

## CHAPTER 1

### Basic definitions and examples

Let  $V$  be a finite-dimensional real vector space,  $n := \dim V$ . We denote by  $\mathcal{P}(V)$  (resp.,  $\mathcal{K}(V)$ ) the family of all compact convex non-empty polytopes, resp. convex sets.

1.0.1. DEFINITION. A scalar-valued functional

$$\phi: \mathcal{P}(V) \text{ (resp. } \mathcal{K}(V)) \rightarrow \mathbb{C}$$

is called a valuation if it has the following additivity property:

$$\phi(A \cup B) = \phi(A) + \phi(B) - \phi(A \cap B),$$

whenever  $A, B, A \cup B \in \mathcal{P}(V)$  (resp.  $\mathcal{K}(V)$ ).

1.0.2. REMARK. (1) The main emphasis in these lectures will be on valuations on  $\mathcal{K}(V)$  rather than on  $\mathcal{P}(V)$ . Nevertheless there will be several important purely combinatorial results for valuations on  $\mathcal{P}(V)$ .

(2) We will discuss only translation-invariant valuations, i.e.,  $\phi(A+x) = \phi(A)$  for any  $A$  and any  $x \in V$ . Moreover, in the case of  $\mathcal{K}(V)$  the valuations will be assumed to be continuous (see below).

(3) There are interesting classes of valuations which are not scalar valued, but vector valued. Sometimes we study valuations taking values in a semigroup, e.g. the semigroup of all convex compact sets equipped with the operation of the Minkowski addition.

Let us define the class of continuous valuations on  $\mathcal{K}(V)$  which will be of central importance in these lectures.

1.0.3. DEFINITION. A valuation on  $\mathcal{K}(V)$  is called continuous if it is continuous in the Hausdorff metric on  $\mathcal{K}(V)$ .

Let us recall the definition of the Hausdorff metric. Let us fix a Euclidean metric on  $V$ . Define the Hausdorff distance between two compact convex sets by

$$d_H(A, B) = \inf\{\varepsilon > 0 \mid A \subset B_\varepsilon \text{ and } B \subset A_\varepsilon\},$$

where  $A_\varepsilon$  denotes the  $\varepsilon$ -neighborhood of  $A$ . It is not hard to check that  $d_H$  satisfies all the axioms of a metric. By the Blaschke selection theorem (see, e.g., [45], Thm. 1.8.7),  $\mathcal{K}(V)$  equipped with the metric  $d_H$  is a locally compact space; in fact any closed ball is compact. The topology on  $\mathcal{K}(V)$  induced by  $d_H$  is independent of the choice of Euclidean metric on  $V$ . Usually we will use only the topology rather than the metric.

Let us denote by  $Val(V)$  the set of all translation-invariant continuous ( $\mathbb{C}$ -valued) valuations on  $\mathcal{K}(V)$ . Clearly it is a complex vector space.

Let us give some examples of translation-invariant continuous valuations.

1.0.4. EXAMPLE. (1) A Lebesgue measure  $vol$  belongs to  $Val(V)$  (prove it!).

(2) The Euler characteristic  $\chi$  defined by  $\chi(K) = 1$  for any  $K \in \mathcal{K}(V)$  also belongs to  $Val(V)$  (prove it!).

(3) Let  $\phi$  be a continuous valuation. Let us fix  $A \in \mathcal{K}(V)$ . Then

$$\psi(K) := \phi(K + A)$$

is also a continuous valuation, where  $K + A := \{k + a \mid k \in K, a \in A\}$  is the Minkowski sum. The continuity of  $\psi$  follows from the fact that if  $K_i \rightarrow K$  then  $K_i + A \rightarrow K + A$ . Let us prove the valuation property. It suffices to prove the following two statements for  $K, L \in \mathcal{K}(V)$ :

(a)  $(K \cap L) + A = (K + A) \cap (L + A)$ , provided  $K \cup L$  is convex;

(b)  $(K \cup L) + A = (K + A) \cup (L + A)$ .

Property (b) being obvious, let us prove (a). The inclusion  $(K \cap L) + A \subset (K + A) \cap (L + A)$  is obvious. To show the converse, let  $x \in (K + A) \cap (L + A)$ . Then for some  $k \in K, l \in L, a_1, a_2 \in A$  one can write

$$x = k + a_1 = l + a_2.$$

Let  $I$  be the segment connecting  $k$  and  $l$ . Then  $I \subset K \cup L$  since  $K \cup L$  is convex.  $I$  is a union of two closed segments  $I \cap K$  and  $I \cap L$ . Since the segment  $I$  is connected, these segments must intersect; namely there exists  $0 \leq \lambda \leq 1$  such that  $z := \lambda k + (1 - \lambda)l \in K \cap L$ . Hence

$$\begin{aligned} x &= \lambda(k + a_1) + (1 - \lambda)(l + a_2) = (\lambda k + (1 - \lambda)l) + (\lambda a_1 + (1 - \lambda)a_2) = \\ &= z + (\lambda a_1 + (1 - \lambda)a_2) \in (K \cap L) + A. \end{aligned}$$

The result is proved.

(4) As a special case of the previous example we have that the valuation  $K \mapsto vol(K + A)$  belongs to  $Val(V)$  where  $A \in \mathcal{K}(V)$  is fixed.

(5) This example will be formulated precisely below. Let  $n := \dim V$ . Fix an integer  $0 \leq i \leq n$  and  $A_1, \dots, A_{n-i} \in \mathcal{K}(V)$ . Then the mixed volume

$$K \mapsto V(\underbrace{K, \dots, K}_i \text{ times}, A_1, \dots, A_{n-i})$$

belongs to  $Val(V)$ .

There is a modified definition of valuation when the additivity property from Definition 1.0.1 is replaced by the following stronger *inclusion-exclusion* property.

1.0.5. DEFINITION. We say that a functional  $\phi: \mathcal{P}(V)$  ( resp.,  $\mathcal{K}(V)$ )  $\rightarrow \mathbb{C}$  satisfies the inclusion-exclusion property if, for any  $A \in \mathcal{P}(V)$  ( resp.,  $\mathcal{K}(V)$ )

represented as a finite union  $A = \cup_{i=1}^s A_i$  with all  $A_i \in \mathcal{P}(V)$  (resp.,  $\mathcal{K}(V)$ ), one has

$$(1.0.1) \quad \phi(A) = \sum_{\emptyset \neq I \subset \{1, \dots, s\}} (-1)^{|I|-1} \phi(\cap_{j \in I} A_j),$$

where the sum is over all non-empty subsets of  $\{1, \dots, s\}$ .

Notice that Definition 1.0.1 of valuation is equivalent to the inclusion-exclusion property with  $s = 2$  only. One can show that all the above examples of valuations do satisfy the inclusion-exclusion property (see part (2) of Theorem 1.0.6 below). Moreover, there are general sufficient conditions on a valuation to satisfy this property. We have the following theorem.

1.0.6. THEOREM. (1) (*Volland 1957*, [47]) *Every valuation on  $\mathcal{P}(V)$  satisfies the inclusion-exclusion property.*

(2) (*Groemer 1978*, [35]) *Every continuous valuation on  $\mathcal{K}(V)$  satisfies the inclusion-exclusion property.*

**Proof.** (1) Let  $A = \cup_{i=1}^s A_i$  where all sets are convex polytopes. Let us prove the statement by induction on  $n + s$ . We may assume that  $s > 2$ . Let us fix an affine hyperplane  $H$  such that  $A_s$  is contained in one of the closed half-spaces of it, say in  $H^+$ . We are going to show first that (1.0.1) holds if and only if it holds when all sets are replaced with their intersections with  $H^+$ . Let us denote  $A^\pm = A \cap H^\pm$ ,  $A_i^\pm = A_i \cap H^\pm$ .

Obviously

$$(1.0.2) \quad A^\pm = \cup_{i=1}^s (A_i \cap H^\pm)$$

$$(1.0.3) \quad A \cap H = \cup_{i=1}^s (A_i \cap H).$$

Since  $\dim(A \cap H) < n$ , the assumption of induction and (1.0.3) imply

$$(1.0.4) \quad \phi(A \cap H) = \sum_{\emptyset \neq I \subset \{1, \dots, s\}} (-1)^{|I|-1} \phi((\cap_{j \in I} A_j) \cap H).$$

Let us now show that

$$(1.0.5) \quad \phi(A^-) = \sum_{\emptyset \neq I \subset \{1, \dots, s\}} (-1)^{|I|-1} \phi(\cap_{j \in I} A_j^-).$$

If  $\dim A^- < n$  then the result holds by the assumption of induction. Hence let us assume that  $\dim A^- = n$ . Notice that  $\dim A_s^- < n$ . It follows that  $A^- = \cup_{i=1}^{s-1} A_i^-$  (union of only  $s - 1$  sets); hence  $A_s^- = \cup_{i=1}^{s-1} (A_i^- \cap A_s^-)$ . By the induction assumption

$$(1.0.6) \quad \phi(A^-) = \sum_{\emptyset \neq I \subset \{1, \dots, s-1\}} (-1)^{|I|-1} \phi(\cap_{j \in I} A_j^-),$$

$$(1.0.7) \quad 0 = \phi(A_s^-) - \sum_{\emptyset \neq I \subset \{1, \dots, s-1\}} (-1)^{|I|-1} \phi((\cap_{j \in I} A_j^-) \cap A_s^-).$$

Adding the last two equalities we get (1.0.5).

Furthermore for any non-empty subset  $I \subset \{1, \dots, s\}$  we have by the definition of valuation

$$(1.0.8) \quad \phi(\cap_{j \in I} A_j) = \phi(\cap_{j \in I} A_j^+) + \phi(\cap_{j \in I} A_j^-) - \phi(\cap_{j \in I} (A_j \cap H)).$$

Hence we have

$$\begin{aligned} & \sum_{\emptyset \neq I \subset \{1, \dots, s\}} (-1)^{|I|-1} \phi(\cap_{j \in I} A_j) - \sum_{\emptyset \neq I \subset \{1, \dots, s\}} (-1)^{|I|-1} \phi(\cap_{j \in I} A_j^+) \\ (1.0.8) \quad & \sum_{\emptyset \neq I \subset \{1, \dots, s\}} (-1)^{|I|-1} \phi(\cap_{j \in I} A_j^-) - \sum_{\emptyset \neq I \subset \{1, \dots, s\}} (-1)^{|I|-1} \phi(\cap_{j \in I} (A_j \cap H)) \\ (1.0.5)^+ (1.0.4) \quad & \phi(A^-) - \phi(A \cap H) = \phi(A) - \phi(A^+), \end{aligned}$$

where the last equality is by the definition of valuation. Thus we see that (1.0.1) is equivalent to this equality with  $A, A_i$  replaced with  $A^+, A_i^+$  respectively.

Now let us present  $A_s$  as an intersection of finitely many closed half-spaces

$$A_s = \cap_{m=1}^t H_m^+.$$

Applying  $t$  times the above argument we may replace each set  $A, A_i$  by its intersection with all  $H_m^+$ 's. Equivalently we may assume that  $A = A_s$ . In this case the right hand side of (1.0.1) is equal to  $\sum_{I \not\ni s} \dots + \sum_{I=J \sqcup \{s\}} \dots$ , which is in turn equal to

$$\begin{aligned} & \left[ \sum_{I \subset \{1, \dots, s-1\}} (-1)^{|I|-1} \phi(\cap_{i \in I} A_i) \right] \\ (1.0.9) \quad & + \left[ \phi(A_s) - \sum_{\emptyset \neq J \subset \{1, \dots, s-1\}} (-1)^{|J|-1} \phi((\cap_{j \in J} A_j) \cap A_s) \right] \\ & = \phi(A_s) = \phi(A), \end{aligned}$$

where the equality (1.0.9) is just cancellation of two summands since  $A_i \subset A_s = A$  for all  $i$ . Part (1) is proved.

(2) This statement will not be used in these lectures, so we only indicate the argument. The proof is very similar to part (1) with only the difference that  $A_s$  should be represented as a countable union of closed half-spaces  $A_s = \cap_{m=1}^{\infty} H_m^+$ . As above, at each step one should intersect each set with  $H_m^+$  and then go to the limit. Q.E.D.

### 1.1. Hadwiger's decomposition of a simplex

In this section we prove a technical result on a decomposition of a simplex which we will use later. Let  $\Delta_n$  be an  $n$ -dimensional simplex in  $V$ ,  $n = \dim V$ . We can choose a coordinate system such that

$$\Delta_n = \{0 \leq x_1 \leq \dots \leq x_n \leq 1\}.$$

1.1.1. THEOREM (Hadwiger's decomposition). *Let  $\lambda, \mu > 0$ . Then*

$$(1.1.1) \quad (\lambda + \mu)\Delta_n = \cup_{k=0}^n \left( \mu \cdot \Delta_k \times (\lambda \cdot \Delta_{n-k} + (\underbrace{\mu, \dots, \mu}_{n-k \text{ times}})) \right),$$

where  $\Delta_k = \{0 \leq x_1 \leq \dots \leq x_k \leq 1\}$  and  $\Delta_{n-k} = \{0 \leq x_{k+1} \leq \dots \leq x_n \leq 1\}$ . Clearly the interiors of these bodies are pairwise disjoint.

**Proof.** Let  $x := (x_1, \dots, x_n) \in (\lambda + \mu) \cdot \Delta_n$ . Then there exists a unique  $0 \leq k \leq n$  such that

$$0 \leq x_1 \leq \dots \leq x_k \leq \mu < x_{k+1} \leq \dots \leq x_n \leq \lambda + \mu.$$

That means that  $x \in \mu \cdot \Delta_k \times (\lambda \cdot \Delta_{n-k} + (\underbrace{\mu, \dots, \mu}_{n-k \text{ times}}))$ . Q.E.D.

1.1.2. REMARK. By the abuse of notation, we will often omit the shift by  $(\mu, \dots, \mu)$  in the right hand side of (1.1.1), thus writing just  $\mu \cdot \Delta_k \times \lambda \cdot \Delta_{n-k}$ .