sup_{t\in I}\|(u(t),\partial_t u(t))\| \lesssim \|(u_0,u_1)\|_{\dot H^1\times L^2}$. However, because of the Krieger-Schlag-Tataru [**71**] example, this does not suffice, and this is typical of critical problems.

b) *Concentration-Compactness Procedure.*

We recall the norms, introduced in “the local theory of the Cauchy problem”, $\|u\|_{S(I)} = \|u\|_{L^8(\mathbb{R}^3\times I)}, \left\|D^{\frac12}u\right\|_{W(I)} = \left\|D^{\frac12}u\right\|_{L^4(\mathbb{R}^3\times I)}$. Recall that if I is the maximal interval, if $T_+ < \infty$, $\|u\|_{S(I_+)} = \infty$. Also if $T_+ = \infty$, u does not scatter, iff $\|u\|_{S(I_+)} = \infty$. Because of a), if $\|\nabla u_0\|^2 < \|\nabla W\|^2$ and $E(u_0,u_1) \leq \eta_0$, η_0 small, then $\|(u_0,u_1)\|_{\dot H^1\times L^2}$ is small, so that u exists globally in time and scatters, from the “local theory of the Cauchy problem”. Consider now

$$\begin{aligned} G = \{E : 0 \leq E < E(W,0),\ &\text{with the property that if} \\ &\|\nabla u_0\|^2 < \|\nabla W\|^2 \text{ and } E(u_0,u_1) < E, \text{ then } \|u\|_{S(I)} < \infty\Big\}. \end{aligned}$$

Let $E_c = \sup G$, so that $0 < \eta_0 \leq E_c \leq E(W,0)$ and, if $\|\nabla u_0\|^2 < \|\nabla W\|^2$, $E(u_0,u_1) < E_c, I = (-\infty,+\infty), u$ scatters and E_c is optimal with this property. Theorem 2.6 i) is the same as $E_c = E(W,0)$. Assume $E_c < E(W,0)$, to reach a contradiction. Fix $\delta_0 > 0$ such that $E_c = (1-\delta_0)E(W,0)$. If $\|\nabla u_0\|^2 < \|\nabla W\|^2, E(u_0,u_1) < E$, with $E < E_c$, then $\|u\|_{S(I)} < \infty$, while if $E > E_c, E < E(W,0), \exists (u_0,u_1), \|\nabla u_0\|^2 < \|\nabla W\|^2, E_c \leq E(u_0,u_1) \leq E$ and $\|u\|_{S(I)} = \infty$. The concentration - compactness procedure allows us to prove:

Proposition 2.14. $\exists (u_{0,c},u_{1,c}) \in \dot H^1 \times L^2 : \|\nabla u_{0,c}\|^2 < \|\nabla W\|^2, E(u_{0,c},u_{1,c}) = E_c, \|u_c\|_{S(I)} = \infty$, *where* u_c *solves* (NLW) *with data* $(u_{0,c},u_{1,c}), I = I_{\max}(u_c)$.

Proposition 2.15. *Let* u_c *be as in Prop* 2.14, *with (say),* $\|\nabla u_c\|_{S(I_+)} = \infty$ *with* $I_+ = I \cap [0,\infty)$. *Then* $\exists x(t) \in \mathbb{R}^3, \lambda(t) \in \mathbb{R}_+, t \in I_+$, *such that*

$$K = \left\{\left(\frac{1}{\lambda(t)^{\frac12}}u_c\left(\frac{x-x(t)}{\lambda(t)},t\right), \frac{1}{\lambda(t)^{\frac32}}\partial_t u_c\left(\frac{x-x(t)}{\lambda(t)},t\right)\right) : t \in I_+\right\}$$

has compact closure in $\dot H^1 \times L^2$.

The proofs of the propositions use our variational estimates and a "profile decomposition" due to Bahouri-Gérard ([**4**]). A corresponding "profile decomposition" for NLS in the mass-critical case was obtained independently by Merle-Vega ([**80**]).

THEOREM 2.16 (Concentration-compactness, profile decomposition, Bahouri-Gérard 99). *Let* $\{(v_{0,n}, v_{1,n})\}_{n=1}^{\infty} \in \dot H^1 \times L^2$*, with* $\|(v_{0,n}, v_{1,n})\|_{\dot H^1 \times L^2} \leq A$*. Assume that* $\|S(t)\,(v_{0,n}, v_{1,n})\|_{S(-\infty,+\infty)} \geq \delta > 0$*, where* $\delta = \delta(A)$ *is as in "the local theory of Cauchy problem". Then, there exists a sequence* $\{(V_{0,j}, V_{1,j})\}_{j=1}^{\infty}$ *in* $\dot H^1 \times L^2$*, a subsequence of* $\{(v_{0,n}, v_{1,n})\}$ *(which we still call* $(v_{0,n}, v_{1,n})$*) and a triple* $(\lambda_{j,n}; x_{j,n}; t_{j,n}) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}$*, with the orthogonality property:*

$$\frac{\lambda_{j,n}}{\lambda_{j',n}} + \frac{\lambda_{j',n}}{\lambda_{j,n}} + \frac{|t_{j,n} - t_{j',n}|}{\lambda_{j,n}} + \frac{|x_{j,n} - x_{j',n}|}{\lambda_{j,n}} \overset{n\to\infty}{\to} +\infty.$$

for $j \neq j'$*, such that*

i) $\|(V_{0,1}, V_{1,1})\|_{\dot H^1 \times L^2} > \alpha_0(A) > 0$.

ii) *If* $V_j^l = S(t)\,((V_{0,j}, V_{1,j}))$*, then given* $\varepsilon_0 > 0$*,* $\exists J = J(\varepsilon_0)$ *such that*

$$\begin{aligned} v_{0,n} &= \sum_{j=1}^{J} \frac{1}{\lambda_{j,n}^{\frac12}} V_j^l\left(\frac{x - x_{j,n}}{\lambda_{j,n}}, -\frac{t_{j,n}}{\lambda_{j,n}}\right) + w_{0,n}^J \\ v_{1,n} &= \sum_{j=1}^{J} \frac{1}{\lambda_{j,n}^{\frac32}} \partial_t V_j^l\left(\frac{x - x_{j,n}}{\lambda_{j,n}}, -\frac{t_{j,n}}{\lambda_{j,n}}\right) + w_{1,n}^J \end{aligned}$$

with $\left\|S(t)\left(w_{0,n}^J, w_{1,n}^J\right)\right\|_{S(-\infty,+\infty)} \leq \varepsilon_0$*, for* n *large.*

iii)$_a$

$$\|\nabla_x v_{0,n}\|^2 = \sum_{j=1}^{J} \left\|\nabla_x V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right)\right\|^2 + \left\|\nabla w_{0,n}^J\right\|^2 + o(1)$$

$$\|v_{1,n}\|^2 = \sum_{j=1}^{J} \left\|\partial_t V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right)\right\|^2 + \left\|w_{1,n}^J\right\|^2 + o(1)$$

iii)$_b$

$$E\left((v_{0,n}, v_{1,n})\right) = \sum_{j=1}^{J} E\left(\left(V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right), \partial_t V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right)\right)\right) + E\left((w_{0,n}^J, w_{1,n}^J)\right) + o(1)$$

as $n \to \infty$*.*

A first consequence of the "profile decomposition", which already appears in Bahouri-Gérard [**4**] (implicitly, since they only treat the defocusing case) is the following:

COROLLARY 2.17. *There exists a decreasing function* $g : (0, E_c] \to [0, \infty)$*, such that for every* (u_0, u_1) *with* $\|\nabla u_0\|^2 < \|\nabla W\|^2, E\left((u_0, u_1)\right) = E_c - \eta$*, we have* $\|u\|_{S(-\infty,+\infty)} \leq g(\eta)$*.*

REMARK. A precise form of g was obtained in work of Duyckaerts-Merle ([**38**]). The proof of the Corollary also follows from the arguments that we will use in the proof of Proposition 2.18 below.

In order to apply the linear theorem above to the non-linear Propositions 2.14, 2.15, we need the notion of a "non-linear profile". Thus, let $(v_0, v_1) \in \dot{H}^1 \times L^2$, $v(x,t) = S(t)(v_0, v_1)$, let $\{t_n\}_{n=1}^{\infty}$ be a sequence with $\lim_{n\to\infty} t_n = \bar{t} \in [-\infty, +\infty]$. We say that $u(x,t)$ is a non-linear profile associated with $((v_0, v_1), \{t_n\}_{n=1}^{\infty})$ if there exists an interval I, with $\bar{t} \in \mathring{I}$ (if $t = \pm\infty$, then $I = [a, \infty)$ or $I = (-\infty, a]$)) such that u is a solution of the Cauchy problem in I and

$$\lim_{n\to\infty} \|(u(t_n), \partial_t u(t_n)), (v(t_n), \partial_t v(t_n))\|_{\dot{H}^1 \times L^2} = 0.$$

There always exists a non-linear profile associated with $((v_0, v_1), \{t_n\})$. Indeed, if $\bar{t} \in (-\infty, +\infty)$, we solve (NLW) with data $\big(v(x,\bar{t}), \partial_t v(x,\bar{t})\big)$ at $\bar{t}$. If $\bar{t} = +\infty$ (say), we solve the integral equation

$$u(t) = S(t)((v_0, v_1)) + \int_t^{+\infty} \frac{\sin\big((t-t')\sqrt{-\Delta}\big)}{\sqrt{-\Delta}} F(u)(t')dt',$$

using the fact that $w(t) = \int_t^{\infty} \frac{\sin\left((t-t')\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} h(t')dt'$ verifies the same Strichartz estimates as before, working now on $\mathbb{R}^3 \times [t_{n_0}, +\infty)$, where n_0 is so large that $\|S(t)(v_0, v_1)\|_{S(t_{n_0}, +\infty)} < \delta$. Then, $u(t_n) - v(t_n) = \int_{t_n}^{+\infty} \frac{\sin\left((t-t')\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} F(u)(t')dt' \to 0$ in $\dot{H}^1 \times L^2$ since $D^{\frac{1}{2}} F(u) \in L^{\frac{4}{3}}(t > t_{n_0}) L_x^{\frac{4}{3}}$. It is easy to see that if $u^{(1)}, u^{(2)}$ are non-linear profiles associated to $((v_0, v_1), \{t_n\})$, on $I \ni \bar{t}$, $u^{(1)} \equiv u^{(2)}$ on I. Hence, there exists a maximal interval I of existence for the non-linear profile. Note that it might not contain 0. Near finite end-points of I, the S norm is infinite, while if $\bar{t} = +\infty$ (say), $I = (a, +\infty)$, the S norm is finite near $+\infty$ by construction. In order to use these concepts to prove Proposition 2.14, Proposition 2.15, we will need:

Proposition 2.18. *Let $\{(z_{0,n}, z_{1,n})\} \in \dot{H}^1 \times L^2$, with $\|\nabla z_{0,n}\|^2 < \|\nabla W\|^2$ and $E((z_{0,n}, z_{1,n})) \to E_c < E((W,0))$. Assume that $\|S(t)(z_{0,n}, z_{1,n})\|_{S(-\infty,+\infty)} \geq \delta > 0$.*

Let $(V_{0,j}, V_{1,j})_{j=1}^{\infty}$ be as in the profile decomposition. Assume that one of

a) $\underline{\lim}_{n\to\infty} E\left(\left(V_1^l\left(-\frac{t_{1,n}}{\lambda_{1,n}}\right), \partial_t V_1^l\left(-\frac{t_{1,n}}{\lambda_{1,n}}\right)\right)\right) < E_c$

b) $\underline{\lim}_{n\to\infty} E\left(\left(V_1^l\left(-\frac{t_{1,n}}{\lambda_{1,n}}\right), \partial_t V_1^l\left(-\frac{t_{1,n}}{\lambda_{1,n}}\right)\right)\right) = E_c$, *and*

for $s_n = -\frac{t_{1,n}}{\lambda_{1,n}}$, after passing to a subsequence so that $s_n \to \bar{s} \in [-\infty, +\infty]$ and $E\left(\left(V_1^l\left(-\frac{t_{1,n}}{\lambda_{1,n}}\right), \partial_t V_1^l\left(-\frac{t_{1,n}}{\lambda_{1,n}}\right)\right)\right) \to E_c$, if U_1 is the non-linear profile associated to $((V_{0,1}, V_{1,1}), \{s_n\})$, then $I = (-\infty, +\infty)$, $\|U_1\|_{S(-\infty,+\infty)} < \infty$.

Then, (after passing to a subsequence) if $\{z_n\}$ solves (NLW) *with data $(z_{0,n}, z_{1,n})$, we have $\|z_n\|_{S(-\infty,+\infty)} < \infty$ for n large (and in fact is uniformly bounded in n).*

We first assume Proposition 2.18, and use it to prove Proposition 2.14, 2.15

Proof of Proposition 2.14. Find $(u_{0,n}, u_{1,n}) \in \dot{H}^1 \times L^2, \int |\nabla u_{0,n}|^2 < \int |\nabla W|^2, E((u_{0,n}, u_{1,n})) \to E_c, \|u_n\|_{S(I_n)} = +\infty, I_n =$ max interval . We must have

$$\|S(t)(u_{0,n}, u_{1,n})\|_{S(-\infty,+\infty)} \geq \delta > 0,$$

by "the local theory of Cauchy problem". Since $E_c = (1 - \delta_0)E((W,0))$, for n large $E((u_{0,n}, u_{1,n})) \leq \left(1 - \frac{\delta_0}{2}\right) E((W,0))$. By energy trapping, $\exists \bar{\delta}$ such that

$\|\nabla u_n(t)\| \leq (1-\overline{\delta})\|\nabla W\|^2, t \in I_n$. Fix $J \geq 1$, applying the profile decomposition to $\{(u_{0,n}, u_{1,n})\}$, after passing to a subsequence, we have

$$\|\nabla u_{0,n}\|^2 = \sum_{j=1}^{J} \left\|\nabla V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right)\right\|^2 + \left\|\nabla w_{0,n}^J\right\|^2 + o(1) \tag{2.19}$$

$$\|u_{1,n}\|^2 = \sum_{j=1}^{J} \left\|\partial_t V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right)\right\|^2 + \left\|w_{1,n}^J\right\|^2 + o(1) \tag{2.20}$$

(2.21)

$$E\left((u_{0,n}, u_{1,n})\right) = \sum_{j=1}^{J} E\left(\left(V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right), \partial_t V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right)\right)\right) + E\left((w_{0,n}^J, w_{1,n}^J)\right) + o(1)$$

From (2.21), for n large,

$$\left\|\nabla w_{0,n}^J\right\| \leq (1-\frac{\overline{\delta}}{2})\|\nabla W\|^2, \left\|\nabla V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right)\right\|^2 \leq (1-\frac{\overline{\delta}}{2})\|\nabla W\|^2, 1 \leq j \leq J.$$

Hence, by energy trapping, for large n we have

$$E\left(\left(V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right), \partial_t V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right)\right)\right) \geq 0, E\left((w_{0,n}^J, w_{1,n}^J)\right) \geq 0.$$

Thus, by (2.21), $E\left(\left(V_1^l\left(-\frac{t_{1,n}}{\lambda_{1,n}}\right), \partial_t V_1^l\left(-\frac{t_{1,n}}{\lambda_{1,n}}\right)\right)\right) \leq E\left((u_{0,n}, u_{1,n})\right) + o(1)$ and so,

$$\varliminf_{n\to\infty} E\left(\left(V_1^l\left(-\frac{t_{1,n}}{\lambda_{1,n}}\right), \partial_t V_1^l\left(-\frac{t_{1,n}}{\lambda_{1,n}}\right)\right)\right) \leq E_c.$$

Assume first that we have strict inequality. Then, Proposition 2.18 a) gives a contradiction for large n. Thus, we must have $\varliminf_{n\to\infty} E\left(\left(V_1^l\left(-\frac{t_{1,n}}{\lambda_{1,n}}\right), \partial_t V_1^l\left(-\frac{t_{1,n}}{\lambda_{1,n}}\right)\right)\right) = E_c$. Let U_1 be the non-linear profile associated to $\left\{s_n = -\frac{t_{1,n}}{\lambda_{1,n}}\right\}$, $((V_{0,1}, V_{1,1}), \{s_n\})$. The first observation is that $(V_{0,j}, V_{1,j}) = (0,0), j > 1$. Indeed, by (2.21) and $E\left((u_{0,n}, u_{1,n})\right) \to E_c, E\left(\left(V_1^l(s_n), \partial_t V_1^l(s_n)\right)\right) \to E_c$ (after passing to a subsequence), we see that $E\left((w_{0,n}^J, w_{1,n}^J)\right) \to 0$, and $E\left(\left(V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}, \partial_t V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right)\right)\right)\right) \to 0, j \geq 2$. Hence, using coercivity in the x variable, ii) in Lemma 2.8, we see that $\left\|\nabla w_{0,n}^J\right\|^2 + \sum_{j=2}^J \left\|\nabla V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right)\right\|^2 \to 0$. But then, $\left\|w_{1,n}^J\right\|^2 + \sum_{j=2}^J \left\|\partial_t V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right)\right\|^2 \to 0$. Finally, since $\|\nabla V_{0,j}\|^2 + \|V_{1,j}\|^2 = \left\|\nabla V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right)\right\|^2 + \left\|\partial_t V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right)\right\|^2$, we conclude $(V_{0,j}, V_{1,j}) = (0,0), 2 \leq j \leq J$. In addition, $\left\|\nabla w_{0,n}^J\right\|^2 + \left\|w_{1,n}^J\right\|^2 \to 0$, so that

$$\begin{aligned} u_{0,n} &= \frac{1}{\lambda_{1,n}^{\frac{1}{2}}} V_1^l\left(\frac{x - x_{1,n}}{\lambda_{1,n}}, s_n\right) + w_{0,n}^J \\ u_{1,n} &= \frac{1}{\lambda_{1,n}^{\frac{3}{2}}} \partial_t V_1^l\left(\frac{x - x_{1,n}}{\lambda_{1,n}}, s_n\right) + w_{1,n}^J \end{aligned}$$

with $\left\|\left(w_{0,n}^J, w_{1,n}^J\right)\right\|_{\dot H^1\times L^2} \to 0$. Renormalize, setting $v_{0,n} = \lambda_{1,n}^{\frac{1}{2}} u_{0,n}\left(\lambda_{1,n}\left(x + x_{1,n}\right)\right), v_{1,n} = \lambda_{1,n}^{\frac{3}{2}} u_{1,n}\left(\lambda_{1,n}\left(x + x_{1,n}\right)\right)$. By scaling, translation invariance, $(v_{0,n}, v_{1,n})$ has the same properties as $(u_{0,n}, u_{1,n})$ and

$$v_{0,n} = V_1^l\left(s_n\right) + \widetilde{w}_{0,n}^J,\ v_{1,n} = \partial_t V_1^l\left(s_n\right) + \widetilde{w}_{1,n}^J$$

where $\|(\widetilde{w}_{0,n}, \widetilde{w}_{1,n})\|_{\dot H^1\times L^2} \to 0$. Let $I_1 =$ max interval of U_1. By definition of nonlinear profile, $E\left(U_1(s_n), \partial_t U_1(s_n)\right) = E\left(\left(V_1^l(s_n), \partial_t V_1^l(s_n)\right)\right) + o(1) = E_c + o(1)$, $\|\nabla U_1(s_n)\|^2 = \left\|\nabla V_1^l(s_n)\right\|^2 + o(1) = \|\nabla u_{0,n}\|^2 + o(1) < \|\nabla W\|^2$, for n large. Fix $\overline{s} \in I_1$, then $E\left(\left(U_1(\overline{s}), \partial_t U_1(\overline{s})\right)\right) = E\left(\left(U_1(s_n), \partial_t U_1(s_n)\right)\right) \to E_c$, so that $E\left(\left(U_1(\overline{s}), \partial_t U_1(\overline{s})\right)\right) = E_c$. Also, $\|\nabla U_1(s_n)\|^2 < \|\nabla W\|^2$ for n large, so that, by energy trapping, $\|\nabla U_1(\overline{s})\|^2 < \|\nabla W\|^2$. If $\|\nabla U_1\|_{S(I_1)} < +\infty$, Proposition 2.18 b) gives a contradiction. Hence, $\|U_1\|_{S(I_1)} = +\infty$, we take $u_c = U_1$. □

Proof of Proposition 2.15. : (By contradiction). Let $u(x,t) = u_c(x,t)$. If not, $\exists \eta_0 > 0, \{t_n\}_{n=1}^{\infty}, t_n \geq 0$ such that $\forall \lambda_0 \in \mathbb{R}^+, x_0 \in \mathbb{R}^3$ we have (after rescaling)

$$\left\|\frac{1}{\lambda_0^{\frac{1}{2}}} u\left(\frac{x - x_0}{\lambda_0}, t_n\right) - u\left(\frac{x}{\lambda_0}, t_n'\right)\right\|_{\dot H^1}^2 + \left\|\frac{1}{\lambda_0^{\frac{3}{2}}} \partial_t u\left(\frac{x - x_0}{\lambda_0}, t_n\right) - \partial_t u\left(\frac{x}{\lambda_0}, t_n'\right)\right\|_{\dot H^1}^2$$
$$\geq \eta_0 > 0,$$

for $n \neq n'$.

After passing to a subsequence, $t_n \to \overline{t} \in [0, T_+(u_0, u_1)]$ so that by continuity of the flow, $\overline{t} = T_+\left(u_0, u_1\right)$. By the local theory of the Cauchy problem, we can also assume $\|S(t)\left(u(t_n), \partial_t u(t_n)\right)\|_{S(0,+\infty)} \geq \delta > 0$.

We apply the profile decomposition to $(v_{0,n}, v_{1,n}) = (u(t_n), \partial_t u(t_n))$. We have $E\left(\left(u(t), \partial_t u(t)\right)\right) = E\left(\left(u_{0,c}, u_{1,c}\right)\right) = E_c < E\left((W, 0)\right), \|\nabla u_{0,c}\|^2 < \|\nabla W\|^2$, so that $\|\nabla u(t)\|^2 \leq (1 - \overline{\delta})\|\nabla W\|^2, t \in I_+$. Then, $\underline{\lim}_{n\to\infty} E\left(\left(V_1^l\left(-\frac{t_{1,n}}{\lambda_{1,n}}\right), \partial_t V_1^l\left(-\frac{t_{1,n}}{\lambda_{1,n}}\right)\right)\right) \leq E_c$. If we have strict inequality, Propositoin 2.18 a) gives a contradiction. Hence we have equality and as in the previous proof, $(V_{0,j}, V_{1,j}) = 0, j > 1, \left\|\left(w_{0,n}^J, w_{1,n}^J\right)\right\|_{\dot H^1\times L^2} \to 0$.

Thus,

$$\begin{aligned} u(t_n) &= \frac{1}{\lambda_{1,n}^{\frac{1}{2}}} V_1^l\left(\frac{x - x_{1,n}}{\lambda_{1,n}}, -\frac{t_{1,n}}{\lambda_{1,n}}\right) + w_{0,n}^J, \\ \partial_t u(t_n) &= \frac{1}{\lambda_{1,n}^{\frac{3}{2}}} \partial_t V_1^l\left(\frac{x - x_{1,n}}{\lambda_{1,n}}, -\frac{t_{1,n}}{\lambda_{1,n}}\right) + w_{1,n}^J, \end{aligned}$$

$\left\|\left(w_{0,n}^J, w_{1,n}^J\right)\right\|_{\dot H^1\times L^2} \to 0$. Let $s_n = -\frac{t_{1,n}}{\lambda_{1,n}}$. We claim that s_n must be bounded. In fact, if $\frac{t_{1,n}}{\lambda_{1,n}} \leq -C_0, C_0$ a large positive constant, since for n large, $\left\|S(t)\left(w_{0,n}^J, w_{1,n}^J\right)\right\|_{S(-\infty,+\infty)} \leq \frac{\delta}{2}$ and

$$\left\|\frac{1}{\lambda_{1,n}^{\frac{1}{2}}} V_1^l\left(\frac{x - x_{1,n}}{\lambda_{1,n}}, \frac{t - t_{1,n}}{\lambda_{1,n}}\right)\right\|_{S(0,+\infty)} \leq \left\|V_1^l\right\|_{S(C_0,+\infty)} \leq \frac{\delta}{2},$$

we reach a contradiction by the Perturbation Theorem (Theorem 1.12). If on the other hand, $\frac{t_{1,n}}{\lambda_{1,n}} \geq C_0, C_0$ large positive, for n large we have

$$\left\| \frac{1}{\lambda_{1,n}^{\frac{1}{2}}} V_1^l \left(\frac{x - x_{1,n}}{\lambda_{1,n}}, \frac{t - t_{1,n}}{\lambda_{1,n}} \right) \right\|_{S(-\infty,0)} \leq \|V_1^l\|_{S(,-\infty,-C_0)} \leq \frac{\delta}{2},$$

for C_0 large. Thus, for n large, we would have $\|S(t)\left(u(t_n), \partial_t u(t_n)\right)\|_{S(-\infty,0)} \leq \delta$, so that Theorem 1.4 gives $\|u\|_{S(-\infty,t_n)} \leq 2\delta$. But, $t_n \uparrow T_+\left((u_0, u_1)\right)$, a contradiction. Thus, after passing to a subsequence, $\frac{t_{1,n}}{\lambda_{1,n}} \to t_0 \in (-\infty, +\infty)$. But then,

$$\left\| \left(w_{0,n}^J, w_{1,n}^J \right) \right\|_{\dot{H}^1 \times L^2} \to 0 \tag{2.22}$$

gives that for $n \neq n'$, both large,

$$\begin{aligned} &\left\| \frac{1}{\lambda_0^{\frac{1}{2}}} \frac{1}{\lambda_{1,n}^{\frac{1}{2}}} V_1^l \left(\frac{\frac{x-x_0}{\lambda_0} - x_{1,n}}{\lambda_{1,n}}, -\frac{t_{1,n}}{\lambda_{1,n}} \right) - \frac{1}{\lambda_{1,n'}^{\frac{1}{2}}} V_1^l \left(\frac{x - x_{1,n'}}{\lambda_{1,n'}} - \frac{t_{1,n'}}{\lambda_{1,n'}} \right) \right\|_{\dot{H}^1}^2 \\ &+ \left\| \frac{1}{\lambda_0^{\frac{3}{2}}} \frac{1}{\lambda_{1,n}^{\frac{3}{2}}} \partial_t V_1^l \left(\frac{\frac{x-x_0}{\lambda_0} - x_{1,n}}{\lambda_{1,n}}, -\frac{t_{1,n}}{\lambda_{1,n}} \right) - \frac{1}{\lambda_{1,n'}^{\frac{3}{2}}} \partial_t V_1^l \left(\frac{x - x_{1,n'}}{\lambda_{1,n'}} - \frac{t_{1,n'}}{\lambda_{1,n'}} \right) \right\|_{L^2}^2 \\ &\geq \frac{\eta_0}{2}. \end{aligned}$$

for all λ_0, x_0. After changing variables, this gives, for all $\lambda_0, \widetilde{x}_0$, that

$$\begin{aligned} &\left\| \left(\frac{\lambda_{1,n'}}{\lambda_0 \lambda_{1,n}} \right)^2 V_1^l \left(\frac{\lambda_{1,n'} y}{\lambda_0 \lambda_{1,n}} + x_{n,n'} - \widetilde{x}_0, -\frac{t_{1,n}}{\lambda_{1,n}} \right) - V_1^l \left(y, -\frac{t_{1,n'}}{\lambda_{1,n'}} \right) \right\|_{\dot{H}^1}^2 \\ &+ \left\| \left(\frac{\lambda_{1,n'}}{\lambda_0 \lambda_{1,n}} \right)^{\frac{3}{2}} \partial_t V_1^l \left(\frac{\lambda_{1,n'} y}{\lambda_0 \lambda_{1,n}} + x_{n,n'} - \widetilde{x}_0, -\frac{t_{1,n}}{\lambda_{1,n}} \right) - \partial_t V_1^l \left(y, -\frac{t_{1,n'}}{\lambda_{1,n'}} \right) \right\| \geq \frac{\eta_0}{2}. \end{aligned}$$

Choosing now $\lambda_0, \widetilde{x}_0$ suitably, this is a contradiction, since $\frac{t_{1,n'}}{\lambda_{1,n'}} \to t_0$, $\frac{t_{1,n}}{\lambda_{1,n}} \to t_0$. $\square$

Proof of Proposition 2.18. Assume first, that

$$\underline{\lim}\, E\left(\left(V_1^l \left(-\frac{t_{1,n}}{\lambda_{1,n}} \right), \partial_t V_1^l \left(-\frac{t_{1,n}}{\lambda_{1,n}} \right) \right) \right) = E_c.$$

Fix $J \geq 1$ and note that, as in the proof of Proposition 2.14, we have $(V_{0,j}, V_{1,j}) = (0,0), j > 1, \left\|(w_{0,n}^J, w_{1,n}^J)\right\|_{\dot{H}^1 \times L^2} \to 0$.

Moreover, if $v_{0,n} = \lambda_{1,n}^{\frac{1}{2}} z_{0,n}\left(\lambda_{1,n}(x + x_{1,n})\right), v_{1,n} = \lambda_{1,n}^{\frac{3}{2}} z_{1,n}\left(\lambda_{1,n}(x + x_{1,n})\right)$, $\widetilde{w}_{0,n}^J = \lambda_{1,n}^{\frac{1}{2}} w_{0,n}^J\left(\lambda_{1,n}(x + x_{1,n})\right), \widetilde{w}_{1,n}^J = \lambda_{1,n}^{\frac{3}{2}} w_{1,n}^J\left(\lambda_{1,n}(x + x_{1,n})\right)$, $\left\|\left(\widetilde{w}_{0,n}^J, \widetilde{w}_{1,n}^J\right)\right\|_{\dot{H}^1 \times L^2} \to 0, v_{0,n} = V_1^l(s_n) + \widetilde{w}_{0,n}^J, v_{1,n} = \partial_t V_1^l(s_n) + \widetilde{w}_{1,n}^J$, with $E\left((v_{0,n}, v_{1,n})\right) \to E_c < E\left((W,0)\right)$, $\|\nabla v_{0,n}\|^2 < \|\nabla W\|^2$. By definition of non-linear profile,

$$\left\|\left(V_1^l(s_n) - U_1(s_n), \partial_t V_1^l(s_n) - \partial_t U_1(s_n)\right)\right\|_{\dot{H}^1 \times L^2} \to 0,$$

so that $v_{0,n} = U_1(s_n) + \widetilde{\widetilde{w}}_{0,n}^J, v_{1,n} = \partial_t U_1(s_n) + \widetilde{\widetilde{w}}_{1,n}^J, \left\|\left(\widetilde{\widetilde{w}}_{0,n}^J, \widetilde{\widetilde{w}}_{1,n}^J\right)\right\|_{\dot{H}^1 \times L^2} \to 0$. From this, we see that $E\left((U_1, \partial_t U_1)\right) = E_c < E\left((W,0)\right), \|\nabla U_1(s_n)\|^2 < \|\nabla W\|^2$, for n large, so that, by Lemma 2.8, $\sup_{t\in I_1}\|\nabla U_1(t)\|^2 < \|\nabla W\|^2$. Since $\left\|\left(\nabla \widetilde{\widetilde{w}}_{0,n}^J, \widetilde{\widetilde{w_{1,n}}}^J\right)\right\|_{L^2\times L^2} \to 0$, Theorem 1.12 now gives the case b). Assume next that

$$\underline{\lim}\, E\left(\left(V_1^l\left(-\frac{t_{1,n}}{\lambda_{1,n}}\right), \partial_t V_1^l\left(-\frac{t_{1,n}}{\lambda_{1,n}}\right)\right)\right) < E_c$$

and, passing to a subsequence, $\lim E\left(\left(V_1^l\left(-\frac{t_{1,n}}{\lambda_{1,n}}\right), \partial_t V_1^l\left(-\frac{t_{1,n}}{\lambda_{1,n}}\right)\right)\right) < E_c$. We next show that $\underline{\lim}\, E\left(\left(V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right), \partial_t V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right)\right)\right) < E_c, j = 2, \dots, J$. In fact,

$$\|\nabla z_{0,n}\|^2 = \sum_{j=1}^J \left\|\nabla V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right)\right\|^2 + \left\|\nabla w_{0,n}^J\right\|^2 + o(1),$$

$$\|z_{1,n}\|^2 = \sum_{j=1}^J \left\|\partial_t V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right)\right\|^2 + \left\|w_{1,n}^J\right\|^2 + o(1),$$

and since $E\left((z_{0,n}, z_{1,n})\right) \to E_c < E\left((W,0)\right)$, for n large, $E\left((z_{0,n}, z_{1,n})\right) \le (1-\delta_0)E\left((W,0)\right)$. Since $\|\nabla z_{0,n}\|^2 < \|\nabla W\|^2$, Lemma 2.8 gives that $\|\nabla z_{0,n}\|^2 \le (1-\overline{\delta})\|\nabla W\|^2$. Thus, for all n large, $\left\|\nabla V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right)\right\|^2 \le \left(1-\frac{\overline{\delta}}{2}\right)\|\nabla W\|^2$. Corollary 2.9 now shows that $E\left(\left(V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right), \partial_t V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right)\right)\right) \ge 0, E\left(\left(w_{0,n}^J, w_{1,n}^J\right)\right) \ge 0$, $E\left(V_1^l(-s_n), \partial_t V_1^l(-s_n)\right) \ge C\alpha_0 = \overline{\alpha}_0 > 0$, for n large (this fact follows from Lemma 2.8 ii)). Thus,

$$E\left((z_{0,n}, z_{1,n})\right) \ge \overline{\alpha_0} + \sum_{j=2}^J E\left(\left(V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right), \partial_t V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right)\right)\right) + o(1),$$

so our claim follows from $E\left((z_{0,n}, z_{1,n})\right) \to E_c$.

Next, note that if U_j is the non-linear profile associated to $\left((V_{0,j}, V_{1,j}), \left\{-\frac{t_{j,n}}{\lambda_{j,n}}\right\}\right)$, (after passing to a subsequence in n), then U_j exists for all time and $\|U_j\|_{S(-\infty,+\infty)} < \infty$, $1 \le j \le J$. In fact, for n large, $E\left(\left(V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right), \partial_t V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right)\right)\right) < E_c$, so $E\left((U_j, \partial_t U_j)\right) < E_c$ by definition of non-linear profile. Moreover, $\left\|\nabla V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right)\right\|^2 \le \|\nabla z_{0,n}\|^2 + o(1) \le \left(1-\overline{\delta}\right)\|\nabla W\|^2 + o(1)$, so by Lemma 2.8 we have $\|\nabla U_j(t)\| < \|\nabla W\|, \forall t \in I_j$. But then, by definition of E_c, $I_j = (-\infty, +\infty), \|U_j\|_{S(-\infty,+\infty)} < \infty$. Next, note that $\exists j_0$ such that for $j \ge j_0$ we have

$$\|U_j\|_{S(-\infty,+\infty)}^2 \le C\,\|(V_{0,j}, V_{1,j})\|_{\dot{H}^1\times L^2}^2 .$$

In fact, for J fixed, choosing n large, we have

$$\begin{aligned}\sum_{j=1}^J \|\nabla V_{0,j}\|^2 + \|V_{1,j}\|^2 \quad &= \quad \sum_{j=1}^J \left\|\nabla V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right)\right\|^2 \\ &\quad + \left\|\partial_t V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}}\right)\right\|^2 \le \|(z_{0,n}, z_{1,n})\|^2 + o(1).\end{aligned}$$

Note that $\|\nabla z_{0,n}\|^2 < \|\nabla W\|^2$, $E\left(z_{0,n}, z_{1,n}\right) < E\left((W,0)\right)$, so that the right hand side $< C\|\nabla W\|^2$. Hence, for $j \ge j_0$, $\|\nabla V\|_{0,j}^2 + \|V_{1,j}\|^2 \le \widetilde{\delta}$, where $\widetilde{\delta}$ is so small that $\|S(t)\,(V_{0,j},V_{1,j})\|_{S(-\infty,+\infty)} \le \delta$. From the definition of non-linear profile, this gives that $\|U_j\|_{S(-\infty,+\infty)} \le 2\delta$, and that

$$\sup_t \|(U_j(t),\partial_t U_j(t))\|_{\dot H^1\times L^2} + \left\| D^{\frac12} U_j \right\|_{W(-\infty,+\infty)} \le C\,\|(V_{0,j},V_{1,j})\|_{\dot H^1\times L^2}.$$

But then, the integral equation for U_j gives $\|U_j\|_{S(-\infty,+\infty)} \le C\,\|(V_{0,j},V_{1,j})\|_{\dot H^1\times L^2}$, as desired. Next, for $\varepsilon_0>0$, to be chosen, define

$$H_{n,\varepsilon_0} = \sum_{j=1}^{J(\varepsilon_0)} \frac{1}{\lambda_{j,n}^{\frac12}} U_j\left(\frac{x-x_{j,n}}{\lambda_{j,n}}, \frac{t-t_{j,n}}{\lambda_{j,n}}\right).$$

Then, we claim that $\|H_{n,\varepsilon_0}\|_{S(-\infty,+\infty)} \le C_0$, uniformly in ε_0, for $n \ge n(\varepsilon_0)$. In fact,

$$\begin{aligned}
&\|H_{n,\varepsilon_0}\|^8_{S(-\infty,+\infty)} = \iint \left[\sum_{j=1}^{J(\varepsilon_0)} \frac{1}{\lambda j, n^{\frac12}} U_j\left(\frac{x-x_{j,n}}{\lambda_{j,n}}, \frac{t-t_{j,n}}{\lambda_{j,n}}\right)\right]^8 \\
&\le \sum_{j=1}^{J(\varepsilon_0)} \iint \left| \frac{1}{\lambda_{j,n}^{\frac12}} U_j\left(\frac{x-x_{j,n}}{\lambda_{j,n}}, \frac{t-t_{j,n}}{\lambda_{j,n}}\right)\right|^8 \\
&+C_{J(\varepsilon_0)} \sum_{j\neq j'} \iint \left| \frac{1}{\lambda_{j,n}^{\frac12}} U_j\left(\frac{x-x_{j,n}}{\lambda_{j,n}}, \frac{t-t_{j,n}}{\lambda_{j,n}}\right)\right| \left| \frac{1}{\lambda_{j,n}^{\frac12}} U_j\left(\frac{x-x_{j,n}}{\lambda_{j,n}}, \frac{t-t_{j,n}}{\lambda_{j,n}}\right)\right|^7 \\
=\ & I + II.
\end{aligned}$$

For n large, $II \overset{n}{\to} 0$ by orthogonality of $(\lambda_{j,n}, x_{j,n}, t_{j,n})$. Thus, for n large, $II \le I$. But,

$$\begin{aligned}
I &\le \sum_{j=1}^{j_0} \|U_j\|^8_{S(-\infty,+\infty)} + \sum_{j=j_0+1}^{J(\varepsilon_0)} \|U_j\|^8_{S(-\infty,+\infty)} \\
&\le \sum_{j=1}^{j_0} \|U_j\|^8_{S(-\infty,+\infty)} + C \sum_{j=j_0+1}^{J(\varepsilon_0)} \|(V_{0,j},V_{1,j})\|^8_{\dot H^1\times L^2} \\
&\le \sum_{j=1}^{j_0} \|U_j\|^8_{S(-\infty,+\infty)} + C \sup_{j>j_0} \|(V_{0,j},V_{1,j})\|^6_{\dot H^1\times L^2} \cdot \sum_{j>j_0}^{J(\varepsilon_0)} \|(V_{0,j},V_{1,j})\|^2_{\dot H^1\times L^2} \\
&\le \frac{C_0}{2},
\end{aligned}$$

as desired.

Let now $R_{n,\varepsilon_0} = H^5_{n,\varepsilon_0} - \sum_{j=1}^{J(\varepsilon_0)} \widetilde U^5_{j,n}$, where $\widetilde U_{j,n} = \frac{1}{\lambda_{j,n}^{\frac12}} U_j\left(\frac{x-x_{j,n}}{\lambda_{j,n}}, \frac{t-t_{j,n}}{\lambda_{j,n}}\right)$. We have $\left\| D_x^{\frac12} R_{n,\varepsilon_0}\right\|_{L_t^{\frac43} L_x^{\frac43}} \overset{n\to\infty}{\longrightarrow} 0$. This uses orthogonality, the chain rule, $\|U_j\|_{S(-\infty,+\infty)} < \infty$, $\left\|D^{\frac12} U_j\right\|_{W(-\infty,+\infty)} < \infty$. We now define $\widetilde u = H_{n,\varepsilon_0}$, $e = R_{n,\varepsilon_0}$. Choose $J(\varepsilon_0)$ so large, that for n large, $\left\| S(t)\left(w_{0,n}^{J(\varepsilon_0)}, w_{1,n}^{J(\varepsilon_0)}\right)\right\|_{S(-\infty,+\infty)} \le \frac{\varepsilon_0}{2}$. Note that by the profile decomposition, the definition of non-linear profile, we have, for n large $z_{0,n} = H_{n,\varepsilon_0}(0) + \widetilde w_{0,n}^{J(\varepsilon_0)}, z_{1,n} = \partial_t H_{n,\varepsilon_0}(0) + \widetilde w_{1,n}^{J(\varepsilon_0)}$, where, for n large

$\left\|S(t)\left(\widetilde{w}_{0,n}^{J(\varepsilon_0)}, \widetilde{w}_{1,n}^{J(\varepsilon_0)}\right)\right\|_{S(-\infty,+\infty)} \leq \varepsilon_0$. Arguments similar to those above also show that $\sup_t \|(H_{n,\varepsilon_0}(t), \partial_t H_{n,\varepsilon_0}(t))\|_{\dot{H}^1\times L^2} \leq \widetilde{C_0}$, uniformly in ε_0, for n large, and $\left\|\left(\widetilde{w}_{0,n}^{J(\varepsilon_0)}, \widetilde{w}_{1,n}^{J(\varepsilon_0)}\right)\right\|_{\dot{H}^1\times L^2} \leq C\|\nabla W\|$. Choose now $\varepsilon_0 < \varepsilon_0\left(C_0, \widetilde{C}_0, C\|\nabla W\|\right)$ as in Theorem 1.12, and n so large that $\left\|D_x^{\frac{1}{2}} R_{n,\varepsilon_0}\right\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}} \leq \varepsilon_0$. Then, Theorem 1.12 gives Proposition 2.18 a). This concludes the concentration - compactness procedure. $\square$

CHAPTER 9

Soliton Resolution for Radial Solutions to (NLW), I

In this chapter, we start our discussion of the recent proof of the soliton resolution conjecture for radial solutions of (NLW), by Duyckaerts, Kenig and Merle, in [**30**] and [**33**]. The proofs in Chapters 9–11 are from [**33**]. Notice that we have already had a preliminary discussion of soliton resolution in Remark 6.15.

For a long time there has been a widespread belief that global in time solutions of dispersive equations, asymptotically in time, decouple into a sum of finitely many modulated solitons, a free radiation term and a term that goes to 0 at infinity. Such a result should hold for globally well-posed equations, or in general, with the additional condition that the solution does not blow up. When dealing with an equation for which blow-up can occur, such decompositions are always expected to be unstable. So far, the only cases where results of the type have been proved are for the integrable KdV and NLS equations in one space dimension. For $\partial_t u + \partial_x^3 u + u\partial_x u = 0$, for data with regularity and decay, this has been established by Eckhaus-Schuur ([**40**]). Corresponding results for the other integrable KdV equation, the modified KdV, $\partial_t u + \partial_x^3 u + u^2\partial_x u = 0$, were also obtained by the same authors via the Miura transform. Heuristic arguments for this conjecture, in the case of the cubic NLS in 1-d, $i\partial_t u + \partial_x^2 u + |u|^2 u = 0$, another integrable model, were given by Ablowitz-Segur [**91**] and Zakharov-Shabat [**103**]. For a rigorous proof in this case, see Novoksenov [**86**]. All of these equations are globally well-posed and so the decompositions are expected to be stable, unlike the case of equations for which blow-up may occur. For more general dispersive equations, so far results have only been found, for subcritical nonlinearities, for data close to the soliton. (Buslaev-Perelman [**9**], [**10**] for NLS with specific nonlinearities in $1d$, Soffer-Weinstein [**96**], in higher dimensions, Martel-Merle for gKdV (generalized KdV equations) [**76**], ...). Corresponding results near the soliton, in the case of finite time blow-up for critical problems, are in the works of Martel-Merle for gKdV [**77**], Merle-Raphael [**78**] for mass critical NLS, etc. There have also been large solution results for critical equivariant wave maps into the sphere, due to Christodoulou-Tahvildar-Zadeh, Shatah-Tahvilder-Zadeh, Struwe,[**11**],[**95**] and [**98**]. These are results for finite time blow-up, which show convergence along some sequence of times converging to the blow-up time, locally in space-time, to a soliton (harmonic map). Recently, this has been strengthened (with size restrictions) in works of Côte-Kenig-Lawrie-Schlag [**14**], [**15**] and by Côte [**13**] without size restriction, but only for a sequence of times.

In the finite time blow-up case, for the 1-d nonlinear wave equation, Merle-Zaag have obtained results of the "resolution" type, through the use of a global Lyapunov functional in self-similar variables [**82**]. Also, in critical problems of elliptic type, there have been "towering bubbles" detected in asymptotic problems, where the size of an excluded hole goes to 0, see [**84**], etc.

The first general results for radial solutions of (NLW), were obtained in [**30**]. They held for extended type II solutions, for a specific sequence of times. We now have the full soliton resolution for radial solutions of (NLW), in the two asymptotic regimes, finite time type II blow-up, and global in time. Our result here is, [**33**]:

THEOREM 9.1. *Let u be a radial solution of* (NLW). *Then, one of the following holds:*

a) *Type I blow-up:* $T_+ < \infty$ *and*

$$\lim_{t\uparrow T_+} \|(u(t), \partial_t u(t))\|_{\dot{H}^1\times L^2} = \infty.$$

b) *Type II blow-up:* $T_+ < \infty$ *and* $\exists (v_0, v_1) \in \dot{H}^1 \times L^2, J \in \mathbb{N}\setminus\{0\}$ *and* $\forall j \in \{1,\dots,J\}, i_j \in \{\pm 1\}$ *and* $\lambda_j(t) > 0$ *such that* $0 < \lambda_1(t) \ll \lambda_2(t) \ll \cdots \ll \lambda_J(t) \ll T_+ - t$, *and* $(u(t), \partial_t u(t)) = \left(\sum_{j=1}^{J} \frac{i_j}{\lambda_j(t)^{\frac{1}{2}}} W\left(\frac{x}{\lambda_j(t)}\right), 0\right) + (v_0, v_1) + o(1)$ *in* $\dot{H}^1 \times L^2$.

c) $T_+ = \infty$ *and* $\exists$ *a solution* v_L *of* (LW), $J \in \mathbb{N}$ *and for all* $j \in \{1,\dots,J\}, i_J \in \{\pm 1\}, \lambda_j(t) > 0$ *such that* $0 < \lambda_1(t) \ll \lambda_2(t) \ll \cdots \ll \lambda_J(t) \ll t$, *and* $(u(t), \partial_t u(t)) = \left(\sum_{j=1}^{J} \frac{i_j}{\lambda_j(t)^{\frac{1}{2}}} W\left(\frac{x}{\lambda_j(t)}\right), 0\right) + (v_L(t), \partial_t v_L(t)) + o(1)$ *in* $\dot{H}^1 \times L^2$.

Here, $a(t) \ll b(t)$ *as* $t \to T$ ($T < \infty$, *or* $T = \pm\infty$) *means* $\lim_{t\to T} \frac{a(t)}{b(t)} = 0$.

REMARK 9.2. When $T_+ < \infty$, a), b) imply that $\lim_{t\uparrow T_+} \|(u(t), \partial_t u(t))\|_{\dot{H}^1\times L^2} = l$ exist, $l \in [\|\nabla W\|^2, +\infty]$, i.e., solutions split into type I, II, no mixed asymptotics exist. Recall that both type I, II blow-up exist. We expect that solutions as in b), with $J > 1$, exist. For the 1-d nonlinear wave equation this has been shown by Côte-Zaag [**18**].

As mentioned earlier, in the elliptic setting, "towering bubbles" do exist [**84**].

REMARK 9.3. When $T_+ = \infty$, c) in particular implies that $\sup_{t>0} \|(u(t), \partial_t u(t))\|_{\dot{H}^1\times L^2} < \infty$. More precisely, $\limsup_{t\uparrow\infty} \|(u(t), \partial_t u(t))\|^2_{\dot{H}^1\times L^2} = l$ and $2E(u_0, u_1) \le l \le 3E(u_0, u_1)$. Also, $J \le \frac{E(u_0,u_1)}{E(W,0)}$.

In this case we also expect that solutions with $J > 1$ exist.

REMARK 9.4. It is known that the set S_1 of initial data such that the corresponding solution scatters to a linear solution is open. It is believed that the set S_2 of initial data leading to type I blow-up is also open. Theorem 9.1 gives a description of solutions whose data is in S_3, the complement of $S_1 \cup S_2$. We believe that from Theorem 9.1 one can show that S_3 is the boundary of $S_1 \cup S_2$. In particular, we conjecture that the asymptotic behavior of data in S_3 is unstable.

A fundamental new ingredient of the proof of Theorem 9.1 is the following dispersive property that all global in time radial solutions to (NLW) (other than $0, \pm W$ up to scaling) must have:

$$\int_{|x|>R+|t|} |\nabla_{x,t} u(x,t)|^2 dx \ge \eta, \text{ for some } R > 0, \eta > 0 \text{ and all } t \ge 0 \text{ or all } t \le 0. \tag{9.5}$$

We establish this only using the behavior of u in "outside regions", $|x| > R+|t|$, without using any global integral identity of virial or Pohozaev type. (This can also be used to give a new proof of the results of Pohozaev (elliptic) and also of the rigidity theorem, Theorem 4.17, in an important special case, as we will see).

REMARK. With Lawrie and Schlag [**58**], we have used these ideas to give a soliton resolution in a stable situation, for 1-equivariant wave maps from $\mathbb{R}^3\backslash B_1$ into S^3, thus establishing a conjecture of Bizon-Chmaj-Maliborski [**7**]. This shows that the ideas in the proof of Theorem 9.1 can also apply to show stable soliton resolutions. The extension to the general k-equivariant case has been recently carried out by Kenig-Lawrie-Liu-Schlag [**56**], [**57**].

We now turn to the proof of Theorem 9.1.

We start with some notation and preliminary results. We will give the proof of c), the one of a), b) being similar.

Let $(u_0, u_1) \in \dot{H}^1 \times L^2, R > 0$, radial. We define

$$(\widetilde{u_0}, \widetilde{u_1}) = \Psi_R\,(u_0, u_1) \tag{9.6}$$

by:

$$\widetilde{u_0}(r) = \begin{cases} u_0(r) \text{ if } r \geq R \\ u_0(R) \text{ if } 0 < r < R \end{cases}$$

$$\widetilde{u_1}(r) = \begin{cases} u_1(r) \text{ if } r \geq R \\ 0 \text{ if } 0 < r < R \end{cases}$$

Note that $(\widetilde{u_0}, \widetilde{u_1}) \in \dot{H}^1 \times L^2, (u_0(r), u_1(r)) = (\widetilde{u_0}(r), \widetilde{u_1}(r))$ for $r \geq R$ and $\|(\widetilde{u_0}, \widetilde{u_1})\|^2_{\dot{H}^1\times L^2} = \int_{|x|>R} |\nabla u_0|^2 + u_1^2$. We will need the following version of the "local theory of the Cauchy problem", involving potentials.

LEMMA 9.7. *$\exists \delta_0 > 0$ such that if $0 \in I$, $V = V(x,t) \in L^8\left(\mathbb{R}^3 \times I\right)$ and*

$$\begin{aligned} &\|V\|_{L^8(\mathbb{R}^3\times I)} + \left\|D_x^{\frac12} V\right\|_{L^4(\mathbb{R}^3\times I)} + \left\|D_x^{\frac12} V^2\right\|_{L^{\frac83}(\mathbb{R}^3\times I)} + \left\|D_x^{\frac12} V^3\right\|_{L^2(\mathbb{R}^3\times I)} \\ &+ \left\|D_x^{\frac12} V^4\right\|_{L^{\frac85}(\mathbb{R}^3\times I)} \leq \delta_0, \|(h_0, h_1)\|_{\dot{H}^1\times L^2} \leq \delta_0, \end{aligned}$$

then $\exists!$ solution h of

$$\begin{cases} \partial_t^2 - \Delta h = 5v^4h + 10v^3h^2 + 10v^2h^3 + h^5 + 5h^4v = (v+h)^5 - v^5 \\ \quad h|_{t=0} = h_0 \\ \partial_t h|_{t=0} = h_1 \end{cases} \tag{9.8}$$

with $\vec{h} = (h, \partial_t h) \in C\left(I; \dot{H}^1 \times L^2\right), h \in L^8\left(\mathbb{R}^3 \times I\right)$. Also, letting h_L be the solution of the (LW)*, we have*

$$\sup_{t\in I} \left\|\vec{h(t)} - \vec{h_L(t)}\right\|_{\dot{H}^1\times L^2} \leq \frac{1}{10} \|(h_1, h_2)\|_{\dot{H}^1\times L^2}\,.$$

The proof ([**33**]) is the same as the one of the "local theory of the CP" of (NLW) (See Theorem 1.4, Remark 1.6). In our applications, we will use the following remark:

REMARK 9.9.

a) $V(x,t) = W(x)$. Then $\exists$ small $t_0 > 0$ such that the conditions hold, with $I = (-2t_0, 2t_0)$.

b)

$$V(x,t) = \begin{cases} W(x), \text{ if } |x| > R_0 + |t| \\ W(R_0 + |t|), \text{ if } |x| \le R_0 + |t|, \end{cases}$$

where $R_0 > 0$. Then, for R_0 large, the conditions hold with $I = (-\infty, +\infty)$.

Remark 9.9 is proved using the Leibniz rule for fractional derivatives (See [**33**], Appendix A).

To motivate what follows, we start out by pointing out the following "dispersive property" of non-zero solutions v to (LW): $\exists R > 0, \eta > 0$ such that for all $t \ge 0$, or for all $t \le 0$,

$$\int_{|x| \ge R + |t|} |\nabla v(x,t)|^2 + (\partial_t v(x,t))^2 \, dx \ge \eta > 0.$$

Indeed, if $\int |\nabla v_0|^2 + v_1^2 \neq 0$, since, as we saw earlier, this equals $\int_0^\infty [\partial_r(rv_0)]^2 + (rv_0)^2 dr \neq 0$, we can find $R > 0$ such that $\int_R^\infty [\partial_r(rv_0)]^2 + (rv_1)^2 dr \ge 2\eta > 0$. By our outer energy lower bound, Corollary 7.6, for $t \ge 0$ or for $t \le 0$, we have $\int_{|x| \ge R+|t|} |\nabla v(x,t)|^2 + (\partial_t v(x,t))^2 \, dx \ge \eta > 0$, as claimed. We call this property the "channel of energy" property. We will extend this property to non-zero radial solutions of (NLW), which are global in time and which are not scalings of W, thus providing a dynamical characterization of W.

We start out with two simple claims which will clarify the result.

CLAIM 9.10. Let u be a solution of (NLW), which exists for all time (positive). Then, $\lim_{R\to\infty} \sup_{t>0} \int_{|x|>t+R} |\nabla u(t)|^2 + |\partial_t u(t)|^2 = 0$.

PROOF. Let $\eta > 0$ be given, choose R_0 large such that $\int_{|x|>R_0} |\nabla u_0|^2 + u_1^2 \le \eta^2$. Let $\left(\widetilde{u_{0,R_0}}, \widetilde{u_{1,R_0}}\right) = \Psi_{R_0}(u_0, u_1)$. For η small, $\widetilde{u_{R_0}}$ exists for all time, scatters and we have $\sup_t \left\| \vec{\widetilde{u}}_{R_0}(t) \right\|_{\dot H^1 \times L^2} \le C\eta$. But, finite speed of propagation shows that for $|x| \ge R_0 + t, \widetilde{u}_{R_0}(x,t) = u(x,t)$, giving our result. $\square$

CLAIM 9.11. Let u be a global in time solution of (NLW), such that for some $R > 0$, $\overline{\lim}_{t\uparrow\infty} \int_{|x|>R+t} |\nabla u(t)|^2 + |\partial_t u(t)|^2 > 0$. Then, $\exists \eta > 0$ such that $\int_{|x|>R+t} |\nabla u(t)|^2 + |\partial_t u(t)|^2 \ge \eta, \forall t \ge 0$.

PROOF. If not, $\exists \{t_n\}, t_n \ge 0$ such that $t_n \uparrow \bar t \in (0, \infty]$, and

$$\lim_{n\to\infty} \int_{|x| \ge R + t_n} |\nabla u(t_n)|^2 + |\partial_t u(t_n)|^2 = 0.$$

Let u_n be the solution of (NLW) such that

$$(u_n(t_n), \partial_t u_n(t_n)) = \Psi_{R+t_n}(u(t_n), \partial_t u(t_n)).$$

Then, $\lim_n \|(u_n(t_n), \partial_t u_n(t_n))\|_{\dot H^1\times L^2} = 0$. Thus, for large n, u_n exists globally and scatters. By the small data theory, if $\varepsilon > 0$ is given and n is chosen so large that

$$\|\vec{u_n}(t_n)\|_{\dot H^1\times L^2} \le \varepsilon,$$

then for all t, $\|\vec{u_n}(t)\|_{\dot H^1\times L^2} \le C\varepsilon$. By finite speed of propagation, for all t, we have

$$\vec{u_n}(t_n+t) = \vec{u}(t_n+t)$$

for $|x| > R + t_n + |t|$. Hence, $\overline{\lim}_{t\uparrow\infty} \int_{|x|\ge R+t} |\nabla_x u(t)|^2 + |\partial_t u(t)|^2 < C\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we reach a contradiction. $\square$

Remark 9.12. Both claims are also valid for $t \le 0$.

Proposition 9.13. *Let u be a global in time, radial solution of* (NLW) *such that for some $R > 0$,*

$$\lim_{t\uparrow+\infty} \int_{|x|>R+t} |\nabla u(t)|^2 + (\partial_t u(t))^2 = \lim_{t\downarrow-\infty} \int_{|x|>R+|t|} |\nabla u(t)|^2 + (\partial_t u(t))^2 = 0.$$

Then, either (u_0,u_1) is compactly supported, or $\exists\lambda > 0, i \in \{\pm 1\}$ such that $(u_0,u_1) - \left(\frac{i}{\lambda^{\frac12}} W\left(\frac{x}{\lambda}\right), 0\right)$ is compactly supported.

In order to prove Proposition 9.13, we need a couple of lemmas.

Lemma 9.14. *Let u be as in Proposition 9.13. Let $v(r,t) = ru$, $v_0 = ru_0, v_1 = ru_1$. Then, there exists $C_0 > 0$ such that if for some $r_0 > 0$ we have*

$$\int_{r_0}^{\infty} \left[(\partial_r u_0)^2 + u_1^2\right] r^2 dr \le \delta_0,$$

where δ_0 is small, then

$$\int_{r_0}^{\infty} \left[(\partial_r v_0)^2 + v_1^2\right] dr \le C_0 \frac{|v_0(r_0)|^{10}}{r_0^5}.$$

Furthermore, for r, r', $r_0 \le r \le r' \le 2r$, we have

$$|v_0(r) - v_0(r')| \le \sqrt{C_0}\frac{|v_0(r)|^5}{r^2} \le \sqrt{C_0}\delta_0^2 |v_0(r)|.$$

Proof. Assume first the first statement. We then show the second one. By the fundamental theorem,

$$\begin{aligned} |v_0(r) - v_0(r')| &\le \left|\int_r^{r'} \partial_r v_0(s) ds\right| \le \sqrt{r}\sqrt{\int_r^\infty \left[\partial_r v_0(s)\right]^2 ds} \\ &\le \sqrt{C_0 r}\frac{|v_0(r)|^5}{r_0^{\frac52}} = \sqrt{C_0}\frac{|v_0(r)|^5}{r_0^2}. \end{aligned}$$

Also, if $r \ge r_0$, $\frac{1}{r}v_0^2(r) = ru_0^2(r) \le \int_r^\infty \left[\partial_s u_0(s)\right]^2 s^2 ds \le \delta_0$, which gives the second inequality in the last line of the statement.

We now prove the first inequality. Let u_L be the solution of (LW), with data (u_0,u_1) and let $v_L = ru_L$. By Corollary 7.6 (outer energy lower bound), for all $t \ge 0$, or for all $t \le 0$,

$$\int_{r_0+|t|}^{\infty} \left[(\partial_r u_L(t))^2 + (\partial_t u_L(t))^2\right] r^2 dr \ge \frac12 \int_{r_0}^\infty (\partial_r v_0)^2 + v_1^2.$$

Let now $(\widetilde{u_0}, \widetilde{u_1}) = \Psi_{r_0}(u_0, u_1), \widetilde{u_L}$ the solution of (LW) with data $(\widetilde{u_0}, \widetilde{u_1})$. By assumption, $\|(\widetilde{u_0}, \widetilde{u_1})\|^2_{\dot{H}^1\times L^2} \leq \delta_0$. If δ_0 is taken small enough, the "local theory of the Cauchy problem" (Theorem 1.4, Remark 1.6) gives that for all $t \in \mathbb{R}$,

$$\begin{aligned}
&\left\|\left(\vec{\widetilde{u}} - \vec{\widetilde{u_L}}\right)(t)\right\|_{\dot{H}^1\times L^2} \leq C\,\|(\widetilde{u_0}, \widetilde{u_1})\|^5_{\dot{H}^1\times L^2}\\
&= C\left[\int_{r_0}^{\infty}\left([\partial_r u_0]^2 + u_1^2\right) r^2 dr\right]^{\frac{5}{2}}\\
&\overset{\text{(integration by parts)}}{=} C\left[\int_{r_0}^{\infty}\left([\partial_r v_0]^2 + v_1^2\right) dr + r_0 u_0^2(r_0)\right]^{\frac{5}{2}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_{r_0+|t|}^{\infty}\left([\partial_r \widetilde{u}_L(t)]^2 + [\partial_t \widetilde{u}_L(t)]^2\right) r^2 dr \;&\leq\; 2\int_{r_0+|t|}^{\infty}\left([\partial_r \widetilde{u}(t)]^2 + [\partial_t \widetilde{u}(t)]^2\right) r^2 dr\\
&\quad + C\left[\int_{r_0}^{\infty}\left[(\partial_r v_0)^2 + v_1^2\right] dr + r_0 u_0^2(r_0)\right]^5.
\end{aligned}$$

By finite speed of propagation, $\vec{u}(r,t) = \vec{\widetilde{u}}(r,t)$ and $\vec{u}_L(r,t) = \vec{\widetilde{u}}_L(r,t)$, for $r > r_0 + |t|$. Thus,

$$\begin{aligned}
\int_{r_0+|t|}^{\infty}\left([\partial_r u_L(t)]^2 + [\partial_t u_L(t)]^2\right) r^2 dr \;&\leq\; 2\int_{r_0+|t|}^{\infty}\left([\partial_r u(t)]^2 + [\partial_t u(t)]^2\right) r^2 dr\\
&\quad + C\left[\int_{r_0}^{\infty}\left[(\partial_r v_0)^2 + v_1^2\right] dr + r_0 u_0^2(r_0)\right]^5.
\end{aligned}$$

Combining this with our outer energy lower for u_L, we see that, for all $t \geq 0$, or for all $t \leq 0$,

$$\begin{aligned}
\int_{r_0}^{\infty}\left[(\partial_r v_0)^2 + v_1^2\right] dr \;&\leq\; 4\int_{r_0+|t|}^{\infty}\left([\partial_r u(t)]^2 + [\partial_t u(t)]^2\right) r^2 dr\\
&\quad + C\left[\int_{r_0}^{\infty}\left[(\partial_r v_0)^2 + v_1^2\right] dr + r_0 u_0^2(r_0)\right]^5.
\end{aligned}$$

Letting $t \to \pm\infty$, according to whether the above holds for $t \geq 0$, or $t \leq 0$ and using our hypothesis, we obtain

$$\int_{r_0}^{\infty}\left[(\partial_r v_0)^2 + v_1^2\right] dr \leq C\left[\int_{r_0}^{\infty}\left[(\partial_r v_0)^2 + v_1^2\right] dr + r_0 u_0^2(r_0)\right]^5.$$

Since $\int_{r_0}^{\infty}\left[(\partial_r v_0)^2 + v_1^2\right] dr \leq \int_0^{\infty}\left[(\partial_r u_0)^2 + u_1^2\right] r^2 dr \leq \delta_0$, if δ_0 is small we can neglect this term in the right hand side. Noticing that $r_0^5 u_0^{10}(r_0) = \frac{v_0^{10}(r_0)}{r_0^5}$, we obtain

$$\int_{r_0}^{\infty}\left[(\partial_r v_0)^2 + v_1^2\right] dr \leq C\frac{v_0^{10}(r_0)}{r_0^5},$$

as desired. □

LEMMA 9.15. *The function* $v_0(r)$ *has a limit* $l \in \mathbb{R}$ *as* $r \to \infty$. *Furthermore,* $\exists C > 0$ *such that* $\forall r \geq 1, |v_0(r) - l| \leq \frac{C}{r^2}$.

PROOF. First note that $\exists C > 0$ such that

$$|v_0(r)| \leq Cr^{\frac{1}{10}}.$$

Indeed, by the second bound in the second line in Lemma 9.14,

$$\left|v_0(2^{n+1}r_0) - v_0(2^n r_0)\right| \leq \sqrt{C_0}\delta_0^2 \left|v_0(2^n r_0)\right|,$$

so that $\left|v_0(2^{n+1}r_0\right| \leq \left[1+\sqrt{C_0}\delta_0^2\right]|v_0(2^n r_0)|$. Iterating, we obtain $|v_0(2^n r_0)| \leq \left[1+\sqrt{C_0}\delta_0^2\right]^n |v_0(r_0)|$. Choosing a smaller δ_0 if necessary, we can assume that $\left(1+\sqrt{C_0}\delta_0^2\right) \leq 2^{\frac{1}{10}}$, which then shows that

$$|v_0(2^n r_0)| \leq 2^{\frac{n}{10}} |v_0(r_0)|.$$

This shows the inequality for $r = 2^n r_0$. The general case follows from the difference estimate in the second bound in the second line in Lemma 9.14.

Next, we prove that

$$\lim_{r\to\infty} v_0(r) = l \in \mathbb{R}.$$

By the first inequality in the second line of the conclusion in Lemma 9.14, we have, for $n \in \mathbb{N}$,

$$\left|v_0(2^n r_0) - v_0(2^{n+1}r_0)\right| \leq \sqrt{C_0}\frac{|v_0|\,(2^n r_0)^5}{(2^n r_0)^2}.$$

Using our bound on $|v_0(r)|$, we then obtain

$$\left|v_0(2^n r_0) - v_0(2^{n+1}r_0)\right| \leq \frac{C}{[2^n]^{2-\frac{5}{10}}} = \frac{C}{2^{\frac{3n}{2}}}.$$

Hence, $\sum_{n\geq 0}\left|v_0(2^n r_0) - v_0(2^{n+1}r_0)\right| < \infty$, which gives that $\lim_{n\to\infty} v_0(2^n r) = l \in \mathbb{R}$. Using again that $|v_0(r)| \leq Cr^{\frac{1}{10}}$ and our difference estimate, we conclude that $\lim_{r\to\infty} v_0(r) = l$.

Now, since $v_0(r)$ converges as $r \to \infty$, it is bounded. Thus, for $r \geq r_0, n \in \mathbb{N}$,

$$\left|v_0(2^{n+1}r) - v_0(2^n r)\right| \leq \frac{C}{(2^n r)^2},$$

by the first estimate in the second line of Lemma 9.14. Adding, we get

$$|l - v_0(r)| = \left|\sum_{n\geq 0}\left[v_0(2^{n+1}r) - v_0(2^n r)\right]\right| \leq \frac{C}{r^2}\sum_{n\geq 0}\frac{1}{4^n} = \frac{C}{r^2}.$$

as desired. $\square$

We now conclude the proof of Proposition 9.13. We distinguish two cases, $l = 0$ and $l \neq 0$.

Case $l = 0$: In this case we will show that (v_0, v_1) is compactly supported.

Fix a large r and use the second inequality in Lemma 9.14, together with the smallness of δ_0, to see that

$$\left|v_0(2^{n+1}r) - v_0(2^n r)\right| \leq \sqrt{C_0}\delta_0^2 \left|v_0(2^n r)\right| \leq \frac{1}{4}\left|v_0(2^n r)\right|$$

and hence, $\left|v_0(2^{n+1}r)\right| \geq \frac{3}{4}|v_0(2^n r)|$. Iterating, we get $|v_0(2^n r)| \geq \left(\frac{3}{4}\right)^n |v_0(r)|$. Since $l = 0$, Lemma 9.15 gives that $|v_0(2^n r)| \leq \frac{C}{2^{2n}r^2} = \frac{C}{4^n r^2}$.

Hence for all $n \in \mathbb{N}$, $|v_0(r)| \left(\frac{3}{4}\right)^n \leq \frac{C}{4^n r^2}$, which shows that $v_0(r) \equiv 0$ for $r > r_0$. Since, by the first inequality in Lemma 9.14, we have

$$\int_r^\infty \left[\partial_s v_0(s) + v_1^2(s)\right] ds \leq C_0 \frac{|v_0(r)|^{10}}{r^5},$$

we see that v_1 also has compact support.

Case $l \neq 0$: In this case, we show that $\exists \lambda > 0$ and sign $\pm$ such that

$$\left(u_0 \pm \frac{1}{\lambda^{\frac{1}{2}}} W\left(\frac{x}{\lambda}\right), u_1\right)$$

has compact support.

Note that, for large r, $\left|\frac{1}{\lambda^{\frac{1}{2}}} W\left(\frac{r}{\lambda}\right) - \frac{\sqrt{3}\lambda^{\frac{1}{2}}}{r}\right| \leq \frac{C}{r^3}$, which follows from $W(r) = \frac{1}{(1+\frac{r^2}{3})^{\frac{1}{2}}}$. Hence, Lemma 9.15 implies that $\exists C > 0$ such that

$$\left|\pm \frac{1}{\lambda^{\frac{1}{2}}} W\left(\frac{x}{\lambda}\right) - u_0(r)\right| \leq \frac{C}{r^3},$$

where $\lambda = \frac{l^2}{3}$, and the sign $\pm$ is the sign of l (by Lemma 9.15, $|ru_0(r) - l| \leq \frac{C}{r^2}, r \geq 1$).

Rescaling u and possibly replacing u by $-u$, we can assume that $|u_0(r) - W(r)| \leq \frac{C}{r^3}, r \geq 1$. Let $h = u - W, H = rh$.

Claim: For a large $R_0, \forall r_0 > R_0$, we have

$$\int_{r_0}^\infty \left[(\partial_r H_0)^2 + H_1^2\right] dr \leq \frac{1}{16} \frac{H_0^2(r_0)}{r_0},$$

where $(H_0, H_1) = (H, \partial_t H)|_{t=0}$. Let us assume the Claim, and conclude that $(H_0(r), H_1(r)) = (0, 0)$ for large r. Indeed, the claim implies, for large r, $n \in \mathbb{N}$ that

$$\begin{aligned}
\left|H_0(2^{n+1}r) - H_0(2^n r)\right| &\leq 2^{\frac{n}{2}} \sqrt{r} \left(\int_{2^n r}^{2^{n+1} r} \left[\partial_s H_0(s)\right]^2 ds\right)^{\frac{1}{2}} \\
&\leq 2^{\frac{n}{2}} \sqrt{r} \frac{1}{4} \frac{|H_0(2^n r)|}{2^{\frac{n}{2}} \sqrt{r}} = \frac{1}{4} \left|H_0\left(2^n r\right)\right|,
\end{aligned}$$

so that $\left|H_0(2^{n+1}r)\right| \geq \left(\frac{3}{4}\right) |H_0(2^n r)|$ and hence,

$$|H_0(2^n r)| \geq \left(\frac{3}{4}\right)^n |H_0(r)|.$$

Since $|u_0(R) - W(r)| \leq \frac{C}{r^3}, r \geq 1, |H_0\left(2^n r\right)| \leq \frac{C}{4^n r^2}$, which letting $n \to \infty$ gives $H_0(r) \equiv 0$. Thus, H_0 is compactly supported and the claim shows that the same holds for H_1. It remains to show the claim.

To do this, let R_0 be large,

$$V(x,t) = \begin{cases} W(x) \text{ if } |x| > R_0 + |t| \\ W(R_0 + |t|) \text{ if } |x| \leq R_0 + |t| \end{cases}$$

as in Remark 9.9.

Define $(g_0, g_1) = \Psi_{r_0}(h_0, h_1)$. Let g_L be the solution of (LW) with this data. Let g be the solution of

$$\begin{cases} \partial_t^2 g - \Delta g = (V+g)^5 - v^5 \\ g|_{t=0} = g_0 \\ \partial_t g|_{t=0} = g_1 \end{cases}$$

given by Lemma 9.7. Thus, g is globally defined and $\sup_{t\in\mathbb{R}} \left\| \vec{g(t)} - \vec{g_L}(t) \right\|_{\dot{H}^1\times L^2} \leq \frac{1}{10} \|(g_0, g_1)\|_{\dot{H}^1\times L^2}$.

Recall also from our outer energy lower bound (Corollary 7.6) that, for all $t \geq 0$ or for all $t \leq 0$,

$$\int_{|x|>r_0+|t|} \left[|\nabla g_L(t)|^2 + |\partial_t g_L(t)|^2 \right] \geq \frac{1}{2} \int_{r_0}^{\infty} \left[(\partial_r H_0)^2 + H_1^2 \right] dr.$$

Hence, for all $t \geq 0$ or all $t \leq 0$,

$$\begin{aligned} \int_{r_0}^{\infty} \left[(\partial_r H_0)^2 + H_1^2 \right] dr &\leq 2 \int_{|x|\geq r_0+|t|} \left[|\nabla g_L(t)|^2 + (\partial_t g_L(t))^2 \right] \\ &\leq 4 \int_{|x|\geq r_0+|t|} \left[|\nabla g(t)|^2 + (\partial_t g(t))^2 \right] \\ &\quad + \frac{1}{25} \int_{r_0}^{\infty} \left[(\partial_r g_0)^2 + g_1^2 \right] r^2 dr. \end{aligned}$$

By finite speed of propagation, $\vec{g}(r,t) = \vec{h}(r,t), r \geq r_0 + |t|$. Using that

$$\lim_{t\to\pm\infty} \int_{r_0+|t|}^{\infty} |\partial_r W|^2 r^2 dr = 0,$$

and our hypothesis in Proposition 9.13, $(h(r,t) = u(r,t) - W(r))$ and letting $t \to +\infty$ or $-\infty$, according to where the above holds, we obtain

$$\begin{aligned} \int_{r_0}^{\infty} \left[(\partial_r H_0)^2 + H_1^2 \right] &\leq \frac{1}{25} \int_{r_0}^{\infty} \left[(\partial_r g_0)^2 + g_1^2 \right] r^2 dr \\ &= \frac{1}{25} \left[\int_{r_0}^{\infty} \left[(\partial_r H_0)^2 + H_1^2 \right] dr + \frac{1}{r_0} H_0^2(r_0) \right], \end{aligned}$$

since $(H_0, H_1) = (rh_0, rh_1) = (rg_0, rg_1)$ for $r > r_0$ and where we have used integration by parts. This gives the Claim, and thus Proposition 9.13.

Before proceeding towards the proof of Theorem 9.1, we would like to point out that Proposition 9.13 can be used to give a proof of the rigidity Theorem 4.17 (from [**32**]) which says that if a radial solution of (NLW) has the "compactness property", up to scaling , it must be 0 or $\pm W$. This proof comes from [**34**].

THEOREM 9.16 (Rigidity Theorem). *Let u be a non-zero radial solution of* (NLW), $K = \left\{ \left(\lambda^{-\frac{1}{2}}(t) u\left(\frac{x}{\lambda(t)}, t\right), \lambda(t)^{-\frac{3}{2}} \partial_t u\left(\frac{x}{\lambda(t)}, t\right) \right) : t \in I_{\max}(\omega), \lambda(t) > 0 \right\}$. *Assume that for some $\lambda(t)$, with* $\inf_{t\in I} \lambda(t) > 0$ $\overline{K}$ *is compact in* $\dot{H}^1 \times L^2$. *Then,* $\exists \lambda_0 > 0, i_0 \in \{\pm 1\}$ *such that* $u(x,t) = \frac{i_0}{\lambda_0^{\frac{1}{2}}} W\left(\frac{x}{\lambda_0}\right)$.

Theorem 9.16 many times suffices. To obtain the full Theorem 4.17, extra work is needed.

It should be pointed out though, that Theorem 9.16, combined with the "no self-similar compact blow-up" result in [**62**], Property 4.29 (if $T_+ = 1$ and $\widetilde{K} = \left\{(1-t)^{-\frac{1}{2}} u\left((1-t)x, t\right), (1-t)^{\frac{3}{2}} \partial_t u\left((1-t)x, t\right)\right\}$ is precompact in $\dot{H}^1 \times L^2$, then u cannot exist) show that it suffices to prove the full Theorem 4.17, when $I = (-\infty, +\infty)$. This is a "general property" that can be found in [**34**]. In the radial case, the proof of Property 4.29 simplifies considerably (see [**34**] for this). A proof, also in the radial case, of Property 4.29 using the "channel of energy property" can also be obtained, for this, see [**30**]. We will now sketch the proof of Theorem 9.16, $I = (-\infty, +\infty)$, using Proposition 9.13. Let $A_0 = \inf_{t\in(-\infty,+\infty)} \lambda(t) > 0$.

The pre-compactness in $L^2(\mathbb{R}^3)$ of

$$\left\{\vec{v}(t) = \left(\lambda(t)^{-\frac{3}{2}} \nabla u\left(\frac{x}{\lambda(t)}, t\right), \lambda(t)^{-\frac{3}{2}} \partial_t u\left(\frac{x}{\lambda(t)}, t\right)\right), t \in (-\infty, +\infty)\right\}$$

implies that, given $\varepsilon > 0$, there exist $R_0 > 0$, uniformly in t, such that

$$\int_{|x|>R} |\vec{v}(t)|^2 \, dx \le \varepsilon,$$

for $R \ge R_0$, (and all t).

Changing variables, and using that $A_0 > 0$, we see that $\exists \widetilde{R_0} \left(= \frac{R_0}{A_0}\right)$ such that if $R \ge \widetilde{R_0}$, then

$$\int_{|x|>\widetilde{R_0}} |\nabla u(t)|^2 + |\partial_t u(t)|^2 \le \varepsilon.$$

As a consequence, for any $R > 0$, we have that

$$\lim_{t\to\pm\infty} \int_{|x|>R+|t|} |\nabla u(t)|^2 + |\partial_t u(t)|^2 = 0.$$

Hence, Proposition 9.13 says that, either (u_0, u_1) has compact support, or $\exists \lambda_0 > 0, i_0 \in (\pm 1)$ such that $\left(u_0 - \frac{i_0}{\lambda_0^{\frac{1}{2}}} W\left(\frac{x}{\lambda}\right), u_1\right)$ has compact support. To continue with the proof, for (f_0, f_1) radial, $(f_0, f_1) \in \dot{H}^1 \times L^2$, we denote $\rho(f_0, f_1) = \inf\{r > 0 : |\{s > r : (f_0(s), f_1(s)) \neq (0,0)\}| = 0\}$. We make the convention $\rho(f_0, f_1) = \infty$ if the set over which the inf is taken is $\emptyset$.

Assume first that $\rho_0 = \rho(u_0, u_1) > 0, \rho_0 < \infty$. (This means that (u_0, u_1) has compact support, but is not $\equiv (0,0)$). We will reach a contradiction. Let $\varepsilon = \min\left(\frac{1}{2\sqrt{C_0}}, \delta_0\right)$, where C_0, δ_0 come from Lemma 9.14. Using the definition of r_0 and the continuity of u_0 outside the origin, we can choose $r_1 \in (0, r_0)$, r_1 close to r_0, such that $u_0(r_1) \neq 0$ and $\int_{r_1}^{\infty} \left[(\partial_r u_0)^2 + u_1^2\right] r^2 dr + \frac{|v_0(r_1)|^4}{r_1^2} < \varepsilon$, where $v_0(r) = r u_0(r), v_1(r) = r u_1(r)$.

By the estimate from Lemma 9.14, which says that, if $\int_{r_0}^{\infty} \left[(\partial_r u_0)^2 + u_1^2\right] r^2 dr \le \delta_0$, δ_0 small, then $|v_0(r) - v_0(r')| \le \sqrt{C_0} \frac{|v_0(r)|^5}{r^2}$, when $r_0 \le r \le r' \le 2r$, we obtain

$$|v_0(\rho_1)| = |v_0(\rho_1) - v_0(\rho_0)| \le \frac{\sqrt{C_0}\, |v_0(\rho_1)|^5}{\rho_1^2} \le \sqrt{C_0} \varepsilon \, |v_0(\rho_1)|,$$

a contradiction since $\varepsilon\sqrt{C_0} < 1$ and $v_0(\rho_1) \neq 0$.

Next, after rescaling and possible change of sign, we know that $(u_0 - W(x), u_1)$ has compact support. Repeating the proof of Proposition 9.13, for each t, and noticing that the compactness property, with the lower bound on $\lambda(t)$, gives uniform in t estimates, we see that Lemma 9.15 gives that, for each t, $\left|u(r,t)-\frac{l(t)}{r}\right| \le \frac{C}{r^3}, r \ge 1$ where $l(t)$ is bounded in t and C is independent of t. Moreover, our normalization gives $l(0)=\frac{1}{3}$. We next show that $l(t)$ is independent of t. Fix $t_1 < t_2$. Then, $l(t_2)-l(t_1)=\frac{1}{R}\int_R^{2R}\left[u(r,t_2)-u(r,t_2)\right]rdr + O(R^{-2})$ as $R\to\infty$. Thus,

$$\begin{aligned}
|l(t_2)-l(t_1)| &= \left|\frac{1}{R}\int_R^{2R}\int_{t_1}^{t_2}\partial_t u(r,t) r dr dt\right| + O(R^{-2}) \\
&\le \int_{t_1}^{t_2}\left(\frac{1}{R}\int_R^{2R}\left|\partial_t u(r,t)\right|^2 r^2 dr\right)^{\frac{1}{2}} dt + O(R^{-2}) \\
&\le CR^{-\frac{1}{2}}\left|t_1-t_2\right| + O(R^{-2}),
\end{aligned}$$

so that $l(t_1)=l(t_2)$ and hence, by our normalization at $t=0$, $l(t)\equiv\frac{1}{3}$. Following the proof of Proposition 9.13, we see that $\exists R_0$ such that $\operatorname{supp}\ (u(t)-W,\partial_t u(t)) \subset B_{R_0}$, where R_0 is independent of t.

For each $t\in\mathbb{R}$, we let $\rho(t)=\rho\left(u(t)-W,\partial_t u(t)\right)$. We also let $\rho_{\max}=\sup_{t\in\mathbb{R}}\rho(t), r_0=\frac{\rho_{\max}}{2}$. By contradiction, assume that $(u_0,u_1)\not\equiv(W,0)$. Then, $\rho_{\max}>0$, and $\rho_{\max}\le R_0$. Let $V(x,t)=W(x)$, choose t_0 as in Remark 9.9 a). Choosing a smaller t_0 if necessary, we can assume that $\rho_{\max}-\frac{t_0}{2}>0$. Choose $t_1\in\mathbb{R}$ such that $\rho(t_1)\ge\rho_{\max}-\frac{t_0}{2}>0$. Translating in time, we assume $t_1=0$. Choose $r_1\in(0,\rho(0))$ such that $0<\frac{\rho(0)-r_1}{10r_1}<\frac{1}{2}, r_1+t_0>\rho_{\max}$, and

$$0<\int_{r_1}^{\infty}\left[\left(\partial_r h_0\right)^2+h_1^2\right]r^2dr<\delta_0,$$

(where $(h_0,h_1)=(u(0)-W,\partial_t u(0))$). We now apply the argument in the proof of Proposition 9.13, case $l\ne 0$, in the interval $I=[-t_0,t_0]$. Then, for all $t\in[0,t_0]$ or all $t\in[-t_0,0]$, we have

$$\int_{r_1}^{\infty}\left[\left(\partial_r H_0\right)^2+H_1^2\right]\le 5\int_{r_1+|t|}^{\infty}\left[\left(\partial_r g(r,t)\right)^2+\left(\partial_t g(r,t)\right)^2\right]r^2dr+\frac{1}{10r_1}H_0^2(r_1).$$

$(h=u-W, H=rh, g=\Psi_{r_1}(h_0,h_1))$.

Since $r_1+t_0>\rho_{\max}$, $\int_{r_1+|t|}\left[\left(\partial_r g(r,t)\right)^2+\left(\partial_t g(r,t)\right)^2\right]r^2dr=0$ at $t=\pm t_0$. Hence,

$$\begin{aligned}
\int_{r_1}^{\infty}\left[\left(\partial_r H_0\right)^2+H_1^2\right]dr &\le \frac{1}{10r_1}H_0^2(r_1) \\
&\le \frac{1}{10r_1}\left(\int_{r_1}^{\rho(0)}\left|\partial_r H_0\right|dr\right)^2 \\
&\le \frac{1}{10r_1}\left[\rho(0)-r_1\right]\int_{r_1}^{\rho(0)}\left[\partial_r H_0\right]^2dr.
\end{aligned}$$

Since $\frac{1}{10r_1}\left[\rho(0)-r_1\right]\le\frac{1}{2}$, we see that $\int_{r_1}^{\infty}\left[\left(\partial_r H_0\right)^2+H_1^2\right]dr=0$. By the compact support of H_0, it follows that $\int_{r_1}^{\infty}\left[\left(\partial_r h_0\right)^2+h_1^2\right]r^2dr=0$, which contradicts the fact that $0<\int_{r_1}^{\infty}\left[(\partial_r h_0)^2+h_1^2\right]r^2dr$. This completes the proof.

We now return to the proof of Theorem 9.1. We will need two propositions:

PROPOSITION 9.17. *Let u be a non-zero radial solution of* (NLW) *such that $\forall\lambda > 0$ and all $\pm$ signs, $\left(u_0 \pm \frac{1}{\lambda^{\frac{1}{2}}} W\left(\frac{x}{\lambda}\right), u_1\right)$ is not compactly supported. Then, there exist $R > 0$, $\eta > 0$ and $\widetilde{u}$ a globally defined solution of* (NLW), *such that $\widetilde{u}$ scatters in both time directions and for all $t \geq 0$ or for all $t \leq 0$*

$$\int_{|x|>R+|t|} |\nabla \widetilde{u}(x,t)|^2 + (\partial_t \widetilde{u}(x,t))^2 \, dx \geq \eta,$$

and $\widetilde{u}(x,t) = u(x,t)$ for $|x| > R + |t|$.

PROOF. Assume first that (u_0, u_1) is not compactly supported. Let $(\widetilde{u_0}, \widetilde{u_1}) = \Psi_R(u_0, u_1)$, where $R > 0$ is chosen so large that $0 < \|(\widetilde{u}_0, \widetilde{u}_1)\|_{\dot{H}^1 \times L^2} < \widetilde{\delta}$, where $\widetilde{\delta}$ is given by the Remark 1.6. By Claim 9.11, the conclusion is verified for $\widetilde{u}$ unless $\lim_{t\uparrow+\infty} \int_{|x|>R+|t|} |\nabla_x \widetilde{u}(t)|^2 + (\partial_t \widetilde{u}(t))^2 = \lim_{t\downarrow-\infty} \int_{|x|>R+|t|} |\nabla_x \widetilde{u}(t)|^2 + (\partial_t \widetilde{u}(t))^2 = 0$.

But, in this case, by Proposition 9.13, $(\widetilde{u}_0, \widetilde{u}_1)$ is either compactly supported (which is excluded since we assumed that (u_0, u_1) is not compactly supported), or $\exists \lambda > 0, i \in \{\pm 1\}$ such that $(\widetilde{u}_0, \widetilde{u}_1) - \left(\frac{i}{\lambda^{\frac{1}{2}}} W\left(\frac{x}{\lambda}\right), 0\right)$ is compactly supported, which contradicts our hypothesis.

Thus, let us assume that (u_0, u_1) is compactly supported, and not $(0,0)$. Thus, $0 < \rho(u_0, u_1) < \infty$. Let $0 < R < \rho(u_0, u_1)$ and let $(\widetilde{u}_0, \widetilde{u}_1) = \Psi_R(u_0, u_1)$. Choose now R so close to $\rho(u_0, u_1)$ that $0 < \|(\widetilde{u}_0, \widetilde{u}_1)\|_{\dot{H}^1 \times L^2} \leq \widetilde{\delta}$, where $\widetilde{\delta}$ is given by Remark 1.6. Let $\widetilde{u}$ be the corresponding solution of (NLW), which exists globally and scatters, and $\widetilde{u}_L$ the solution of (LW). Thus, we have

$$\sup_t \left\| \vec{\widetilde{u}}(t) - \vec{\widetilde{u}}_L(t) \right\|_{\dot{H}^1 \times L^2} \leq \frac{1}{10} \|(\widetilde{u}_0, \widetilde{u}_1)\|,$$

and for all $t \geq 0$, or for all $t \leq 0$,

$$\int_{|x|\geq R+|t|} |\nabla \widetilde{u}_L(t)|^2 + |\partial_t \widetilde{u}_L(t)|^2 \geq \frac{1}{2} \left[\|\nabla \widetilde{u}_0\|^2 + \|\widetilde{u}_0\|^2 - R\widetilde{u}_0^2(R) \right].$$

But, since $\rho(\widetilde{u}_0, \widetilde{u}_1) = \rho(u_0, u_1)$, if R is close enough to $\rho(\widetilde{u}_0, \widetilde{u}_1)$, then $R|\widetilde{u}_0(R)|^2 \leq \frac{1}{4}\|\nabla \widetilde{u}_0\|^2$, so that $\frac{1}{2}\left[\|\nabla \widetilde{u}_0\|^2 + \|\widetilde{u}_1\|^2 - R\widetilde{u}_0^2(R)\right] \geq \frac{3}{8}\left[\|\nabla \widetilde{u}_0\|^2 + \|\widetilde{u}_1\|^2\right]$. Combining our inequalities we obtain the "channel property" for $\widetilde{u}$, as desired. □

PROPOSITION 9.18. *Let $R_0 > 0$ be a large constant to be chosen. Then, the following holds: let u be a radial solution of* (NLW) *such that $(h_0, h_1) = (u_0 \pm W, u_1)$ is compactly supported and not $\equiv 0$. Then,*

a) *$\exists$ a solution $\check{u}$ of* (NLW), *defined for $t \in [-R_0, R_0]$ and $R' \in (0, \rho(h_0, h_1))$ such that*

$$(\check{u}_0(r), \check{u}_1(r)) = (u_0(r), u_1(r))$$

for $r > R'$, and the following holds: for all $t \in [0, R_0]$ or for all $t \in [-R_0, 0]$:

$$\rho(\check{u}(t) \pm W, \partial_t \check{u}(t)) = \rho(h_0, h_1) + |t|.$$

b) *Assume further that* $\rho(h_0,h_1) > R_0$. *Let* $R < \rho(h_0,h_1)$ *be close to* $\rho(h_0,h_1)$. *Then,* $\exists \eta > 0$ *and a global radial solution* $\widetilde{u}$, *which scatters, such that*

$$(\widetilde{u}_0(r), \widetilde{u}_1(r)) = (u_0(r), u_1(r)), \text{ for } r > R$$

and for all $t \geq 0$ *or for all* $t \leq 0$

$$\int_{|x|>R+|t|} |\nabla \widetilde{u}(t)|^2 + [\partial_t \widetilde{u}(t)]^2 \geq \eta.$$

Proof. We first prove a), by linearization around W. By assumption, up to a sign change $(u_0,u_1) = (W,0) + (h_0,h_1)$, where $0 < \rho(h_0,h_1) < \infty$. Since W is globally defined, Theorem 1.12 shows that $\exists \varepsilon > 0$ such that for any U with $\|(W,0) - (U_0,U_1)\|_{\dot{H}^1\times L^2} \leq \varepsilon$, we have $[-R_0,R_0] \subset I_{\max}(U)$.

Let $(\check{h}_0,\check{h}_1) = \Psi_{R'}(h_0,h_1)$, where $R' < \rho(h_0,h_1)$ is chosen so close to $\rho(h_0,h_1)$ that $0 < \left\|(\check{h}_0,\check{h}_1)\right\|_{\dot{H}^1\times L^2} \leq \varepsilon$. Let $\check{u}$ be the solution of (NLW), with initial data $(W+\check{h}_0,\check{h}_1)$. Equivalently, $\check{h} = \check{u} - W$ solves

$$\begin{cases} \partial_t^2 \check{h} - \Delta \check{h} = (W+\check{h})^5 - W^5 \\ (\check{h},\partial_t\check{h})\,|_{t=0} = (\check{h}_0,\check{h}_1)\,. \end{cases}$$

By finite speed, $(\check{h},\partial_t\check{u}) = (W,0)$, $r \geq \rho(h_0,h_1) + |t|$. Thus, $\rho\left(\check{h}(t),\partial_t\check{h}(t)\right) \leq \rho(h_0,h_1) + |t|$, for $t \in [-R_0,R_0]$. We need to show that for all $t \in [-R_0,0]$, or for all $t \in [0,R_0]$,

$$\rho(\check{h}(t),\partial_t\check{h}(t)) = \rho(h_0,h_1) + |t|\,. \tag{9.19}$$

We first do this for a small time interval. We know that $\exists t_0 > 0$, small, such that W verifies Lemma 9.7, $I = [-t_0,t_0]$ (Remark 9.9 a)). In this step, we show that (9.19) holds for all $t \in [-t_0,0]$ or for all $t \in [0,t_0]$. Indeed, let ρ_0 be close to $\rho(h_0,h_1)$ such that $R' < \rho_0 < \rho(h_0,h_1)$, and let $(g_0,g_1) = \Psi_{\rho_0}\left(\check{h}_0,\check{h}_1\right)$. If $\rho\,(h_0,h_1) - \rho_0$ is small enough, $\|(g_0,g_1)\|_{\dot{H}^1\times L^2} \leq \delta_0$, where δ_0 is as in Lemma 9.7. By Lemma 9.7, $\exists!$ solution g to

$$\begin{cases} \partial_t^2 g - \Delta g = (W+g)^5 - W^5 \\ (g,\partial_t g)\,|_{t=0} = (g_0,g_1)\,. \end{cases}$$

Also if g_L solves (LW) with the same initial data, $\sup_{-t_0\leq t\leq t_0} \|\vec{g}(t) - \vec{g}_L(t)\| \leq \frac{1}{10}\|(g_0,g_1)\|_{\dot{H}^1\times L^2}$. By Corollary 7.6, for all $t \in [-t_0,0]$, or all $t \in [0,t_0]$, we have

$$\int_{|x|\geq\rho_0+|t|} |\nabla g_L(t)|^2 + (\partial_t g_L(t))^2 \geq \frac{1}{2}\int_{|x|\geq\rho_0} |\nabla g_0|^2 + g_1^2 - \frac{1}{2}\rho_0 g_0^2(\rho_0).$$

By the argument at the end of the proof of Proposition 9.17, if ρ_0 is close enough to $\rho(h_0,h_1)$, $\rho_0 g_0^2(\rho_0) \leq \frac{1}{4}\|\nabla g_0\|$. Thus, for all $t \geq 0$ or all $t \geq 0$, $t \in [-t_0,t_0]$,

$$\int_{\|x\|\geq\rho_0+|t|} |\nabla g(t)|^2 + |\partial_t g(t)|^2 \geq \frac{1}{40}\int_{|x|\geq\rho_0} |\nabla g_0|^2 + |g_1|^2 > 0.$$

By finite speed, we can replace g by $\check{h}$ in the left hand side. Hence, $\rho\left(\check{h}(t),\partial_t\check{h}(t)\right) \geq \rho_0 + |t|$, $\forall t \in [-t_0,0]$ or $\forall t \in [0,t_0]$. Letting $\rho_0 \to \rho(h_0,h_1)$ we see that $\rho\left(\check{h}(t),\partial_t\check{h}(t)\right) = \rho(h_0,h_1) + |t|\,, t \in [-t_0,0]$ or $t \in [0,t_0]$. It is now easy to conclude the proof.

Assume, for instance that this holds, for $t \in [0, t_0]$, we apply the previous argument to $\check{h}(t + t_0)$, to conclude that $\forall t \in [-t_0, 0]$ or $\forall t \in [0, \min[t_0, R_0 - t_0]]$,

$$\rho\left(\check{h}(t_0 + t), \partial_t \check{h}(t_0 + t)\right) = \rho(h_0, h_1) + t_0 + |t| \,.$$

If the above holds $\forall t \in [-t_0, 0]$, we get a contradiction with $\rho(\check{h}(0), \partial_t \check{h}(0)) = \rho(h_0, h_1)$. Thus, $\forall t \in [0, \min(t_0, R_0 - t_0)], \rho\left(\check{h}(t_0 + t), \partial_t \check{h}(t_0 + t)\right) = \rho(h_0, h_1) + t_0 + t$. Continuing we get the desired result.

To prove b), we use the argument in the proof of Proposition 9.17, (u_0, u_1) compactly supported, using instead of (NLW) the equation in Lemma 9.7 with V as in Remark 9.9 b), which determines who R_0 is. $\square$