CHAPTER 0

An overview of the Elliott program

Operator algebras on Hilbert spaces were first studied by Murray and von Neumann in 1930's. These are also C^*-algebras. A C*-algebra A is a Banach algebra with involution $x \mapsto x^*$ satisfying the C^*-algebra identity $\|xx^*\| = \|x\|^2$ for all $x \in A$.

A finite dimensional C^*-algebra B always has the form $B = \bigoplus_{i=1}^m M_{r(i)}$, where $M_{r(i)}$ is the full matrix algebras over $\mathbb{C}$ with the size of $r(i)$. A C^*-algebra is called AF if it is an inductive limit of finite dimensional C^*-algebras. Using the rank on projections in $M_{r(i)}$, one can associate to it the ordered group $\mathbb{Z}$. Thus B is associated with the ordered group $\mathbb{Z}^m$. The size of matrices can be recovered from the image of the rank of the identity of B which is called the scale of the ordered group. Therefore a unital AF-algebra A is associated with an inductive limit of these scaled ordered abelian groups which Elliott named the dimension group. This group is in fact the same as $K_0(A)$. Together with the order and the scale, it is written as $(K_0(A), K_0(A)_+, [1_A])$.

In 1976, George A. Elliott proved the following theorem:

Theorem 0.0.1. *Let A and B be two unital AF-algebras. Then A is isomorphic to B if and only if $(K_0(A), K_0(A)_+, [1_A])$ is isomorphic to $(K_0(B), K_0(B)_+, [1_B])$.*

A special case of AF-algebras are called uniform hyperfinite (UHF)-algebras which was first studied and classified, using supernatural numbers, by J. Glimm ([**47**]). Dixmier ([**27**]) later classified non-unital versions of these C^*-algebras. O. Bratteli ([**10**]) used the Bratteli diagrams to study AF-algebras.

It was around 1989 that Elliott classified the simple $A\mathbb{T}$-algebras of real rank zero ([**33**]). $A\mathbb{T}$-algebras are inductive limits of C^*-algebras which are finite direct sums of C^*-algebras of the form $C(\mathbb{T}) \otimes M_n$, namely the C^*-algebras of continuous maps from the circle $\mathbb{T}$ to M_n. A unital C^*-algebra has real rank zero if the set of invertible self-adjoint elements is dense in the set of self-adjoint elements. The invariant used in [**33**] are the scaled ordered K_0-groups together with K_1-groups. This is probably the true beginning of the Elliott program, the program to classify separable simple amenable C^*-algebras using the Elliott invariant, a set of K-theory related data.

Simple $A\mathbb{T}$-algebras of real rank zero form a rich class of simple C^*-algebras. Notable examples are the Bunce-Deddens algebras. Later Elliott and Evans ([**36**]) proved that irrational rotation algebras are actually simple $A\mathbb{T}$-algebras of real rank zero (together with a result Putnam).

0.0.1. AH-algebras and the Elliott invariant. 1990's saw the take off of the Elliott program. Let us mention the classification of simple AH-algebras with no dimension growth and with real rank zero. Following the historical term, a homogeneous C^*-algebra is a C^*-algebra of the form $PM_r(C(X))P$, where X is a finite CW complex, $r \geq 1$ is an integer and $P \in M_r(C(X))$ is a projection. An AH-algebra is an inductive limit of homogeneous C^*-algebras. Let $B = PM_r(C(X))P$ be a homogeneous C^*-algebra. We may write

$$B = \bigoplus_{j=1}^{m} P_j M_{r(j)}(C(X_j))P_j,$$

where each X_j is connected and $P_j \in M_{r(j)}(C(X))$ is a non-zero projection. Put

$$d(B) = \max_j \frac{\dim(X_j)}{\mathrm{rank} P_j}. \tag{e 0.0.1}$$

Let $A = \lim_{n\to\infty}(B_n, \varphi_n)$, where each B_n is a homogeneous C^*-algebra. We say A has slow dimension growth if $\limsup_{n\to\infty} d(B_n) = 0$. A is said to have no dimension growth, if $\{\dim(X_n)\}$ is bounded, where $B_n = P_n M_{r(n)}(C(X_n))P_n$, $n = 1, 2, \dots$ It was proved in [**26**] that if A is a unital simple AH-algebra with slow dimension growth then A has stable rank one.

In 1996, Elliott and Gong ([**37**]) proved the following theorem:

THEOREM 0.0.2. *Let A and B be two unital simple AH-algebras with slow dimension growth and with real rank zero. Then $A \cong B$ if and and only if*

$$(K_0(A), K_0(A)_+, [1_A], K_1(A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B)).$$

One should realize that while $A\mathbb{T}$-algebras have torsion free K-theory, there is no such restriction on K-theory of AH-algebras.

Let A be a unital C^*-algebra and let $T(A)$ be the tracial state space of A (it could be an empty set). Suppose that A is stably finite. One has an order preserving homomorphism ρ_A from $K_0(A)$ to $\mathrm{Aff}(T(A))$, the real continuous affine functions on the Choquet simplex $T(A)$, defined by $\rho_A([p])(\tau) := \tau(p)$ for all projections in $M_n(A)$ ($n = 1, 2, \dots$), where $\tau := \tau \otimes \mathrm{Tr}_n$, $\tau \in T(A)$ and Tr_n is the standard trace on M_n. Let $e \in K_0(A)_+ \setminus \{0\}$ be an order unit. Let $S_e(K_0(A))$ denote the state space of $K_0(A)$, i.e., the set of order preserving homomorphism $h : K_0(A) \to \mathbb{R}$ such that $h(e) = 1$. The map ρ_A induces an affine homomorphism $r_A : T(A) \to S_{[1_A]}(K_0(A))$ defined by $r_A(\tau)(x) = \rho_A(x)(\tau)$ for all $\tau \in T(A)$. It was proved in [**7**] under the assumption that A is a separable simple exact stably finite C^*-algebra the map r_A is always surjective.

Given a unital separable simple stably finite C^*-algebra A, Elliott introduced the following set of invariant:

$$\mathrm{Ell}(A) = (K_0(A), K_0(A)_+, [1_A], K_1(A), T(A), r_A). \tag{e 0.0.2}$$

The condition that a simple C^*-algebra has real rank zero and stable rank one implies that $\rho_A(K_0(A))$ is dense in $\mathrm{Aff}(T(A))$. Therefore $\mathrm{Aff}(T(A))$ is determined by $K_0(A)$. So $T(A)$ was not needed in Theorem 0.0.2 and Theorem 0.0.5 but is needed in general.

For simple C^*-algebras which do not have real rank zero, one has the following theorem of Elliott-Gong-Li ([**38**] appeared in 2007).

THEOREM 0.0.3. *Let A and B be two unital simple AH-algebras with no dimension growth. Then $A \cong B$ if and only if* $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.

Moreover, Villadsen showed that, if G_0 is any countable weakly unperforated simple ordered group with the Riesz property and with order unit u such that

$G_0/\mathrm{Inf}(G_0) \ncong \mathbb{Z}$, where $\mathrm{Inf}(G_0)$ is the infinitesmal elements, if G_1 is any countable abelian group, T is any metrizable Choquet simplex and $r : T \to S_u(G_0)$ is a surjective affine continuous map which maps the extremal points $\partial_e(T)$ to the extremal points $\partial_e(S_u(G_0))$, then there exists a unital simple AH-algebra A with no dimension growth such that

$$\mathrm{Ell}(A) = (G_0, (G_0)_+, u, G_1, T, r).$$

By Theorem 0.0.3 such a C^*-algebra A is unique up to isomorphism. This provides the range theorem for the simple AH-algebras with no dimension growth. The next stage of the Elliott program was to study simple C^*-algebras without first assuming that they are inductive limits of certain familiar C^*-algebras. However, since the Elliott program is to use K-theoretic invariant to classify C^*-algebras, it is important to restrict ourselves to the class of C^*-algebras that satisfy the Universal Coefficient Theorem (UCT).

0.0.2. Tracial rank. Denote by $\mathcal{I}^{(0)}$ the class of finite dimensional C^*-algebras. Denote by $\mathcal{I}^{(k)}$ the class of C^*-algebras of the form $PM_r(C(X))P$ such that $\dim X = k$.

To classify simple C^*-algebras without first assuming an inductive limit structure, the following notion was introduced by the author shortly before 2000 ([**72**] and [**77**]).

DEFINITION 0.0.4. Let A be a unital simple C^*-algebra. A has tracial rank at most k if the following holds: Let $\varepsilon > 0$, $\mathcal{F} \subset A$ be a finite subset and let $a \in A_+ \setminus \{0\}$. There exists a projection $p \in A$, a C^*-subalgebra $C \subset A$ with $C \in \mathcal{I}^{(k)}$ and $1_C = p$ such that

$$\|px - xp\| < \varepsilon \text{ for all } x \in \mathcal{F}, \tag{e 0.0.3}$$

$$\mathrm{dis}(pxp, C) < \varepsilon \text{ and} \tag{e 0.0.4}$$

$$1 - p \lesssim a, \tag{e 0.0.5}$$

where $1 - p \lesssim a$ means there is a partial isometry $v \in A$ such that $v^*v = 1 - p$ and $vv^* \in \overline{aAa}$. If A has tracial rank at most k, we write $TR(A) \le k$. If $TR(A) \le k$ and $TR(A) \not\le k - 1$, we write $TR(A) = k$.

Note that $TR(A) = 0$ means that C in the above definition is a finite dimensional C^*-subalgebra.

It is proved in [**76**] the following.

THEOREM 0.0.5. *Let A and B be two unital separable simple amenable C^*-algebras with tracial rank zero which satisfy the UCT. Then $A \cong B$ if and only if*

$$(K_0(A), K_0(A)_+, [1_A], K_1(A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B)).$$

If A has $TR(A) \le k$ for some integer k, then A has stable rank one and weakly unperforated $K_0(A)$ (see [**77**]). If $TR(A) = 0$, then A also has real rank zero. A result in [**37**] shows that every simple AH-algebra A with slow dimension growth and real rank zero has tracial rank zero. It was proved in [**48**] that every unital simple AH-algebra A with very slow dimension growth has $TR(A) \le 1$. It was proved by the author ([**75**]) that a unital simple AH-algebra A has $TR(A) = 0$ if and only if A has real rank zero, stable rank one and has weakly unperforated $K_0(A)$. In [**81**] it is shown that a unital separable simple C^*-algebra A with unique tracial state, real rank zero, stable rank one and with weakly unperforated $K_0(A)$ which is also an inductive limit of type I C^*-algebras must have $TR(A) = 0$. Winter ([**127**] and later [**128**]) showed that a unital separable simple C^*-algebras with finite decomposition rank and with real rank zero has tracial rank zero.

Let X be a compact metric space with finite covering dimension. Suppose that $\alpha : X \to X$ is a minimal homeomorphism. The pair (X, α) is called a dynamical system. It is known that, when X has infinitely many points, $A := C(X) \rtimes_\alpha \mathbb{Z}$, the crossed product C^*-algebra from the minimal dynamical system, is always a simple C^*-algebra which satisfies the UCT. In [**100**], N. C. Phillips and the author proved the following:

THEOREM 0.0.6. *Let (X, α) be a minimal dynamical system and let $A = C(X) \rtimes_\alpha \mathbb{Z}$. Then $TR(A) = 0$ if and only if $\rho_A(K_0(A))$ is dense in* $\mathrm{Aff}(T(A))$.

Suppose that (X, α) is unique ergodic and X is connected. Suppose also that the rotation numbers of (X, α) contains an irrational value. Then $\rho_A(K_0(A)) \subset \mathbb{R}$ has an irrational value as well as integer values. Therefore $\rho_A(K_0(A))$ is dense in $\mathbb{R}$. Theorem above implies that $C(X) \rtimes_\alpha \mathbb{Z}$ has tracial rank zero. This recovers the Elliott-Evans theorem ([**36**]) for irrational rotation algebras.

In [**86**], the author proved the following theorem:

THEOREM 0.0.7. *Let A and B be two separable simple amenable C^*-algebras which satisfy the UCT. Suppose that $TR(A) \le 1$ and $TR(B) \le 1$. Then $A \cong B$ if and only if* $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.

Later the author also proved the following ([**96**]):

THEOREM 0.0.8. *Let A be a unital separable amenable simple C^*-algebra which satisfies the UCT. Suppose $TR(A) \le k$ for some integer k, then $TR(A) \le 1$.*

On the other hand, for the purely infinite simple cases, classification program is essentially completed. A unital simple C^*-algebra A is said to be purely infinite, if $A \ne \mathbb{C}$ and if for any $a \ne 0$, there are $x, y \in A$ such that $xay = 1$. In particular, if A is purely infinite, $K_0(A) = K_0(A)_+$ and $T(A) = \emptyset$. So the Elliott invariant consists of only the abelian groups $K_0(A)$ with a distinguished element u (representing identity of A) and $K_1(A)$. Kirchberg and Phillips proved the following

THEOREM 0.0.9. *([**63**] and [**110**]). Let A and B be two separable purely infinite simple amenable C^*-algebras which satisfy the UCT. Then $A \cong B$ if and only if* $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.

0.0.3. Counterexamples and Jiang-Su stability. Just as the successful story of the Elliott program continues, counterexamples began to emerge. J. Villadsen ([**125**]) found examples of separable simple AH-algebras whose K_0 is not weakly unperforated. B. Blackadar introduced the notion of strict comparison. A unital simple C^*-algebra A is said to have strict comparison for projections (and for positive elements), if, for any two projections (positive elements) $p, q \in M_n(A)$ $(a, b \in M_n(A)_+)$ for some integer $n \ge 1$ such that $\tau(p) < \tau(q)$ $(\lim_{n\to\infty} \tau(a^{1/n}) < \lim_{n\to\infty} \tau(b^{1/n}))$ for all $\tau \in QT(A)$, $p \lesssim q$ (or $a \lesssim b$ in the sense of Cuntz). Villadsen's examples do not have strict comparison for projections. Villadsen ([**126**]) also produced stably finite separable simple C^*-algebras which have stable rank greater than one. It has been a long open question whether or not there are separable simple C^*-algebras which contain both finite and infinite projections. M. Rørdam ([**116**]) discovered that there are indeed separable simple amenable C^*-algebras which have both finite and infinite projections. Then A. Toms ([**123**]) gave examples of unital separable simple amenable C^*-algebras with stable rank one which have the same Elliott invariant but are not isomorphic. Both Rørdam and Toms' examples show that the Elliott invariant as it stood can not be a complete invariant for separable simple amenable C^*-algebras.

Before these examples appeared, Jiang and Su ([**59**]) constructed a unital separable simple amenable C^*-algebra $\mathcal{Z}$ with stable rank one whose Elliott invariant is exactly the same as that of $\mathbb{C}$ (this was also obtained by Elliott (see [**59**])). In fact $\mathcal{Z}$ was constructed as an inductive limit of certain sub-homogeneous C^*-algebras (C^*-subalgebras of homogeneous C^*-algebras). For any unital simple C^*-algebra A, $K_i(A) \cong K_i(A\otimes \mathcal{Z})$ $(i=0,1)$ as abelian groups. It was proved by Gong, Jiang and Su ([**49**]) that if A is a unital simple C^*-algebra with weakly unperforated $K_0(A)$, then $\mathrm{Ell}(A)\cong \mathrm{Ell}(A\otimes\mathcal{Z})$. A C^*-algebra A is said to be $\mathcal{Z}$-stable, if $A\cong A\otimes\mathcal{Z}$. The Gong-Jiang-Su result mentioned above shows that, for the class of unital simple separable C^*-algebras with weakly unperforated K_0, if they can be classified by the Elliott invariant, then they must be $\mathcal{Z}$-stable. Later M. Rørdam proved ([**117**]) that if A is a unital simple exact and finite C^*-algebra which are $\mathcal{Z}$-stable then the Cuntz semi-group of A is weakly unperforated and it has strict comparison. Amenable C^*-algebras with finite tracial rank are $\mathcal{Z}$-stable (see [**96**]). Toms' examples mentioned above are not $\mathcal{Z}$-stable. From these facts, it is only prudent that we study only $\mathcal{Z}$-stable simple C^*-algebras at present.

0.0.4. Classification of $\mathcal{Z}$-stable C^*-algebras. Let $\mathfrak{p}$ and $\mathfrak{q}$ be two supernatural numbers which are relatively prime. Let $M_\mathfrak{p}$ and $M_\mathfrak{q}$ be the UHF-algebras associated with $\mathfrak{p}$ and $\mathfrak{q}$, respectively (as in [**47**]). Define

$$\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}=\{f\in C([0,1],M_\mathfrak{q}\otimes M_\mathfrak{q}): f(0)\in M_\mathfrak{p}\otimes 1_{M_\mathfrak{q}}\ \text{ and }\ f(1)\in 1_{M_\mathfrak{p}}\otimes M_\mathfrak{q}\}.$$

It is proved by Rørdam and Winter ([**118**]) that

$$\mathcal{Z}=\lim_{n\to\infty}(\mathcal{Z}_{\mathfrak{p},\mathfrak{q}},\varphi_n), \tag{e 0.0.6}$$

where each $\varphi_n:\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}\to\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ is a unital embedding, $\varphi_n=\varphi_{n+1}$ for all n and that $\tau_1(\varphi_n)=\tau_2(\varphi_n)$ for any $\tau_1,\tau_2\in T(\mathcal{Z}_{\mathfrak{p},\mathfrak{q}})$.

Winter proved ([**130**]) the following:

THEOREM 0.0.10. *Let A and B be two separable simple amenable C^*-algebras, let $\mathfrak{p}$ and $\mathfrak{q}$ be two relatively prime supernatural numbers. Suppose that there is a $C([0,1])$-isomorphism $\varphi: A\otimes\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}\to B\otimes\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ and there exists a continuous path of unitaries $\{u(t): t\in[0,1)\}\subset B\otimes\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ such that $u(0)=1$ and, for all $t\in[0,1)$,*

$$\varphi_t(a)=\mathrm{Ad}\,u(t)\circ\varphi_0(a)\ \text{ for all }\ a\in A. \tag{e 0.0.7}$$

Then $A\otimes\mathcal{Z}\cong B\otimes\mathcal{Z}$.

Combing Winter's above theorem with some versions of the so-called Basic Homotopy Lemma (in [**89**]) in simple C^*-algebras with tracial rank zero, after establishing asymptotic unitary equivalence ([**88**]), Z. Niu and the author ([**98**] and also [**95**]) obtained the following:

THEOREM 0.0.11. *Let A and B be two unital separable simple amenable C^*-algebras which satisfy the UCT. Suppose that, for some infinite dimensional UHF-algebra U, $TR(A\otimes U)=TR(B\otimes U)=0$. Then $A\cong B$ if and only if $\mathrm{Ell}(A)\cong\mathrm{Ell}(B)$.*

It should be noted that if A is a separable amenable simple C^*-algebra such that $TR(A \otimes U) = 0$ ($TR(A \otimes U) = 1$) for one infinite dimensional UHF-algebra U, then $TR(A \otimes C) = 0$ ($TR(A \otimes C) = 1$) for all infinite dimensional AF-algebras C ([**102**]). One distinguished UHF-algebra is Q the universal UHF-algebra with $(K_0(Q), K_0(Q)_+, [1_Q]) = (\mathbb{Q}, \mathbb{Q}_+, 1)$.

One should also notice that $TR(\mathcal{Z})$ is infinite but $TR(\mathcal{Z} \otimes U) = 0$ for any infinite dimensional UHF-algebra U. In fact Theorem 0.0.11 covers a great deal more C^*-algebras than Theorem 0.0.7 does. For example, Toms and Winter proved ([**124**]) the following:

THEOREM 0.0.12. *Let X be a compact metric space with finite covering dimension and let $\alpha : X \to X$ be a minimal homeomorphism. Then the crossed product $A = C(X) \rtimes_\alpha \mathbb{Z}$ is $\mathcal{Z}$-stable and has the property $TR(A \otimes Q) = 0$ if and only if the projections in A separate the trace. In particular, this is always the case when (X, α) is also unique ergodic.*

Denote by $\mathcal{N}_1$ the class of all unital separable simple amenable C^*-algebras A which satisfy the UCT and $TR(A \otimes U) \le 1$ for some infinite dimensional UHF-algebra U.

We have the following result ([**92**])

THEOREM 0.0.13. *Let A and B be two simple C^*-algebras in $\mathcal{N}_1$ which are $\mathcal{Z}$-stable. Then $A \cong B$ if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.*

This is a considerably larger class of unital simple amenable C^*-algebras. Theorem 0.0.13 also unifies all previous classification results for AH-algebras as well as the so-called dimension drop algebras. Notably, Theorem 0.0.13 also implies that a unital simple AH-algebra has tracial rank at most one if and only if it is $\mathcal{Z}$-stable.

In fact, we have the following range theorem.

THEOREM 0.0.14. ([**99**]) *Let G_0 be a countable weakly unperforated simple ordered group with order unit u such that $S_u(G_0)$ is a metrizable Choquet simplex, let G_1 be a countable abelian group, T be a metrizable Choquet simplex and $\lambda : T \to S_u(G_0)$ be a surjective affine continuous map such that $\lambda(\partial_e(T)) \subset \partial_e(S_u(G_0))$. Then there exists one (and only one) $\mathcal{Z}$-stable C^*-algebra $A \in \mathcal{N}_1$ such that*

$$\mathrm{Ell}(A) = (G_0, (G_0)_+, u, G_1, T, \lambda). \tag{e 0.0.8}$$

In particular, if A is a unital separable amenable simple C^*-algebra with a unique tracial state, then, there exists a unital separable simple C^*-algebra $B \in \mathcal{N}_1$ which is $\mathcal{Z}$-stable such that

$$\mathrm{Ell}(A \otimes \mathcal{Z}) = \mathrm{Ell}(B).$$

However, there is still a restriction on the Elliott invariant for unital separable simple amenable $\mathcal{Z}$-stable C^*-algebras which are covered by 0.0.13, namely, the map r_A have to map extremal points to extremal points.

On the hand, Winter ([**129**]) proved that every unital separable simple C^*-algebra with finite nuclear dimension is $\mathcal{Z}$-stable. One of the purposes of these notes is to intorduce the classification for unital separable simple C^*-algebras with finite nuclear dimension which satisfy the UCT.

CHAPTER 1

An introduction to the Basic Homotopy Lemma

1.1. A taste of the Basic Homotopy Lemma

In this section we will introduce the Basic Homotopy Lemma by presenting some easy related examples.

We begin with the following question:

1.1.1. For any $\varepsilon > 0$, is there a positive number $\delta > 0$ satisfying the following: Suppose that u and v are two unitaries in a unital C^*-algebra A such that

$$\|uv - vu\| < \delta,$$

then does there exist a continuous path of unitaries $\{v(t) : t \in [0,1]\}$ in A such that

$$\|v(t)u - uv(t)\| < \varepsilon \text{ for all } t \in [0,1],$$

$v(0) = v$ and $v(1) = 1_A$?

In order for the above question to make sense one needs to assume that $v \in U_0(A)$ (see 1.1.3 bellow). If the answer is yes, one can also ask how long the path is.

While there is no doubt that the study of question 1.1.1 has a long history, a systematic study of this question can be found in the work of Bratteli, Elliott, Evans and Kishimoto ([**9**]). One of the results in their work could be stated as follows:

Theorem 1.1.2. (The Basic Homotopy Lemma—Bratteli, Elliott, Evans and Kishimoto–1998 ([**9**]))

Let $\varepsilon > 0$. There exists $\delta > 0$ satisfying the following: For any unital simple C^*-algebra A of stable rank one and real rank zero and any pair of unitaries $u, v \in A$ with $u \in U_0(A)$, the connected component of $U(A)$ containing the identity such that

$$\|uv - vu\| < \delta \text{ and } \mathrm{bott}_1(u, v) = 0,$$

there exists a continuous path of unitaries $\{u(t) : t \in [0,1]\}$ in A such that $u(0) = u$, $u(1) = 1_A$ and

$$\|u(t)v - vu(t)\| < \varepsilon \text{ for all } t \in [0,1].$$

Moreover $\mathrm{length}(\{u(t)\}) \leq 4\pi + 1$.

We will explain what $\mathrm{bott}_1(u, v)$ means later but will not present a proof of this theorem. However, we will present similar results most relevant to these notes.

Definition 1.1.3. Let A be a unital C^*-algebra. Denote by $U(A)$ the unitary group of A, and denote by $U_0(A)$ the normal subgroup of $U(A)$ consisting of those unitaries of A which are in the connected component of A containing 1_A. Denote by $DU(A)$ the commutator subgroup of $U_0(A)$ and $CU(A)$ the closure of $DU(A)$ in $U(A)$.

DEFINITION 1.1.4. Let X be a normed space and let $f : [0,1] \to X$ be a continuous map. Suppose that $\mathcal{P} : 0 = t_0 < t_1 < \cdots < t_n = 1$ is a partition of $[0,1]$. Put

$$L(f,\mathcal{P}) = \sum_{i=1}^{n} \|f(t_i) - f(t_{i-1})\|.$$

Define $\mathrm{length}(f) = \sup_{\mathcal{P}} L(f,\mathcal{P})$. We call $\mathrm{length}(f)$ the length of f.

Let A be a unital C^*-algebra and let $u \in U_0(A)$. Suppose that $\{u(t) : t \in [0,1]\} \subset U_0(A)$ is a continuous path of unitaries in A such that $u(0) = u$ and $u(1) = 1_A$. Denote by

$$\mathrm{cel}(u) = \inf\{\mathrm{length}(\{u(t) : t \in [0,1]\} \subset U_0(A)) \text{ and } u(0) = u \text{ and } u(1) = 1_A\}.$$

We begin with the following easy fact.

PROPOSITION 1.1.5. *Let A be a C^*-algebra, let $u \in M_k$ be a unitary for some integer $k \geq 1$ and let $\varphi : A \to M_k$ be a map such that*

$$u\varphi(a) = \varphi(a)u \text{ for all } a \in A. \tag{e 1.1.1}$$

Then there exists a continuous path of unitaries $\{u(t) : t \in [0,1]\}$ in M_k such that

$$u_0 = u, \ u_1 = 1, \ u(t)\varphi(a) = \varphi(a)u(t) \text{ for all } a \in A \text{ and for all } t \in [0,1].$$

Moreover, $\mathrm{length}(\{u(t)\}) \leq \pi$.

PROOF. There is a continuous function h from $\mathrm{sp}(u)$ to $[-\pi,\pi]$ such that

$$\exp(ih(u)) = u.$$

Since $h(u)$ is in the C^*-subalgebra generated by u,

$$\varphi(a)h(u) = h(u)\varphi(a) \text{ for all } a \in A.$$

Note that $h(u) \in (M_k)_{s.a.}$ and $\|h(u)\| \leq \pi$. Define $u(t) = \exp(i(1-t)h(u))$ $(t \in [0,1])$. Then $u(0) = u$ and $u(1) = 1$. Also

$$u(t)\varphi(a) = \varphi(a)u(t)$$

for all $a \in A$ and $t \in [0,1]$. Moreover, since $\|h\| \leq \pi$, one has $\mathrm{length}(\{u(t)\}) \leq \pi$. □

If the commutativity in (e 1.1.1) becomes almost commutativity, one has the following:

LEMMA 1.1.6. (cf. Lemma 2.6.11 [**78**]) *Let $\varepsilon > 0$ and let $d > 0$, there exists $\delta > 0$ satisfying the following: Suppose that A is a unital C^*-algebra and $u \in A$ is a unitary such that $\mathbb{T} \setminus \mathrm{sp}(u)$ contains an arc with length d. Suppose that $a \in A$ with $\|a\| \leq 1$ such that*

$$\|ua - au\| < \delta.$$

Then there exists a self-adjoint element $h \in A$ with $\|h\| \leq 2\pi$ such that $u = \exp(ih)$,

$$\|ha - ah\| < \varepsilon \text{ and } \|\exp(ith)a - a\exp(ith)\| < \varepsilon$$

for all $t \in [0,1]$.

PROOF. By replacing u by $e^{i\theta}\cdot u$ for some $\theta\in(-\pi,\pi)$, possibly increasing the length by π, we may assume that

$$\mathrm{sp}(u)\subset\Omega_d=\{e^{i\pi t}: -1+d/2\le t\le 1-d/2\}\subset\mathbb{T}. \tag{e 1.1.2}$$

There is a continuous function $g:\Omega_d\to(-\pi,\pi)$ such that $u=\exp(ig(u))$. Let $h=g(u)$. Then $\|h\|\le\pi$. Choose an integer $N\ge 1$ such that

$$\sum_{n=N+1}^{\infty}\frac{1}{n!}<\frac{\varepsilon}{4}.$$

There is $\delta>0$ such that

$$\|au-ua\|<\delta \tag{e 1.1.3}$$

implies that $\|h^n a-ah^n\|<\varepsilon/6$ for $n=1,2,\dots,N$. Then for any $t\in[0,1]$

$$\|\exp(ith)a-a\exp(ith)\|\le\|(\sum_{n=0}^{N}\frac{(ith)^n}{n!})a-a(\sum_{n=0}^{N}\frac{(ith)^n}{n!})\|+2\sum_{n=N+1}^{\infty}\frac{1}{n!}$$
$$\le\sum_{n=1}^{N}\frac{\varepsilon}{6n!}+\frac{\varepsilon}{2}<\varepsilon.$$

□

COROLLARY 1.1.7. *Let $n\ge 1$ be an integer. Let C be a unital C^*-algebra and let $\mathcal{F}\subset C$ be a finite subset. For any $\varepsilon>0$, there exists $\delta>0$ satisfying the following: Suppose $L:C\to M_n$ is a contractive map and $u\in M_n$ is a unitary such that*

$$\|L(c)u-uL(c)\|<\delta \text{ for all } c\in\mathcal{F}. \tag{e 1.1.4}$$

Then there exists a continuous path of unitaries $\{u(t): t\in[0,1]\}\subset M_n$ such that $u(0)=u$, $u(1)=1_{M_n}$ and

$$\|L(c)u(t)-u(t)L(c)\|<\varepsilon \text{ for all } c\in\mathcal{F}. \tag{e 1.1.5}$$

Moreover, $\mathrm{length}(u(t))\le 2\pi$.

PROOF. The spectrum of u has a gap with the length at least $d=2\pi/n$. The length statement follows from the fact $\|h\|\le 2\pi$ in the proof of 1.1.6. □

Lemma 1.1.6 and Corollary 1.1.7 may be viewed as elementary versions of the Basic Homotopy Lemma. In the next section we will also give some applications of these.

1.2. Some easy applications

We begin with a few conventions some of which will be used later.

DEFINITION 1.2.1. Let A be a unital C^*-algebra. Denote by $T(A)$ the tracial state space of A, i.e., those positive linear functionals $\tau: A\to\mathbb{C}$ such that $\tau(x^*x)=\tau(xx^*)$ for all $x\in A$ and $\|\tau\|=1$. Let $\tau\in T(A)$. We say that τ is faithful if $\tau(a)>0$ for all $a\in A_+\setminus\{0\}$. Denote by $T_f(A)$ the set of all faithful tracial states.

Denote by $\mathrm{Aff}(T(A))$ the space of all real continuous affine functions on $T(A)$ and denote by $\mathrm{LAff}_b(T(A))$ the set of all bounded lower-semi-continuous real affine functions on $T(A)$.

Suppose that $T(A) \neq \emptyset$. There is an affine map $r_{\rm aff} : A_{s.a.} \to \mathrm{Aff}(T(A))$ by

$$r_{\rm aff}(a)(\tau) = \hat{a}(\tau) = \tau(a) \text{ for all } \tau \in T(A)$$

and for all $a \in A_{s.a.}$. Denote by $A^q_{s.a.}$ the image $r_{\rm aff}(A_{s.a.})$ and $A^q_+ = r_{\rm aff}(A_+)$.

For each integer $n \geq 1$ and $a \in M_n(A)$, write $\tau(a) = (\tau \otimes \mathrm{Tr})(a)$, where Tr is the (non-normalized) standard trace on M_n.

DEFINITION 1.2.2. Let A and B be two C^*-algebras and let $\varphi : A \to B$ be a homomorphism. Denote by φ_{*i} the induced homomorphism from $K_i(A)$ to $K_i(B)$, $i = 0, 1$.

LEMMA 1.2.3. (Hall's Marriage lemma, cf. [**58**], see also [**60**]) *Let X and Y be two sets of n elements (which may not be distinct) and let $R \subset X \times Y$. Suppose, for each subset $S \subset X$,*

$$|\cup_{x \in S} \{y \in Y : (x, y) \in R\}| \geq |S| \tag{e 1.2.1}$$

(counting multiplicities). Then there exists a bijection $\sigma \colon X \to Y$ such that $(x, \sigma(x)) \in R$ for all $x \in X$.

Let $R \subset \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$ be a subset and let $A \subset \{1, 2, \ldots, m\}$. Define $R_A \subset \{1, 2, \ldots, n\}$ to be the subset of those $j's$ such that $(i, j) \in R$, for some $i \in A$.

The following follows from Hall's Marriage lemma.

LEMMA 1.2.4. *If $\{a_i\}_{i=1}^m, \{b_i\}_{j=1}^n \subset \mathbb{Z}_+^k$ with $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$, and $R \subset \{1, \ldots, m\} \times \{1, \ldots, n\}$ satisfying: for any $A \subset \{1, \ldots, m\}$,*

$$\sum_{i \in A} a_i \leq \sum_{j \in R_A} b_j, \tag{e 1.2.2}$$

then there are $\{c_{ij}\} \subset \mathbb{Z}_+^k$ such that

$$\sum_{j=1}^n c_{ij} = a_i \text{ for all } i, \quad \sum_{i=1}^m c_{ij} = b_j, \text{ for all } j \tag{e 1.2.3}$$

and

$$c_{ij} = 0 \textit{ unless } (i, j) \in R. \tag{e 1.2.4}$$

LEMMA 1.2.5. *Let X be a compact metric space with finitely many connected components, let $C = M_r(C(X))$ for some integer $r \geq 1$ and let $n \geq 1$ be an integer. Let $\varepsilon > 0$ and let $\mathcal{F} \subset C$ be a finite subset. There exists $\delta > 0$ and a finite subset $\mathcal{H} \subset C_{s.a.}$ satisfying the following. Suppose that $\varphi, \psi : C \to M_n$ are two unital homomorphisms such that*

$$\varphi_{*0} = \psi_{*0} \textit{ and } |\tau \circ \varphi(g) - \tau \circ \psi(g)| < \delta \text{ for all } g \in \mathcal{H} \tag{e 1.2.5}$$

(τ is the tracial state on M_n). Then there exists a unitary $u \in U(M_n)$ such that

$$\|\mathrm{Ad}\, u \circ \varphi(a) - \psi(a)\| < \varepsilon \text{ for all } a \in \mathcal{F}. \tag{e 1.2.6}$$

Moreover, if $n = kr$, $\varphi(f) = \sum_{i=1}^k e_i \otimes f(x_i)$ and $\psi(f) = \sum_{j=1}^k e'_j \otimes f(y_j)$ for all $f \in C$, where x_i, $y_j \in X$, $\{e_1, e_2, \ldots, e_k\}$ and $\{e'_1, e'_2, \ldots, e'_k\}$ are two sets of mutually orthogonal projections in M_n, and if $d > 0$ is given, then we may assume that there is a permutation σ on $\{1, 2, \ldots, k\}$ such that $u^*(e_i \otimes 1_{M_r})u = e'_{\sigma(i)} \otimes 1_{M_r}$, and $\mathrm{dist}(x_i, y_{\sigma(i)}) < d$, $i = 1, 2, \ldots, k$, where δ depends on ε, $\mathcal{F}$ and d.

PROOF. To simplify the notation, by the assumption that $\varphi_{*0} = \psi_{*0}$ and by considering each summand, without loss of generality, we may assume that X is connected. There exists $\eta > 0$ such that

$$\|f(x) - f(y)\| < \varepsilon/4 \text{ for all } f \in \mathcal{F}, \tag{e 1.2.7}$$

provided that $\text{dist}(x, y) < \eta$. Let $O_1, O_2, \ldots, O_m$ be a finite open cover such that each O_i has diameter $< \eta/4$. Let $S \subset X$ be a finite subset. Put $O_S = \cup_{S \cap O_j \neq \emptyset} O_j$. Define $g_S \in C(X)_+$ such that $0 \leq g_S \leq 1$, $g_S(x) = 1$ if $x \in O_S$, $g_S(x) = 0$ if $\text{dist}(x, O_S) \geq \eta/4$. Let $h_S \in C(X)$ be such that $0 \leq h_S \leq 1$, $h_S(x) = 1$ if $\text{dist}(x, O_S) < \eta/2$ and $h_S(x) = 0$ if $\text{dist}(x, O_S) > \eta$. Let $\delta = \min\{\eta/16n, 1/16n\}$. For each $J \subset \{1, 2, \ldots, m\}$, put $O^J = \cup_{i \in J} O_i$. Note that, for any finite subset $S \subset X$, $O_S = O^J$ for some J. Set

$$\mathcal{H} = \{g_S \cdot 1_{M_r}, h_S \cdot 1_{M_r} : S \text{ finite subset of } X\}. \tag{e 1.2.8}$$

So $\mathcal{H}$ is a finite subset of $M_r(C(X))$. Now suppose that $\varphi, \psi : M_r(C(X)) \to M_n$ are two unital homomorphisms such that

$$|\tau \circ \varphi(c) - \tau \circ \psi(c)| < \delta \text{ for all } c \in \mathcal{H}. \tag{e 1.2.9}$$

Note that $n = kr$ for some integer $k \geq 1$. There are two sets of mutually orthogonal projections $\{p_1, p_2, \ldots, p_{k_1}\}$ and $\{q_1, q_2, \ldots, q_{k_2}\}$ of M_k with $\sum_{i=1}^{k_1} p_i = \sum_{j=1}^{k_2} q_j = 1_{M_k}$, and two subsets $X_0 = \{x_1, x_2, \ldots, x_{k_1}\}$ and $Y_0 = \{y_1, y_2, \ldots, y_{k_2}\}$ of X such that, for all $f \in M_r(C(X))$,

$$\varphi(f) = \sum_{i=1}^{k_1} p_i \otimes f(x_i) \text{ and } \psi(f) = V^*(\sum_{j=1}^{k_2} q_j \otimes f(y_j))V, \tag{e 1.2.10}$$

where $V \in M_n$ is a unitary.

Define a subset $R \subset \{1, 2, \ldots, k_1\} \times \{1, 2, \ldots, k_2\}$ as follows: $(i, j) \in R$ if and only if $\text{dist}(x_i, y_j) < \eta$. Let $a_i = \text{rank}\, p_i$ and $b_j = \text{rank}\, q_j$. Let $S \subset X_0$ be a subset. Put $A_S = \{i \in \{1, 2, \ldots, k_1\} : x_i \in S\}$.

Then

$$\tau(\varphi(g_S \cdot 1_{M_r})) \geq r \sum_{x_i \in S} a_i/n. \tag{e 1.2.11}$$

It follows from (e 1.2.5) that

$$\tau(\psi(g_S \cdot 1_{M_r})) \geq r \sum_{x_i \in S} a_i/n - 1/16n. \tag{e 1.2.12}$$

Let P_S be the range projection of $\psi(g_S \cdot 1_{M_r})$ in M_n. Then

$$\tau(P_S) \geq r \sum_{x_i \in S} a_i/n = r \sum_{i \in A_S} a_i/n. \tag{e 1.2.13}$$

Therefore

$$\tau(\psi(h_S \cdot 1_{M_r})) \geq r \sum_{i \in A_S} a_i/n. \tag{e 1.2.14}$$

It follows that

$$\sum_{i \in A} a_i \leq \sum_{j \in R_A} b_j. \tag{e 1.2.15}$$

This holds for any subset $A \subset \{1, 2, \dots, k_1\}$. By 1.2.4 there are $\{c_{i,j}\} \subset \mathbb{Z}^+$ such that

$$\sum_{j=1}^{k_2} c_{ij} = a_i, \ \ \sum_{i=1}^{k_1} c_{ij} = b_j$$

and $c_{ij} \neq 0$ if and only if $(i, j) \in R$. Therefore there are mutually orthogonal projections p_{ij} and q_{ij} such that

$$\sum_{j=1}^{k_2} p_{ij} = p_i, \ \ \sum_{i=1}^{k_1} q_{ij} = q_j,$$

$\operatorname{rank} p_{ij} = \operatorname{rank} q_{ij}$ and $p_{ij} \neq 0$ and $q_{ij} \neq 0$ if and only if $(i, j) \in R$.

We may write

$$\varphi(f) = \sum_{(i,j)\in R} p_{ij} \otimes f(x_i) \ \text{ and } \ \psi(f) = V^*(\sum_{(i,j)\in R} q_{ij} \otimes f(y_j))V$$

for all $f \in C$. Moreover, $p_{ij} \neq 0$ and $q_{ij} \neq 0$ if and only if $\operatorname{dist}(x_i, y_j) < \eta$. Therefore there exists a unitary $v \in M_k$ such that

$$v^* p_{ij} v = q_{ij}.$$

Choose $u = (v \otimes 1_{M_r})V$. Then we have that

$$\|\operatorname{Ad} u \circ \varphi(f) - \psi(f)\| < \varepsilon$$

for all $f \in \mathcal{F}$. The lemma then follows.

□

REMARK 1.2.6. If X is connected, in Lemma 1.2.5, the condition $\varphi_{*0} = \psi_{*0}$ is not needed since we assume both φ and ψ are unital. Of course, with a standard argument, the assumption that X has only finitely many connected components can also be removed. We omit the discussion of this technicality in this introductory section.

Lemma 1.2.5 may be viewed as an elementary version of some deeper versions of the so called uniqueness theorem. The target algebra in Lemma 1.2.5 is M_n. We now use Lemma 1.1.6 and Corollary 1.1.7 to pass from M_n to $C([0, 1], M_n)$.

THEOREM 1.2.7. *Let X be a compact metric space, let $C = M_r(C(X))$ and let $n \geq 1$ be an integer. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exists $\delta > 0$ and a finite subset $\mathcal{H} \subset C_{s.a.}$ satisfying the following: Suppose that $\varphi, \psi : C \to C([0, 1], M_n)$ are two unital homomorphisms such that*

$$\varphi_{*0} = \psi_{*0} \ \text{ and } \ |\tau \circ \varphi(g) - \tau \circ \psi(g)| < \delta \ \text{ for all } \ g \in \mathcal{H} \tag{e 1.2.16}$$

and for all $\tau \in T(C([0, 1], M_n))$. Then there exists a unitary $u \in C([0, 1], M_n)$ such that

$$\|u^* \varphi(f) u - \psi(f)\| < \varepsilon \ \text{ for all } \ f \in \mathcal{F}. \tag{e 1.2.17}$$

PROOF. We prove only the case that X is connected (see 1.2.6). Let $\delta > 0$ be as required by Corollary 1.1.7 for the given $\varepsilon/16$ and $\mathcal{F}$ and n. Let $\varepsilon_1 = \min\{\varepsilon/64, \delta/16\}$. Let $\delta_1 > 0$ (in place of δ) and $\mathcal{H} \subset C_{s.a.}$ be a finite subset as required by Lemma 1.2.5 for given ε_1 (in place of ε) and $\mathcal{F}$ (as well as n). Choose $\eta > 0$ such that

$$\|\varphi(f)(t) - \varphi(f)(t')\| < \varepsilon_1 \text{ and } \|\psi(f)(t) - \psi(f)(t')\| < \varepsilon_1$$

for all $f \in \mathcal{F}$, whenever $|t - t'| < \eta$.

Let $0 = t_0 < t_1 < \cdots < t_m = 1$ be a partition of $[0, 1]$ with $|t_i - t_{i-1}| < \eta$ for all i. By the assumption and 1.2.5, there is a unitary $u_i \in M_n$ such that

$$\|u_i^* \varphi(f)(t_i) u_i - \psi(f)(t_i)\| < \varepsilon_1 \text{ for all } f \in \mathcal{F}, \ i = 0, 1, 2, \ldots, m.$$

It follows that

$$\begin{aligned} u_{i+1} u_i^* \varphi(f)(t_i) u_i u_{i+1}^* &\approx_{\varepsilon_1} u_{i+1} \psi(f)(t_i) u_{i+1}^* \\ &\approx_{\varepsilon_1} u_{i+1} \psi(f)(t_{i+1}) u_{i+1}^* \approx_{\varepsilon_1} \varphi(f)(t_{i+1}) \approx_{\varepsilon_1} \varphi(f)(t_i). \end{aligned}$$

It follows from 1.1.7 that there exists a continuous path of unitaries $\{w_i(t) : t \in [t_i, t_{i+1}]\} \subset M_n$ such that $w_i(t_i) = 1_{M_n}$ and $w_i(t_{i+1}) = u_{i+1} u_i^*$ and

$$\|w_i(t) \varphi(f)(t_i) - \varphi(f)(t_i) w_i(t)\| < \varepsilon/16 \text{ for all } f \in \mathcal{F},$$

$i = 0, 1, 2 \ldots, m$.

Define $v(t) = w_i(t) u_i$ for $t \in [t_i, t_{i+1}]$, $i = 0, 1, 2 \ldots, m-1$. Then $v(t_i) = u_i$ and $v(t_{i+1}) = u_{i+1}$, $i = 0, 1, 2, \ldots, m-1$, and $v \in C([0, 1], M_n)$. Moreover, for $t \in [t_i, t_{i+1}]$,

$$\begin{aligned} v(t)^* \varphi(f)(t) v(t) &\approx_{\varepsilon_1} u_i^* w_i(t)^* \varphi(f)(t_i) w_i(t) u_i \approx_{\varepsilon/16} u_i^* \varphi(f)(t_i) u_i \\ &\approx_{\varepsilon_1} \psi(f)(t_i) \approx_{\varepsilon_1} \psi(f)(t) \end{aligned}$$

for all $f \in \mathcal{F}$. In other words,

$$\|v^* \varphi(f) v - \psi(f)\| < \varepsilon \text{ for all } f \in \mathcal{F}.$$

□

The following approximate diagonalization is another application of 1.1.7.

THEOREM 1.2.8. *Let X be a compact metric space which is locally path connected and let $C = M_r(C(X))$. Suppose that $\varphi : C \to C([0, 1], M_n)$ is a unital homomorphism, where $n = kr$ for some integer $k \geq 1$. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exists a set of mutually orthogonal rank one projections $p_1, p_2, \ldots, p_k \in C([0, 1], M_k)$ such that*

$$\|\varphi(f) - \sum_{i=1}^{k} p_i \otimes f(\alpha_i)\| < \varepsilon \text{ for all } f \in \mathcal{F}, \tag{e 1.2.18}$$

where $\alpha_i : [0, 1] \to X$ is a continuous map and where $(p_i \otimes f(\alpha_i))(t) := p_i(t) \otimes f((\alpha_i(t)))$ for all $t \in [0, 1]$, $i = 1, 2, \ldots, k$ (and where M_n is identified with $M_k \otimes M_r$).

PROOF. We prove only the case that X is connected (see 1.2.6). Let $\delta > 0$ be required by Corollary 1.1.7 for the given integer n and $\varepsilon/4$ (in place of ε). Let $d > 0$ satisfy the following: if $\mathrm{dist}(x, x') < 2d$,

$$\|f(x) - f(x')\| < \varepsilon/4 \text{ for all } f \in \mathcal{F} \tag{e 1.2.19}$$

and there is an open ball B of radius $< d$ which contains a continuous path in B connecting x and x'. Let $\delta_1 > 0$ (in place of δ) and $\mathcal{H} \subset C$ be a finite subset as required by 1.2.5 for the given $\min\{\varepsilon/4, \delta/2\}$ (in place of ε), $\mathcal{F}$, n and $d/2$. There exists $\eta > 0$ such that

$$\|\varphi(g)(t) - \varphi(g)(t')\| < \min\{\varepsilon/4, \delta_1/2, \delta/2\} \text{ for all } g \in \mathcal{F} \cup \mathcal{H} \tag{e 1.2.20}$$

whenever $|t - t'| < \eta$.

Let $0 = t_0 < t_1 < t_2 < \cdots < t_m = 1$ be a partition of $[0, 1]$ with $|t_i - t_{i-1}| < \eta$, $i = 1, 2, \ldots, m$. Note that $n = kr$ for some integer $k \geq 1$. We may write, for each i, that

$$\varphi(f)(t_{i-1}) = \sum_{j=1}^{k} p_{i-1,j} \otimes f(x_{i-1,j}) \text{ for all } f \in M_r(C(X)),$$

where $x_{i-1,j} \in X$ and $\{p_{i-1,1}, p_{i-1,2}, \ldots, p_{i-1,k}\}$ is a set of mutually orthogonal rank one projections in M_k.

It follows from Lemma 1.2.5 and (e 1.2.20) that there are unitaries $u_i \in M_n$ such that

$$\|u_i^* \varphi(f)(t_{i-1})u_i - \varphi(f)(t_i)\| < \min\{\delta/2, \varepsilon/4\} \text{ for all } f \in \mathcal{F}, \tag{e 1.2.21}$$

$i = 1, 2, \ldots, m$. Moreover, we may assume, without loss of generality, that there is a permutation σ_i such that

$$u_i^* p_{i-1,j} u_i = p_{i,\sigma_i(j)} \text{ and } \operatorname{dist}(x_{i-1,j}, x_{i,\sigma_i(j)}) < d/2,$$

$j = 1, 2, \ldots, k$, $i = 1, 2, \ldots, m$. By (e 1.2.21) and (e 1.2.20),

$$\|\varphi(f)(t_{i-1})u_i - u_i\varphi(f)(t_{i-1})\| < \delta \text{ for all } f \in \mathcal{F},$$

$i = 1, 2, \ldots, m$. It follows from 1.1.7 that there exists a continuous path of unitaries $\{v_i(t) : t \in [t_{i-1}, t_i]\} \subset M_n$ such that $v_i(t_{i-1}) = 1$ and $v_i(t_i) = u_i$ and

$$\|v_i(t)\varphi(f)(t_{i-1}) - \varphi(f)(t_{i-1})v_i(t)\| < \varepsilon/4 \text{ for all } f \in \mathcal{F},$$

$i = 1, 2, \ldots, m$. Define $p_j(t) = v_i(t)^* p_{i-1,j} v_i(t)$ for $t \in [t_{i-1}, t_i]$, $i = 1, 2, \ldots, m$. Then $p_j(t_0) = p_{0,j}$, $p_j(t_i) = p_{i,\sigma_i(j)}$, $i = 1, 2, \ldots, m$. Since $\operatorname{dist}(x_{i-1,j}, x_{i,\sigma_i(j)}) < d/2$, there exists a continuous path $\alpha_{j,i-1} : [t_{i-1}, t_i] \to B_i$ such that $\alpha_{j,i-1}(t_{i-1}) = x_{i-1,j}$ and $\alpha_{j,i-1}(t_i) = x_{i,\sigma_i(j)}$, where B_i is an open ball with radius d which contains both $x_{i-1,j}$ and $x_{i,\sigma_i(j)}$. Define $\alpha_j : [0, 1] \to X$ by $\alpha_j(t) = \alpha_{j,i-1}(t)$ if $t \in [t_{i-1}, t_i]$, $i = 1, 2, \ldots, m$. Define

$$\psi(f) = \sum_{i=1}^{k} p_i \otimes f(\alpha_i) \text{ for all } f \in M_r(C(X)).$$

On $[t_{i-1}, t_i]$,

$$\begin{aligned}
\|\varphi(f)(t) - \psi(f)(t)\| \quad &\leq \quad \|\varphi(f)(t) - \sum_{j=1}^{k} p_{i-1,j} \otimes f(x_{i-1,j})\| \\
&+\|\sum_{j=1}^{k} p_{i-1,j} \otimes f(x_{i-1,j}) - \sum_{j=1}^{k} p_j(t) \otimes f(\alpha_{j,i-1}(t))\| \\
&< \varepsilon/4 + \|\sum_{j=1}^{k} p_{i-1,j} \otimes f(x_{i-1,j}) - \sum_{j=1}^{k} v_i^*(t) p_{i-1,j} v_i(t) \otimes f(x_{i-1,j})\| + \varepsilon/4 \\
&= \|\varphi(f)(t_{i-1}) - v_i^*(t)\varphi(f)(t_{i-1})v_i(t)\| + \varepsilon/2 < \varepsilon
\end{aligned}$$

for all $f \in \mathcal{F}$. □

1.3. The Voiculescu example and an Exel-Loring invariant

In Lemma 1.1.6, the constant δ depends not only on ε but also on d. One also notices that, in 1.1.7, δ depends on the integer n. For application purposes, as well as theoretical reasons, one may ask whether δ can be chosen independent of n. This section will explain that this cannot be done.

1.3.1. Suppose that $u, v \in M_n$ are two unitaries such that $\|uv - vu\| < 1$. Then $\|v^*uvu^* - 1\| < 1$. One has

$$(1/2\pi i)\mathrm{Tr}(\log(v^*uvu^*)) \in \mathbb{Z}.$$

If there is a continuous path of unitaries $\{v(t) : t \in [0,1]\} \subset M_n$ such that $v(0) = v$ and $v(1) = 1_{M_n}$ and

$$\|v^*(t)uv(t)u^* - 1\| < 1,$$

then $(1/2\pi i)\mathrm{Tr}(\log(v^*(t)uv(t)u^*)$ is a continuous function on $[0,1]$. But it is zero at $t = 1$. Therefore

$$(1/2\pi i)\mathrm{Tr}(\log(v^*(t)uv(t)u^*) = 0 \ \text{ for all } \ t \in [0,1].$$

This implies that, in order to have such a continuous path of unitaries $\{v(t)\}$ as above, one has that

$$(1/2\pi i)\mathrm{Tr}(\log(v^*uvu^*)) = 0.$$

In other words, if $(1/2\pi i)\mathrm{Tr}(\log(v^*uvu^*)) \neq 0$, then such a continuous path of unitaries $\{v(t)\}$ does not exist.

EXAMPLE 1.3.2. Let

$$u_n = \begin{pmatrix} e^{2\pi i/n} & 0 & 0 & \cdots & 0 \\ 0 & e^{4\pi i/n} & 0 & \cdots & 0 \\ & & \ddots & & \\ & & & & e^{2n\pi i/n} \end{pmatrix}$$

and

$$v_n = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}$$

be two $n \times n$ unitary matrices. This is the Voiculescu pair.

One computes that

$$v_n^* u_n v_n u_n^* = \begin{pmatrix} e^{2\pi i/n} & 0 & 0 & \cdots & 0 \\ 0 & e^{2\pi i/n} & 0 & \cdots & 0 \\ & & \ddots & & \\ & & & & e^{2\pi i/n} \end{pmatrix}.$$

In particular

$$\lim_{n\to\infty} \|u_n v_n - v_n u_n\| = \lim_{n\to\infty} |e^{2\pi i/n} - 1| = 0.$$

However,

$$\mathrm{Tr}(\log(v_n^* u_n v_n u_n^*)) = 2\pi i.$$

In other words, there is **no** $\delta > 0$ satisfying the following:

For any integer $n \geq 1$, any pair of unitaries $u, v \in M_n$ with $\|uv - vu\| < \delta$, there is a continuous path of unitaries $\{v(t) : t \in [0,1]\} \subset M_n$ such that $v(0) = v$ and $v(1) = 1_{M_n}$ and

$$\|uv(t) - v(t)u\| < 1 \text{ for all } t \in [0,1].$$

1.3.3. A closely related question to 1.1.1 may be formulated as follows:

Let $\varepsilon > 0$. Does there exist $\delta > 0$ satisfying the following?

For any two unitaries u and v such that

$$\|uv - vu\| < \delta,$$

then there exists a pair of commuting unitaries U and V such that

$$\|U - u\| < \varepsilon \text{ and } \|V - v\| < \varepsilon.$$

1.3.4. In the case that $A = M_n$, if U commutes with V, then one may write $V = \exp(iH)$ for some self adjoint element in A such that $UH = HU$. Define

$$V(t) = \exp(2i(1-t)H) \text{ for all } t \in [1/2, 1].$$

Then $V(t)$ is a continuous path of unitaries with $V(1/2) = V$ and $V(1) = 1_{M_n}$. Moreover $UV(t) = V(t)U$ for all $t \in [1/2, 1]$. If $\|V - v\| < \varepsilon_0$ with $0 < \varepsilon_0 < \arcsin(\varepsilon/8)$, then

$$\|v^*V - 1_{M_n}\| < \varepsilon_0.$$

Hence $v^*V = \exp(ih)$ for some self adjoint element $h \in M_n$ with $\|h\| < \varepsilon/4$. Define $V(t) = v\exp(i2th)$ for $t \in [0, 1/2]$. Then $V(0) = v$ and $V(1/2) = V$. Therefore $\{V(t) : t \in [0,1]\}$ is a continuous path of unitaries such that $V(0) = v$ and $V(1) = 1_{M_n}$. Moreover,

$$\|V(t) - v\| < \sin(\varepsilon/4) \text{ for all } t \in [0, 1/2].$$

Suppose that $\|u - U\| < \varepsilon/4$. Then, for $t \in [1/2, 1]$,

$$\|uV(t) - V(t)u\| = \|uV(t) - UV(t)\| + \|UV(t) - V(t)U\| + \|u - U\| < \varepsilon.$$

If $t \in [0, 1/2]$, then

$$\|uV(t) - V(t)u\| \leq \|uV(t) - uv\| + \|uv - vu\| + \|vu - V(t)u\| < \varepsilon/4 + \delta + \varepsilon/4.$$

Thus, in the case that $A = M_n$, if the answer to 1.3.3 is in the affirmative then so is the answer to 1.1.1.

Definition 1.3.5. Let

$$f(e^{2\pi it}) = \begin{cases} 1 - 2t & \text{if } 0 \leq t \leq 1/2, \\ -1 + 2t & \text{if } 1/2 < t \leq 1, \end{cases}$$

$$g(e^{2\pi it}) = \begin{cases} (f(e^{2\pi it}) - f(e^{2\pi it})^2)^{1/2} & \text{if } 0 \leq t \leq 1/2, \\ 0 & \text{if } 1/2 < t \leq 1, \end{cases}$$

and

$$h(e^{2\pi it}) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1/2, \\ (f(e^{2\pi it}) - f(e^{2\pi it})^2)^{1/2} & \text{if } 1/2 < t \leq 1. \end{cases}$$

These are non-negative continuous functions defined on $\mathbb{T}$. Suppose that u and v are unitaries with $uv = vu$. Define

$$e(u, v) = \begin{pmatrix} f(v) & g(v) + h(v)u^* \\ g(v) + uh(v) & 1 - f(v) \end{pmatrix}.$$

Then $e(u, v)$ is a projection.

If u and v do not actually commute, $e(u, v)$ is merely a positive element. There exists a $\delta_0 > 0$ such that $\|uv - vu\| < \delta_0$ implies that the spectrum of $e(u, v)$ has a gap at 1/2. The bott element $\mathrm{bott}_1(u, v)$ as defined by Exel and Loring (which had appeared in 1.1.2) is

$$[\chi_{[1/2,\infty]}(e(u,v))] - [\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}]$$

as an element in $K_0(A)$. Moreover, by choosing a possibly smaller δ_0, one has the following the Exel trace formula (in the case that $A = M_n$):

$$\text{(e 1.3.1)} \qquad \mathrm{tr}(\mathrm{bott}_1(u, v)) = \frac{1}{2\pi\sqrt{-1}} \mathrm{tr}(\log(u^* v u v^*)),$$

where tr is the normalized trace on M_n.

1.3.6. Let A be a unital C^*-algebra. If u, $v \in A$ are two commuting unitaries, then the C^*-subalgebra of A generated by u and v is a quotient of $C(\mathbb{T}^2)$. Let X be a compact metric space. Let $d_X : K_0(C(X)) \to \mathbb{Z}$ be the (order preserving) homomorphism defined by taking the rank of projections. Then there is a short exact sequence:

$$0 \to \ker d_X \to K_0(C(X)) \overset{d}{\to} \mathbb{Z} \to 0.$$

In the case of $X = \mathbb{T}^2$, one has $\mathrm{ker} d_{\mathbb{T}^2} = \mathbb{Z}$ and $K_0(C(\mathbb{T}^2)) = \mathbb{Z} \oplus \mathbb{Z}$. In fact $\mathrm{ker} d_{\mathbb{T}^2}$ is generated by $\mathrm{bott}_1(u, v)$, where u and v are standard generators of $C(\mathbb{T}^2)$.

Consider C^*-algebras $\prod_{n=1}^\infty M_n$, $\bigoplus_{n=1}^\infty M_n$ and $\prod_{n=1}^\infty M_n / \bigoplus_{n=1}^\infty M_n$. Note that

$$\text{(e 1.3.2)} \qquad \prod_{n=1}^{\infty} M_n = \{\{a_n\} : a_n \in M_n \text{ and } \sup \|a_n\| < \infty\} \text{ and}$$

$$\text{(e 1.3.3)} \qquad \bigoplus_{n=1}^{\infty} M_n = \{\{a_n\} : a_n \in M_n \text{ and } \lim_{n\to\infty} \|a_n\| = 0\}.$$

Denote by $\Pi : \prod_{n=1}^\infty M_n \to \prod_{n=1}^\infty M_n / \bigoplus_{n=1}^\infty M_n$ the quotient map.

Now consider a pair of sequence of unitaries $\{u_n\}$ and $\{v_n\}$, where for each n u_n, $v_n \in M_n$. Suppose that

$$\text{(e 1.3.4)} \qquad \lim_{n\to\infty} \|u_n v_n - v_n u_n\| = 0.$$

Then $\{u_n\}$, $\{v_n\} \in \prod_{n=1}^\infty M_n$ and $\{u_n v_n - v_n u_n\} \in \bigoplus_{n=1}^\infty M_n$. Let $U = \Pi(\{u_n\})$ and $V = \Pi(\{v_n\})$. Then U and V are unitaries in $\prod_{n=1}^\infty M_n / \bigoplus_{n=1}^\infty M_n$ that *commute*! Thus these two unitaries induce a unital homomorphism $\varphi : C(\mathbb{T}^2) \to \prod_{n=1}^\infty M_n / \bigoplus_{n=1}^\infty M_n$.

Let us choose u_n and v_n as in 1.3.2. Then, one can compute, by the Exel trace formula, $\mathrm{bott}_1(u_n, v_n) = 1$. Therefore $\varphi_{*0}(\mathrm{ker} d_{\mathbb{T}^2}) \neq 0$. On the other hand, any homomorphism $\varphi_n : C(\mathbb{T}^2) \to M_n$ must have $(\varphi_n)_{*0}(\mathrm{ker} d_{\mathbb{T}^2}) = 0$. Suppose that $\Psi : C(\mathbb{T}^2) \to \prod_{n=1}^\infty M_n$ is a homomorphism. Then $\Psi = \{\varphi_n\}$, where each φ_n is a homomorphism from $C(\mathbb{T}^2)$ to M_n. Therefore $\Psi_{*0}(\mathrm{ker} d_{\mathbb{T}^2}) = 0$. Therefore $\varphi \neq \Pi \circ \Psi$. In other words, φ cannot be lifted to a homomorphism from $C(\mathbb{T}^2)$ to $\prod_{n=1}^\infty M_n$ which also implies that $\{u_n\}$ and $\{v_n\}$ cannot be asymptotically approximated by a sequence of commuting pairs of unitaries.

To end this section, we present the following theorem without proof.

THEOREM 1.3.7. (Loring 1998([**103**])) *Let $\varepsilon > 0$, there is $\delta > 0$ satisfying the following: For any pair of unitaries $u, v \in M_n$ (for any integer $n \geq 1$) with the property*

$$\|uv - vu\| < \delta \text{ and } \mathrm{bott}_1(u, v) = 0,$$

there exists a continuous path of unitaries $\{v(t) : t \in [0,1]\} \subset M_n$ such that $v(0) = v$, $v(1) = 1_{M_n}$ and

$$\|uv(t) - v(t)u\| < \varepsilon \text{ for all } t \in [0,1]. \tag{e 1.3.5}$$

Moreover, $\mathrm{length}(\{v(t)\}) \leq \pi$.

1.4. Exercises

1.4.1. Recall that a unital C^*-algebra A has stable rank one if the set of invertible elements of A is dense in A. Prove that $C([0,1], M_n)$ has stable rank one.

1.4.2. Let $f \in C([0,1], M_n)$. Suppose that, for some $\varepsilon > 0$, there are two elements $a_0, a_1 \in M_n$ such that $\|a_0 - f(0)\| < \varepsilon$ and $\|a_1 - f(1)\| < \varepsilon$. Then there is $g \in C([0,1], M_n)$ such that $g(0) = a_0$, $g(1) = a_1$ and $\|g - f\| < \varepsilon$. If $f \in U(C([0,1], M_n))$ and $a_0, a_1 \in U(M_n)$, in addition, one can choose $g \in U(C([0,1], M_n))$.

1.4.3. Let $n \geq 1$ be an integer and $u \in C([0,1], M_n)$. Then for any $\varepsilon > 0$, there is a set of mutually orthogonal rank one projections $p_1, p_2, \ldots, p_n \in C([0,1], M_n)$ and continuous functions $\lambda_1, \lambda_2, \ldots, \lambda_n \in C([0,1])$ such that $|\lambda_i(t)| = 1$ for all $t \in [0,1]$, $i = 1, 2, \ldots, n$, and

$$\|u - \sum_{i=1}^{n} \lambda_i p_i\| < \varepsilon.$$

Moreover, if $\det(u(t)) = 1$ for all $t \in [0,1]$, then $\det(\sum_{i=1}^{n} \lambda_i p_i)(t) = 1$ for all $t \in [0,1]$.

1.4.4. Let $n \geq 1$ be an integer and $u \in C([0,1], M_n)$. Then for any $\varepsilon > 0$, there exists a self-adjoint $h \in C([0,1], M_n)$ such that

$$\|u - \exp(ih)\| < \varepsilon. \tag{e 1.4.1}$$

Moreover, if $\det(u(t)) = 1$ for each $t \in [0,1]$, h can be chosen so that $\det(\exp(ih(t))) = 1$ for each $t \in [0,1]$.

1.4.5. Let $n \geq 1$ be an integer and let $M > 0$. For any $\varepsilon > 0$, there exists $\delta > 0$ satisfying the following: Suppose that h_1, h_2 are two self-adjoint elements such that $h_i = \sum_{j=1}^{n} \alpha_{i,j} e_{i,j}$, where $\{e_{i,1}, e_{i,2}, \ldots, e_{i,n}\}$ is a set of mutually orthogonal rank one projections, $|\alpha_{i,j}| \leq M$, $\alpha_{i,j'} \neq \alpha_{i,j}$ if $j \neq j'$, $1 \leq i, j \leq n$.

Suppose also that $\|h_1 - h_2\| < \delta$ and $\alpha_{1,1} < \alpha_{1,2} < \cdots < \alpha_{1,n}$. Then there is a permutation σ on $\{1, 2, \ldots, n\}$ such that $\alpha_{2,\sigma(1)} < \alpha_{2,\sigma(2)} < \cdots < \alpha_{2,\sigma(n)}$ and

$$|\alpha_{2,\sigma(j)} - \alpha_{1,j}| < \varepsilon, \ \ j = 1, 2, \ldots, n.$$

Moreover, if $\sum_{j=1}^{n} \alpha_{i,j} = \beta$, $i = 1, 2$, then $\sum_{j=1}^{n} t\alpha_{1,j} + (1-t)\alpha_{2,j} = \beta$ for all $t \in (0,1)$.

1.4.6. Let u be a unitary in $C([0,1], M_n)$. Then, for any $\varepsilon > 0$, there exist continuous functions $h_j \in C([0,1])_{s.a.}$ such that

$$\|u - u_1\| < \varepsilon,$$

where $u_1 = \exp(i\pi H)$, $H = \sum_{j=1}^{n} h_j p_j$ and $\{p_1, p_2, \ldots, p_n\}$ is a set of mutually orthogonal rank one projections in $C([0,1], M_n)$, and $\exp(i\pi h_j(t)) \neq \exp(i\pi h_k(t))$ if $j \neq k$ for all $t \in (0,1)$. Moreover, if $u(0)$ and $u(1)$ have distinct eigenvalues, then one may choose $u_1(0) = u(0)$ and $u_1(1) = u(1)$.

Furthermore, if $\det(u(t)) = 1$ for all $t \in [0,1]$, then we may also assume that $\det(u_1(t)) = 1$ for all $t \in [0,1]$.