CHAPTER 1

Three views of Ramsey theory

There are a number of viewpoints which can be taken when studying various classes of Ramsey theorems. We mention several of these now.

Let $(S, \prec)$ be a (finite) set partially ordered by $\prec$ and having a unique minimal element 0. We say that S is *graded* if all maximal chains from any element $x \in S$ to 0 have the same length. In this case we call this length the *rank* of x, and denote it by $\rho(x)$. We usually denote the set of rank k elements of S by $\begin{bmatrix} S \\ k \end{bmatrix}$. Examples of this are:

(a) $S = 2^{[n]}$, the collection of subsets of the set $[n] := \{1, 2, \dots, n\}$ partially ordered by inclusion, and for $x \in S$, $\rho(x) := |x|$, the cardinality of x.

(b) $S =$ the lattice of subspaces of a given n-dimensional vector space V over a fixed finite field $GF(q)$ partially ordered by inclusion, and for $x \in S$, $\rho(x) :=$ dimension of x.

(c) $S =$ collection of partitions of $[n]$ partially ordered by refinement, and for the partition $x = B_1 \cup B_2 \cup \dots \cup B_k$ of $[n]$, $\rho(x) := n - k$.

Let $\mathcal{S} = (S_n, \prec)$, $n \in \mathbb{N} := \{1, 2, 3, \dots\}$, be a sequence of graded partially ordered sets (where we will use the convention $\mathbb{N} = \{1, 2, 3, \dots\}$). We say that $\mathcal{S}$ has the Ramsey property if for any $k, \ell, r \in \mathbb{N}$ there is an n such that if the rank k elements of S_n are arbitrarily partitioned into r classes, there is always a rank ℓ element $y \in S_n$ such that all rank k elements x with $x \prec y$ belongs to a single class. More symbolically:

For all $k, \ell, r \in \mathbb{N}$ there exists n such that for all $\lambda : \begin{bmatrix} S_n \\ k \end{bmatrix} \longrightarrow [r]$ there exists $y \in \begin{bmatrix} S_n \\ \ell \end{bmatrix}$ and $i \in [r]$ so that $\left\{ x \in \begin{bmatrix} S_n \\ k \end{bmatrix} : x \prec y \right\} \subseteq \lambda^{-1}(i)$.

The reader is invited to try this statement out for various families $\mathcal{S}$, for example, with S_n taken to be sets S of maximum rank n in (a), (b), or (c) (as well as for other families). We will see proofs for these particular cases in later chapters.

For another viewpoint, let us consider a bipartite graph G with vertex sets A and B and edge set $E \subseteq A \times B$. We say that G is r-Ramsey if for all mappings $\lambda : B \longrightarrow [r]$ there is an $x \in A$ such that, for some $i \in [r]$, $\{y \in B : (x, y) \in E\} \subseteq \lambda^{-1}(i)$. Much of Ramsey theory can be reduced to determining whether particular graphs are r-Ramsey. However, while conceptually simple, this formulation has not (so far) contributed very much to the solution of specific questions in Ramsey theory. The reason may be simply that it is so general that one is usually not able to take advantage of the special structure of the particular problem at hand. For example, consider the bipartite graph G shown in Figure 1, with top part B and bottom part A, respectively.

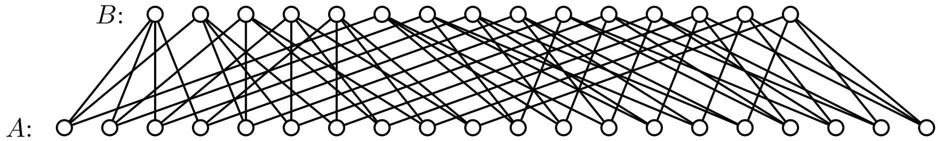

FIGURE 1

It is a fact that G is 2-Ramsey. However a direct verification of this could involve checking each of the 2^{15} mappings $\lambda : B \longrightarrow [2]$. In fact, G is exactly the graph obtained by (suitably) identifying A and B with the 3-sets and 2-sets, respectively of [6] and placing an edge between $x \in A$ and $y \in B$ if $y \subset x$, i.e., we are coloring the edges (B) of a complete graph on six vertices so that some triangle (A) is monochromatic. With this interpretation the fact that G is 2-Ramsey is almost immediate.

EXERCISE 1.1. (a) Verify the claim that G is 2-Ramsey.
(b) Does G remain 2-Ramsey if a vertex of A is deleted? What about deleting two vertices of A? What about deleting one vertex of B? (See Golomb [**Gol**].)

A final point of view we mention is that of hypergraphs. By a hypergraph $\mathcal{H} = \mathcal{H}(V, E)$ we mean a set V together with a family E of subsets of V, each containing at least two elements. The *chromatic number* of $\mathcal{H}$, denoted by $\chi(\mathcal{H})$, is defined to be the least integer t such that there is a mapping $\lambda : V \longrightarrow [t]$ so that there is no $e \in E$ and $i \in [t]$ with $e \subseteq \lambda^{-1}(i)$. The term "chromatic" comes from the following interpretation. We imagine the mapping λ to be an assignment of *colors* to the points of $\mathcal{H}$. If all the points of some edge $e \in E$ are assigned the same color, we say that e is *monochromatic*[1] (or monoχ). Thus, $\chi(\mathcal{H}) = t$ if t is the least integer for which there is a t-coloring of V forming no monochromatic edge of E.

It is not difficult to see the connection between a t-chromatic hypergraph $\mathcal{H}$ and the corresponding (appropriately constructed) $(t-1)$-Ramsey bipartite graph $G = G(\mathcal{H})$.

A fundamental tool which is used quite often in Ramsey theory is (some version of) the compactness theorem of de Bruijn and Erdős [**BrE**]. Before stating it we need one more bit of terminology. A hypergraph $\mathcal{G} = \mathcal{G}(V', E')$ is said to be a subhypergraph of $\mathcal{H} = \mathcal{H}(V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.

COMPACTNESS THEOREM. *If $\chi(\mathcal{H}) > t$ and all edges of $\mathcal{H}$ are finite then there is a finite subhypergraph $\mathcal{G}$ of $\mathcal{H}$ with $\chi(\mathcal{G}) > t$.*

PROOF. (Countable case.) Without loss of generality we can take $V = \mathbb{N}$. Define $\mathcal{H}_n$ to be the subhypergraph with vertex set $V_n = [n]$ and edge set $E_n = \{e \in E : e \subseteq [n]\}$. Suppose $\chi(\mathcal{H}_n) \leq t$ for all $n \in \mathbb{N}$. Thus, there exists $\lambda_m : [n] \longrightarrow [t]$, such that no mono$\chi$ edge is formed. Consider the images $\lambda_n(1)$, $n = 1, 2, 3, \ldots$. By the pigeonhole principle, some value, say $i_1 \in [t]$, occurs infinitely often. Define $\lambda^*(1) = i_1$ and let $n_1 < n_2 < \cdots$ be the indices such that $\lambda_{n_i}(1) = i_1$. Consider the images $\lambda_{n_i}(2)$, $i \geq 2$. As before, some value, say $i_2 \in [t]$, occurs infinitely often.

[1]A synonym for this in common use (especially by set theorists) is homogeneous.

Define $\lambda^*(2) = i_2$ and let $n'_1 < n'_2 < \cdots$ be the subsequence of the n_i such that $\lambda_{n'_i}(2) = i_2$. Consider the images $\lambda_{n'_i}(3)$, $i \geq 3$. Once again, some value, say $i_3 \in [t]$, occurs infinitely often. Define $\lambda^*(3) = i_3$ and let $n''_1 < n''_2 < \cdots$ be the subsequence of the n'_i such that $\lambda_{n''_i}(3) = i_3$. Continuing this argument (which really is an application of the König infinity lemma[2]) we define a mapping $\lambda^* : \mathbb{N} \longrightarrow [t]$ with the property that *no edge of* $\mathcal{H}$ *is* λ^**-monochromatic*, i.e., for all $e \in E$, $i \in [t]$, $e \not\subseteq {\lambda^*}^{-1}(i)$. This follows immediately from the construction of λ^* since for any n there is an n' (in fact, infinitely many) such that

$$\lambda^*(i) = \lambda_{n'}(i), \quad i \in [n].$$

However this contradicts the assumption that $\chi(\mathcal{H}) > t$ and the proof (for the countable case) is completed. □

The proof in the general case requires the use of the Axiom of Choice or something equivalent, such as Tychonoff's theorem, and will not be given here.[3] We note that the Axiom of Choice does matter for some problems, for example in 2-colorings of $\mathbb{R}$ looking for monochromatic solutions to $x + y + z = 4w$; we will discuss this in a later chapter.

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[2]Which asserts that an infinite tree with all vertices having finite degree contains an infinite path.

[3]This result also follows from the compactness theorem for propositional calculus.

CHAPTER 11

Current trends

Throughout this book we have given open problems and have mentioned some of the current trends that are active in Ramsey theory. In this final chapter we will briefly indicate some additional directions along which Ramsey theory is being developed. Space limitations prevent us from giving more than passing acknowledgment to these exciting new areas but we hope that the reader may at some time have the inclination and opportunity to explore some of these more fully.[1]

One of the greatest achievements in the area of combinatorial number theory has been Szemerédi's theorem. There now exist a number of different proofs of this result, but one of the earliest was given by Furstenberg using ergodic theory. These ergodic techniques were used by Furstenberg and Katznelson [**FuK1**] to prove the n-dimensional analogue of Szemerédi's theorem (constellations), which was previously unattainable by other methods. Indeed, rather elementary ideas from ergodic theory and topological dynamics have been employed to give topological proofs of van der Waerden's theorem and Hindman's theorem. While more modern methods have since been discovered these give important connections between mathematical fields.

As examples of the type of results we are referring to, we mention the following.

THEOREM (Furstenberg and Weiss [**FuW**]). *Let $T : X \longrightarrow X$ be a bijective homomorphism of a compact metric space (X, d) into itself. Then for all $\varepsilon > 0$, $k \in \mathbb{Z}^+$, there exist $x \in X$, $n \in \mathbb{Z}^+$, such that $d(x, T^{in}X) < \varepsilon$, for all $1 \le i \le k$. (This implies van der Waerden's theorem.)*

THEOREM (Furstenberg [**Fu1**]). *Let $T : X \longrightarrow X$ be a measure preserving bijection of a finite measure space (X, μ) into itself. Then for all $A \subseteq X$ of positive measure and all k, there exists $n \in \mathbb{Z}^+$ such that*

$$\mu(A \cap T^n A \cap \cdots \cap T^{kn} A) > 0.$$

(This implies Szemerédi's theorem.)

A complete discussion of these results appears in [**Fu2**].

One of the extensions of Szemerédi's theorem that grew out of the ergodic approach was to establish a polynomial version of the result.

THEOREM (Bergelson and Leibman [**BerLe1**]). *Let $(X, \mathcal{B}, \mu)$ be a probability space, let $T_1, \ldots, T_k$ be commuting measure preserving invertible transformations of X, let $p_1(n), \ldots, p_k(n)$ be polynomials with rational coefficients taking integer values on the integers and satisfying $p_i(0) = 0$, $i = 1, \ldots, k$, and let $A \in \mathcal{B}$ with*

[1] We have maintained the original title of this chapter. A more appropriate title might be "Current trends, then and now" or "Current and current-er trends".

$\mu(A) > 0$. *Then*

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu\big(T_1^{-p_1(n)}(A) \cap T_2^{-p_2(n)}(A) \cap \cdots T_k^{-p_k(n)}(A)\big) > 0.$$

Using this theorem one can show, for example, that any subset of $\mathbb{N}$ with upper density contains long AP's whose difference is a perfect square. Similar polynomial variations on van der Waerden's theorem and the Hales-Jewett theorem have also been obtained (see [**BerLe1, BerLe2**]).

We recall that a system $\mathcal{L}$ of homogeneous linear equations is called *regular* if it has monoχ solutions in every finite coloring of $\mathbb{Z}^+$. Let us call a subset $X \subseteq \mathbb{Z}^+$ regular if every regular system $\mathcal{L}$ can be solved in X.

Exercise 11.1 (Rado). Show that if $\mathbb{Z}^+ = C_1 \cup \cdots \cup C_r$ then some C_i is regular. (Hint: Use the Product theorem.)

In 1973, Deuber [**D1**] succeeded in settling a 40-year-old conjecture of Rado by proving the following.

Theorem. *If $X \subseteq \mathbb{Z}^+$ is regular and $X = C_1 \cup \cdots \cup C_r$ then some C_i is regular.*

Deuber's proof introduced the concept of the so-called (n, p, c)-sets and provided a very satisfying step in this area of Ramsey theory.

Quite a different thrust has been made by Paris and Harrington [**PH**]. By making a slight(!) modification in the statement of (the finite version of) Ramsey's theorem, they have provided the first example of a "natural" theorem in first-order Peano Arithmetic which is unprovable there. Perhaps the simplest statement is the following. Let us call a set $A \subseteq \mathbb{Z}^+$ *large* if $|A| \geq \min(A)$.

Theorem (Paris-Harrington [**PH**]). *For all $k, r \in \mathbb{Z}^+$ there is an integer $m =$ PH(k, r) such that for all r-colorings of the k-subsets of $[m]$, there is a large set $X \subseteq [m]$ with $|X| > k$ such that all the k-subsets of X have the same color. However, this result cannot be proved in first-order Peano Arithmetic.*

The existence of PH(k, r) follows at once from the infinite version of Ramsey's theorem (since every infinite set contains a large subset). However, Paris and Harrington showed that this cannot be deduced strictly within the framework of first-order Peano Arithmetic. One explanation for why this happens is that the *lower bounds* on PH(k, r) grow so rapidly that they cannot even be proved in first-order Peano Arithmetic to be well defined. This led Solovay [**Sol**] and others to suspect that perhaps the known Ackermann-like bounds on the van der Waerden and Hales-Jewett function are really a reflection of the true state of affairs. However, the work of Gowers and of Polymath has shown that this is not the case.

In this connection we mention the following two results.

1. Suppose $n(k)$ has the property that any set

$$\{\lfloor \alpha x + \beta \rfloor : x = 1, 2, \ldots, n(k)\}$$

contains a k-term AP. Such sets share many common properties with AP's (although they are much more numerous). How large must $n(k)$ be to guarantee this? It turns out[2] that $n(k)$ need be only on the order of ck^2 to guarantee this.

[2] Unpublished work of Abramson, Burr and Erdős.

2. Suppose $A = \{a_1 < a_2 < \cdots\}$ has bounded gaps, say, $a_{n+1} - a_n \leq B$. How many terms of A, say, $w(k, B)$, are needed to guarantee that $\{a_1, \ldots, a_{w(k,B)}\}$ contains a k-term AP? It might be expected that because of the strong regularity condition imposed on A, a reasonable upper bound on $w(k, B)$ might be provable, e.g., one that was primitive recursive. However, as observed by Nathanson,[3] $w(k, B)$ behaves essentially the same way as $W(k, B)$, the ordinary van der Waerden function. On one hand, we can B-color $\mathbb{Z}^+$ by coloring x with color i if $a_n + i = x < a_{n+1}$ for some $i \in \{0, 1, \ldots, B-1\}$. Any mono$\chi$ k-term AP in color i must be a translate (by i) of a k-term AP in color 0, i.e., it is in A and consequently $w(k, B) \leq W(k, B)$. On the other hand, let $\chi : \mathbb{Z}^+ \longrightarrow [B]$. Define a sequence $A(\chi) = \{a_1, a_2, \ldots\}$ by letting $a_k = (k-1)B + \chi(k)$, $k \in \mathbb{Z}^+$. Thus, $a_{n+1} - a_n < 2B$. However, if $c + d, c + 2d, \ldots, c + kBd$ is a kB-term AP in $A(\chi)$ then every Bth term, say $c + Bd, c + 2Bd, \ldots, c + kBd$, has the same χ-color. Therefore,
$$W(k, B) \leq w(kB, 2B).$$

Work of Mills [**Mi**] shows that certain functions associated with labelled trees unexpectedly (to the authors) show a similar hyperexplosive growth (and in fact can be used to bound on both sides the Paris-Harrington function).

We have hinted earlier at some of the really remarkable results achieved by Nešetřil and Rödl. They have been very successful in proving large classes of what might be called "restricted" Ramsey results, e.g., if G is a K_n-free graph then there exists a K_n-free H with $H \longrightarrow (G, G)$. Or, for any $k, r \in \mathbb{Z}^+$ there is a set $A \subseteq \mathbb{Z}^+$ containing no $(k+1)$-term AP such that any r-coloring of A must always contain a monoχ k-term AP. The reader is referred to [**NeR1**] and [**NeR3**] for a survey of their results.

One of the biggest developments in graph Ramsey theory since the first edition of this book has been the development of the theory to random graphs (this being different from using random graphs, which has a long association with the field). For parameters n and m, let $F_{n,m}$ denote the family of all (labeled) graphs on n vertices and m edges and $R_{G,r}(n, m)$ denote the set of graphs in $F_{n,m}$ so that any r-coloring of the edges of the graph contain a monochromatic copy of G. Then we have the following.

Theorem (Rödl and Ruciński [**RR**]). *For all $r \geq 2$ and G which is not a forest of stars, there exist constants c_1, c_2 so that*
$$\lim_{n\to\infty} \frac{|R_{G,r}(n,m)|}{|F_{n,m}|} = \begin{cases} 0 & \text{if } N < c_1 n^{2-1/\theta(G)}, \\ 1 & \text{if } N > c_2 n^{2-1/\theta(G)}, \end{cases}$$
where $\theta(G) = \max\limits_{H \subseteq G, |V(H)| \geq 3} \dfrac{|E(H)| - 1}{|V(H)| - 2}$.

In particular, this theorem shows that for a given graph G if a particular graph H is sufficiently dense then with high probability any coloring of H contains a monochromatic copy of G and if a particular graph h is not sufficiently dense then with high probability there is a coloring of H that does not contain a monochromatic copy of G. This density of $n^{2-1/\theta(G)}$ thus represents a threshold between whether

[3] During the meeting.

or not a graph satisfies a given property. This behavior is not limited to graphs, but can apply to any structure, for example [**RR**] also contains a threshold result for van der Waerden's theorem.

This result marked an important milestone in the exploration of finding combinatorial structures inside of sparse random sets. For more on this area look at the work of Conlon and Gowers [**CoGo**] and the references contained therein.

A number of people, including in particular, Hindman, Baumgartner, Glazer, Taylor and Galvin, have attacked many of the infinite Ramsey theorems by means of ultrafilters. In particular, Glazer has obtained a remarkable one-page proof of Hindman's theorem through this approach. A very readable account of this research can be found in [**Hi2**]. In this connection Hindman has investigated the existence of stronger versions of his theorem in infinite sets with all finite sums monoχ. By restricting attention to powers of 2 it is easy to see that every finite coloring of $\mathbb{Z}^+$ contains an infinite set with all its finite *products* having one color. Hindman has shown that we cannot be guaranteed of having all products *and* sums monoχ. In particular he has constructed in [**Hi3**] a 7-coloring $\chi : \mathbb{Z}^+ \longrightarrow [7]$ such that there is no infinite set A with $A \subseteq \chi^{-1}(i)$, $\{a + a' : a \neq a' \in A\} \subseteq \chi^{-1}(i)$, $\{a \cdot a' : a \neq a' \in A\} \subseteq \chi^{-1}(i)$ for some i. Perhaps such an A exists if the condition $A \subseteq \chi^{-1}(i)$ is dropped.

A particularly nice open question here is the finite version:

Question. Is it true that for any finite coloring of $\mathbb{Z}^+$ there are arbitrarily large (finite) sets A with all subset sums *and* product sums from A having a single color?

Probably the branch of Ramsey theory in which the most papers are currently appearing is graph Ramsey theory. Many different concepts have been introduced here, including, for example, the Ramsey functions on graphs of Buckley [**Buc**], many kinds of mixed Ramsey numbers of Lesniak and others [**LeR**], size Ramsey number (Erdős, et al. [**EFRS**]), connected Ramsey numbers, "k-out-of-ℓ" Ramsey numbers of Chung and Liu [**ChuL**], etc. The reader can find most of this discussed in the survey papers of Burr [**Bu1**], [**Bu2**], Lesniak (see [**BCL-F**]), and Parsons [**Pa**].

A particularly large and important area of Ramsey theory which we have completely ignored is the so-called Partition Calculus of Erdős and Rado. This is basically Ramsey theory for infinite cardinals and ordinals. A typical result is the following theorem of Chang [**Cha**] (subsequently supplied with a very elegant proof by Larson [**La**]).

Theorem. $(\mathbb{N}^{\mathbb{N}}) \longrightarrow (\mathbb{N}^{\mathbb{N}}, 3)^2$.

That is, if the pairs of a set of order type $\mathbb{N}^{\mathbb{N}}$ are 2-colored then *either* there is a set of order type $\mathbb{N}^{\mathbb{N}}$ with all pairs having color 1 *or* a 3-element set with all pairs having color 2.

One trend which has become more prominent is the use of computational tools to explore Ramsey theory problems. One of the hallmarks of Ramsey theory is the combinatorial explosion involved in investigating parameters, for example even though we have $43 \leq R(2, 5, 2) \leq 49$ the number of graphs that have to be checked is prohibitive even with the best computers. Nevertheless computation can help guide exploration through small cases. In addition the flag algebra method is highly computational, and SAT solvers can be used to establish important bounds.

As an example of the use of SAT solvers we can look at what has happened with Graham's number. This is a bound on a combinatorial problem, namely to find the least such n so that if we color all the edges between vertices of the form $\{\pm 1\}^n$ with 2 colors, then there are 4 coplanar points with all 6 edges in the plane monochromatic. The current lower bound is 13 while the original upper bound, Graham's number, is so large that new notation had to be used to express the number.[4] Lavrov, Lee and Mackey [**LLM**] gave a new upper bound[5] which dropped it from unimaginably large to almost imaginably large (which is still quite large), and a key step in the proof was the use of a SAT solver.

Looking at the amazing progress that has been made in the 35 years since the first edition of this book has come out, it is hard to predict what will happen in the next 35 years. Ramsey theory is as active an area of research as it has been at any time in the past, and we look forward to seeing what new amazing discoveries await.

[4]Graham's number also held the Guinness record for the largest number in a mathematical proof.

[5]The title of the paper "Graham's number is less than $2 \uparrow\uparrow\uparrow 6$" is misleading. Graham's number will always be the same number! This only marks a much better upper bound to the original problem.