Introduction

The theory of valuations on convex sets is a classical part of convex geometry which goes back at least to the positive solution of the third Hilbert problem by M. Dehn in 1900. Dehn showed that in a 3-dimensional Euclidean space, a cube and tetrahedron of equal volume are not scissor congruent, namely the cube cannot be dissected into finitely many convex polytopes such that after independent translation and rotation of each part one can construct the tetrahedron. In our language, Dehn has constructed a finitely additive measure (i.e., a valuation) on polytopes which is isometry invariant, vanishes on polytopes of dimension less than 3, and takes different values on the cube and the tetrahedron. This purely combinatorial theory is rather rich now, see, e.g., the surveys [**43**], [**44**], and the book [**29**]. Then theory of valuations was forgotten for a while until Blaschke, motivated by the applications to integral geometry, started to study valuations again in 1930's. His results on valuations seem to be somewhat partial.

Systematic study of valuations on polytopes and convex compact sets was initiated by Hadwiger in 1940s and 1950s, see, e.g., his book [**38**]. Hadwiger has obtained a number of fundamental results in the area, some of which are discussed in these lecture notes. He seems to be the first to realize the crucial importance of the class of valuations on convex compact sets continuous in the Hausdorff metric. This class of valuations is also central to these notes. We would like to make a loose analogy with the classical measure theory. There is no rich systematic theory of finitely additive measures on, say, Borel sets. Usually one imposes the additional analytic condition of countable additivity. Under this extra assumption one has a rich general theory which also has many applications in analysis, probability, etc. Similarly there is no rich theory of all valuations on arbitrary convex compact sets. However, if one imposes the analytic condition of continuity in the Hausdorff metric then, as it was shown by Hadwiger, there is a rich general theory of such valuations (especially translation-invariant ones) which covers many examples of geometric interest.

Since Hadwiger, valuations have been studied systematically. We will not give here any historical overview of these developments. However, we find it appropriate to mention that in our opinion the modern period in the study of valuations begins in 1995 with the Klain–Schneider classification (see [**39**], [**46**]) of continuous translation-invariant simple valuations. At the time this was a major breakthrough in the field, which led subsequently to many further developments. After it was proved, the theory of valuations underwent rapid development in the last 20 years; many of the more recent results are based on the Klain–Schneider theorem. This theorem is discussed in detail in these lectures.

The Klain–Schneider theorem was used by the author in his proof [**4**] of McMullen's conjecture about a description of all translation-invariant continuous valuations. These lectures are focused around the proof of the McMullen's conjecture which has also had several further applications in the theory of continuous valuations. However, the proof presented here is not completely self-contained: at some moment we use computations from the theory of infinite dimensional representation of the group $GL(n, \mathbb{R})$. We decided not to discuss these computations here since this requires a background beyond the scope of these lectures. Let us notice nevertheless that we believe that representation theoretical tools might be useful in other problems in convexity related to harmonic analysis on Grassmannians or partial flag manifolds. To illustrate this in a simpler situation we used similar arguments, again without detailed computations, in the proof of the Klain–Schneider theorem which originally was proved using the more elementary technique of spherical harmonics.

In section 10 we give an overview of more recent developments in the theory of translation-invariant continuous valuations. In particular, we describe the structures of product, convolution, and Fourier type transforms on the dense subspace of so-called smooth valuations. Very recently these structures turned out to be useful in applications of the valuations theory to integral geometry; these applications will not be discussed in the lectures. In subsection 10.8 we review a construction of continuous translation-invariant valuations using various Monge-Ampère operators.

In these notes, we do not assume any preliminary knowledge of valuations theory. However, we will use several basic results about general convexity. We will formulate these results precisely and provide references to proofs. As a general reference to convexity we use Schneider's book [**45**] (the numeration corresponds to the second edition). Notice that this second edition contains a new chapter (Chapter 6) about valuations which significantly overlaps with the material of these lectures.

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