Preface

C*-algebras are algebras of operators on Hilbert spaces. Operator algebras were introduced by John von Neumann in connection with the study of quantum mechanics in 1929 followed by the work of F. Murray and J. von Neumann which developed the basic theory of von Neumann algebras. Later, Gelfand and Naimark introduced the notion of C*-algebras. The Gelfand transformation shows that every unital commutative C*-algebra is isomorphic to the algebra of continuous functions on a compact Hausdorff space. Therefore it is also natural to view C*-algebras as non-commutative topological spaces. While original motivational examples are limited to the spectral theorem for a normal operator and unitary representations of locally compact groups, today, with the development of K-theory and KK-theory, its study grows to a diversified field which includes or is closely related to the index theory for subfactors, abstract harmonic analysis, group representation theory and non-commutative geometry, as well as the classification of amenable C*-algebras.

The classification of simple amenable C*-algebras is often known as the Elliott program. Classification of amenable C*-algebras was initiated by G. A. Elliott through his paper [**33**] and was much publicized through his ICM lecture [**34**], though the Elliott program is predated by works of Glimm ([**47**]), Dixmier ([**27**]) and Bratteli ([**10**]), as well as Elliott's classification of AF-algebras ([**31**]). The main goal is to use the Elliott invariant (a set of K-theoretic invariants) to completely determine amenable C*-algebras up to isomorphism. These notes will discuss some of the current developments.

In the study of topology, one studies continuous maps between spaces. In C*-algebra theory, which is often regarded as non-commutative topology, one studies homomorphisms from one C*-algebra to another. Given two homomorphisms φ_1 and φ_2 from a C*-algebra A into another C*-algebra B, one asks when these two homomorphisms are approximately or asymptotically unitarily equivalent. The latter means that there is a continuous path of unitaries $\{u(t) : t \in [1, \infty)\}$ in B such that φ_1 is conjugate to φ_2 eventually via the path. One of the features of these notes is that it presents a study of the asymptotic unitary equivalence of monomorphisms from a given unital separable C*-algebra to a unital separable simple C*-algebra.

The so-called Basic Homotopy Lemma first appeared in a paper of O. Bratteli, G. A. Elliott, D. Evans and A. Kishimoto [**9**]. It studied a pair of unitaries in a unital simple C*-algebra. It was later developed to the following type of problem: Let A be a unital simple C*-algebra, let C be another unital C*-algebra and let $\varphi : C \to A$ be a unital homomorphism. Suppose that u is a unitary in A such that u almost commutes with the image of φ and there exists a continuous path of unitaries $\{u(t) : t \in [0, 1]\}$ with $u(0) = u$ and $u(1) = 1_A$. The question is whether there exists a continuous path of unitaries $\{v(t) : t \in [0, 1]\}$ in A such that $v(0) = u$

and $v(1) = 1$ such that $v(t)$ almost commutes with the image of φ for all $t \in [0, 1]$. This problem appeared more frequently in the study of the structure of C*-algebras and their homomorphisms. It turns out that this type of technical problem studies a profound mathematical phenomenon. In these notes some versions of this will be exploited. We will also present some applications of the Basic Homotopy Lemma in the study of asymptotic unitary equivalence of homomorphisms among other related subjects.

Chapter 1 serves as an introduction to these notes. Chapter 2 studies homomorphisms from subhomogeneous C^*-algebras to finite dimensional C^*-algebras. Chapter 3 provides the stable version of the Basic Homotopy Lemma. Moreover, a concrete version of the Bott map is discussed. Chapter 4 presents a version of the Basic Homotopy Lemma in finite dimensional C^*-algebras. Chapter 5 introduces the notation $\mathrm{gTR}(A) \leq 1$. A brief discussion of the classification of C^*-algebras with $\mathrm{gTR(A)} \leq 1$ is presented. Chapter 6 presents the Basic Homotopy Lemma in C^*-algebras A with $\mathrm{gTR(A)} \leq 1$. It also presents a theorem concerning how to lift KK-elements to homomorphisms. Chapter 7 introduces the notation of asymptotic unitary equivalence. Chapter 8 presents some current developments in the Elliott program without giving full proofs. We also include an overview of the development of the Elliott program as these notes are written as Chapter 0.

These notes were based on the lecture notes from the CBMS lecture sequence at the University of Wyoming in the summer of 2015. Part of these lecture notes were based on a joint paper with Guihua Gong and Zhuang Niu. However, a large part of that paper is not mentioned here due to the limitations on these lecture notes. Materials are also taken from recent developments in related research including the author's other related papers. The author is infinitely in debt, but his words must be few; with these he can only begin to express his great gratitude, especially to Guihua Gong and Zhuang Niu for their generosity and to George Elliott for his valuable comments during the lectures, as well as the audience attending the lecture sequence. It is also the author's great pleasure to acknowledge Michael Yuan Sun for his kind help reading, checking and editing.

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