Preface

> Mathematics is an experimental science, and definitions do not come first, but later on.
>
> — Oliver Heaviside (1850–1925)

Traditionally the notion of harmonic analysis has centered around analysis of transforms and expansions, and involving *dual variables.* The area of partial differential equations (PDE) has been a source of motivation and a key area of applications; dating back to the days of Fourier. Or rather, the applications might originate in such neighboring areas as signal processing, diffusion equations, and in more general applied inverse problems. The dual variables involved are typical notions of *time* vs *frequency*, or *position* vs *momentum* (in quantum physics). As most students know, Fourier series and Fourier transforms have been a mainstay in analysis courses we teach. In the case of Fourier series, and Fourier transforms, we refer to the variables involved as dual variables. If a function in an x-domain admits a Fourier expansion, the associated transform will be a function in the associated dual variable, often denoted λ, in what is to follow.

Now the x-domain may involve a suitable subset Ω in $\mathbb{R}^d$. The aim of Fourier harmonic analysis is *orthogonal* L^2 *Fourier expansions*, at least initially. Alternatively, the x-domain may refer to a prescribed *measure*, say μ with compact support in $\mathbb{R}^d$. These settings are familiar, at least in the classical case, which we shall here refer to as the "smooth case." Now if the measure μ might be *fractal*, say a Cantor measure, an *iterated function system* (IFS) *measure*, e.g., a Sierpinski construction, then it is not at all clear that the familiar setting of Fourier duality will yield useful orthogonal L^2 decompositions. Take for example the Cantor IFS constructions, arising from scaling by 3, and one gap; or the corresponding IFS measure resulting from scaling by 4, but now with two gaps. In a paper in 1998, Jorgensen and Pedersen showed that the first of these Cantor measures does not admit any orthogonal L^2 Fourier series, while the second does. In the two decades that followed, a rich theory of harmonic analysis for fractal settings has ensued.

If a set Ω, or a measure μ, admits an L^2 spectrum, we shall talk about *spectral pairs* (Ω, Λ), or (μ, Γ), where the second set in the pair will be called a *spectrum.* If the first variable arises as a *fractal in the small*, we will see that associated spectra will arise as *fractals in the large*; in some cases as lacunary Fourier expansions, series with large gaps, or lacunae, between the non-zero coefficients in expansions. We will focus on Fourier series with similar gaps between non-zero coefficients, gaps being a power of a certain scale number. There is a slight ambiguity in modern usage of the term lacunary series. When needed, clarification will be offered in the notes.

In addition to harmonic analyses via *Fourier duality*, there are also *multiresolution wavelet* approaches; work by Dutkay and Jorgensen. In the notes, both will be developed systematically, and it will be demonstrated that the wavelet tools are more flexible but perhaps not as precise for certain fractal applications. A third tool for our fractal harmonic analysis will be L^2 spaces derived from appropriate *Gaussian processes* and their analysis.

In our development of some of these duality approaches, or multiresolution analysis constructions, we shall have occasion to rely on certain *non-commutative* harmonic analyses. They too will be developed from scratch (as needed) in the notes.

The present book is based on a series of 10 lectures delivered in June 2018 at Iowa State University. I am extremely grateful to Feng Tian, who was a big help organizing both the visual material used in the 10 lectures and the text we ended up using for the book.

In addition to my 10 lectures, this CBMS also included the following three featured speakers: Kasso Okoudjou (University of Maryland), Marius Ionescu (United States Naval Academy), and Daniel Alpay (Chapman University). These speakers highlighted connections between the present central themes and a variety of neighboring, current areas of mathematics and its applications. Some relevant references are the following: [**AS17**, **ACQS17**, **AL18**, **AG18**, **IRT12**, **IRS13**, **IR14**, **IOR17**, **WO17**, **BBCO17**].

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