Preface

These lecture notes are an expanded version of the author's CBMS ten Lectures at the University of Kentucky in June 20–24, 2011. The lectures were devoted to eigenfunctions of the Laplacian and of Schrödinger operators, in particular to their L^p-norms and nodal sets. The lecture notes have undergone extensive revisions in the intervening years, due in part to progress in the field and also to the new publications on related topics, which made some of the original lecture notes obsolete. In particular, the new book [**So2**] of Chris Sogge and the author's 2013 Park City Lecture notes [**Ze7**] are also devoted to eigenfunctions and includes extensive background on pseudo-differential operators and harmonic analysis. (References for the preface can be found at the end of §1.) The book of Maciej Zworski [**Zw**] contains a systematic introduction to semi-classical Fourier integral operators and includes applications to quantum ergodicity of eigenfunctions. The recent book [**GS**] of V. Guillemin and S. Sternberg also gives background on the global theory of Fourier integral operators and in particular on their symbols. Fanghua Lin and Qing Han also have a book in progress on eigenfunctions from viewpoint of local elliptic equations. For this reason, we do not feel it is useful in these lecture notes to provide any systematic background on these techniques, although their properties will be used freely. We do include some background on symplectic geometry, pseudo-differential and Fourier integral operators to establish notation and links to other references. But overall we assume that the reader is willing to consult these other references for the basic techniques.

The purpose of these lecture notes is to convey inter-related themes and results, and so we rarely give detailed proofs. Rather we aim to outline key ideas and how they are related to other results. The lectures concentrate on the following themes:

- Local versus Global analysis of eigenfunctions. The Local analysis of eigenfunctions belongs to the theory of elliptic equations, and pertains to local solutions of the eigenvalue problem $(\Delta + \lambda)\varphi = 0$ on small balls of radius $\frac{C}{\sqrt{\lambda}}$. The global analysis belongs to hyperbolic equations, i.e., studies the eigenfunctions through the wave equation $\cos t\sqrt{-\Delta}\varphi = \cos t\sqrt{\lambda}\varphi$ and their relations to geodesics as $\lambda \to \infty$. One of the aims of these lectures is to survey both local and global methods, and to discuss how they interact. For instance, the main existence theorem that there exists a zero of φ_λ in each ball $B(p, \frac{A_g}{\sqrt{\lambda}})$ whose radius is a certain number C_g of wavelengths is a local result and global methods are not particularly useful in proving it. On the other hand, the basic sup-norm estimates of eigenfunctions are most easily proved using the wave equation. It often seems that researchers on eigenfunctions split into two disjoint groups, exclusively using local or global methods. It is likely that many problems require both types of methods. In §5.3 we review the elliptic methods that have been applied to eigenfunctions by Donnelly-Fefferman, F. H. Lin, Nazarov-Sodin, Colding-Minicozzi, and many others.

- Quantum analogues of classical dynamical methods for ergodic or completely integrable systems. For instance, Birkhoff normal forms are local normal forms on both the classical and quantum level around invariant sets such as closed geodesics, which are useful in study concentration on submanifolds.
- L^p bounds on eigenfunctions and their source in the global dynamics of the geodesic flow.
- Restriction theorems for eigenfunctions under dynamical assumptions mainly in the ergodic setting.
- Nodal geometry in the complex domain. Considerable space is devoted to analytic continuation of eigenfunctions of Laplacians of real analytic Riemannian manifolds to the complexification of the manifold. The rationale for analytic continuation is that the nodal sets are better behaved and easier to study in the complex domain than the real domain. From the viewpoint of quantum mechanics, both the real and complex domains are equally good representations.

0.1. Organization

Let us go over the sequence of events in these lectures and explain what is and what is not contained in them and what is the logic of the presentation.

We introduce the subject of eigenfunctions in terms of vibrating membranes and quantum energy eigenstates. The rich phenomenology of examples developed over the last two hundred years is rapidly surveyed. In Chapter 3 we give an overview of the principal new results that will be discussed in detail. The model surfaces of constant curvature are introduced in §4. Harmonic analysis begins with the Euclidean eigenfunctions $e^{i\langle x,\mathbf{k}\rangle}$ on $\mathbb{R}^n$ or T^n, yet they have very unusual properties compared to eigenfunctions on other Riemannian manifolds. The eigenfunctions of S^2 illustrate virtually the entire range of behavior of eigenfunctions of any Riemannian metric with regard to size and concentration. On the other hand, they are restrictions of harmonic polynomials on $\mathbb{R}^3$ and their nodal sets are potentially tamer than for a general C^∞ metric. Eigenfunctions of hyperbolic surfaces $\mathbb{H}^2/\Gamma$ come next. They are the material of quantum chaos and are the subject of intense investigation over the last 30 years. In §5-5.3 the local elliptic analysis of eigenfunctions is surveyed. This leads §6 on the wave equation on a Riemannian manifold and the Hadamard-Riesz construction of parametrices. This construction parallels the Minakshisundaraman-Pleijel parametrix construction for heat kernels. In some ways, the original presentations of Hadamard and Riesz remain the best expositions, in particular in their presentations of the convergence of the parametrix construction in the real analytic case. It was a precursor to the Fourier integral operator theory, which is rapidly reviewed in §2.1, §2.5, §7. As mentioned above, this material is contained in many other references and is principally used to establish notation. In §8.2 classical results on the pointwise and local Weyl laws are reviewed, and the results presented give the universal sup-norm estimates on eigenfunctions and their gradients. The author is not aware of a proof of such estimates using elliptic estimates. Geometric analysts who are more familiar with elliptic estimates might want to compare their methods to the small time wave equation methods used in the proofs. In §9, the asymptotics and limits of matrix elements $\langle A\varphi_j, \varphi_j\rangle$ of pseudo-differential operators with respect to eigenfunctions are introduced. Matrix elements are the fundamental quantities in quantum mechanics. They are quadratic in the eigenfunctions and thus are related to energy estimates. There exist some results on multilinear eigenfunction estimates but they are not covered in these lectures. In §9.5 the basic facts about quantum ergodic systems are reviewed. At this point in the lecture notes, the global long-time dynamics of the geodesic flow takes over as the dominant player. In §11 some parallel results for quantum integrable systems are presented. At this time there exist only a few results on quantizations of mixed systems, and despite the great interest in mixed systems we do not present these results but only record the existence of several articles devoted to them. L^p norms of eigenfunctions are studied in §10. Sogge's books [**So1**, **So2**] also concern L^p norms but the material presented here contains both less and more on them. Less, because the universal Sogge estimates are not presented, and more because the more advanced results due to Sogge and the author are given in some detail. In §11.6, L^p norms of eigenfunctions in the quantum integrable case are reviewed. One of the motivations to include this material is the belief that such QCI eigenfunctions are extremals for L^p norms and restrictions of eigenfunctions. Although it is very relevant the restriction theorems of Burq-Gérard-Tzvetkov are not discussed here. Rather we turn to quantum ergodic restriction theorems in §12.21. They have proved useful in the study of nodal sets and that is the main topic for the rest of the lectures. Nodal sets in the real domain are discussed in §13, in particular bounds on hypersurface volumes and counting nodal domains. Starting in §14, the analytic continuation of eigenfunctions to Grauert tubes and their complex zeros are studied. Complex nodal sets and their intersections with complexified geodesics are studied in §14.30. Use of the complexified wave kernel gives a simplified proof of the Donnelly-Fefferman upper bound on the hypersurface measure of nodal sets. The lower bound seems to be disconnected from global methods. In §14.33, Alex Brudnyi has contributed a simplified proof of the Donnelly-Fefferman lower bound. In §14.37, the author's results on equidistribution of complexified nodal sets in the ergodic case are presented. There are parallel results in the completely integrable case which are still in progress. Other results in this section are those of John Toth and the author giving upper bounds on numbers of intersection points of nodal lines with curves in dimension two.

0.2. Topics which are not covered

There are many important topics on eigenfunctions which are not discussed in these lecture notes, due to time and length constraints. A more comprehensive treatment of eigenfunctions would include the following topics:

- Arithmetic quantum chaos. These lecture notes are devoted to PDE methods and therefore we do not get into the special methods available for Hecke-Maass forms on arithmetic quotients. The sharpest results on L^p norms or nodal sets of eigenfunctions are for these special joint eigenfunctions of Δ and of Hecke operators. One might compare their special properties to those of joint eigenfunctions of a quantum integrable system although they are much more complicated and the dynamics is in the opposite chaotic regime.

- Entropy of quantum limits. The breakthrough results of Anantharaman and the subsequent work of Anantharaman-Nonnenmacher and Rivière are very relevant to the theme of these lectures.
- General L^p bounds on restrictions of eigenfunctions, multilinear estimates and Kakeya-Nikodym bounds.
- Gaussian random spherical harmonics and more general random linear combinations of eigenfunctions.
- Spectral and scattering theory for non-compact complete Riemannian manifolds.

0.3. Topics which are double covered

It is impossible to avoid overlaps with the author's prior expository articles, such as the article on local and global analysis of eigenfunctions [**Ze3**], on nodal sets [**Ze6**] or Park City Lecture notes [**Ze7**] and other expository articles on eigenfunctions and nodal sets.

Another double-coverage is with regard to cited references. Each chapter has a bibliography of the references cited in it. Many references are cited in several chapters. Although this results in duplicated references it seems preferable to only listing hundreds of references at the end of the lecture notes.

0.4. Notation

Notation regarding eigenvalue parameters is given in §1.2 and notation for geometric and dynamical objects is given in §2.

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