CHAPTER 1

Introduction. Smooth vs the non-smooth categories

> I hope that posterity will judge me kindly, not only as to the things which I have explained, but also to those which I have intentionally omitted so as to leave to others the pleasure of discovery.
> — Descartes, René (1596-1650)

There is a recent and increasing interest in understanding the harmonic analysis of non-smooth geometries, typically fractal like. They are unlike the familiar smooth Euclidean geometry. In the non-smooth case, nearby points are not locally connected to each other. Real-world examples where these types of geometry appear include large computer networks, relationships in datasets, and fractal structures such as those found in crystalline substances, light scattering, and other natural phenomena where dynamical systems are present.

The book is based on a series of lectures on smooth and non-smooth harmonic analysis by the author. It aims to demonstrate surprising connections between the two domains of geometry and Fourier spectra, and to bring both experienced and new researchers together to stimulate collaboration on this timely topic. It also aims to advance representation and participation of underrepresented minorities within mathematics, and the development of a globally competitive STEM workforce.

1.1. Preview

Smooth harmonic analysis refers to harmonic analysis over a connected or locally connected domain — typically Euclidean space or locally connected subsets of Euclidean space. The classical example of this is the existence of Fourier series expansions for square integrable functions on the unit interval. *Non-smooth harmonic analysis then refers to harmonic analysis on discrete or disconnected domains* — typical examples of this setting are *Cantor like subsets* of the real line and analogous *fractals in higher dimensions.* In 1998, Jorgensen and Steen Pedersen proved a result: there exists a Cantor like set (of Hausdorff dimension 1/2) with the property that the uniform measure supported on that set is spectral, meaning that there exists a sequence of frequencies for which the exponentials form an orthonormal basis in the Hilbert space of square integrable functions with respect to that measure. This surprising result, together with results of Robert Strichartz, has lead to a plethora of new research directions in non-smooth harmonic analysis.

Research that has been inspired by this surprising result includes: *fractal Fourier analyses* (fractals in the large), spectral theory of Ruelle operators; representation theory of Cuntz algebras; convergence of the cascade algorithm in wavelet theory; reproducing kernels and their boundary representations; Bernoulli convolutions and Markov processes. The remarkable aspect of these broad connections is that they often straddle both the smooth and non-smooth domains. This is particularly evident in Jorgensen's research on the *cascade algorithm*, as wavelets already possess a "dual" existence in the *continuous* and *discrete* worlds, and also his research on the *boundary representations* of *reproducing kernels*, as the non-smooth domains appear as boundaries of smooth domains. In work with Dorin Dutkay, Jorgensen showed that the general affine IFS-systems, even if not amenable to Fourier analysis, in fact do admit wavelet bases, and so in particular can be analyzed with the use of multiresolutions. In recent work with Herr and Weber, Jorgensen has shown that fractals that are not spectral (and so do not admit an orthogonal Fourier analysis) still admits a harmonic analysis as boundary values for certain subspaces of the Hardy space of the disc and the corresponding reproducing kernels within them.

The book covers the following overarching themes: the *harmonic analysis* of Cantor spaces (and measures) arising as *fractals* (including fractal dust) and iterated function systems (IFSs), as well as the methods used to study their harmonic analyses that span both the smooth and non-smooth domains. A consequence of the fact that these methods form a bridge between the smooth and non-smooth domain is that the topics to be discussed — while on the surface seem largely unrelated — actually are closely related and together form a tightly focused theme. Hopefully, the breadth of topics will attract a broader audience of established researchers, while the interconnectedness and sharply focused nature of these topics will prove beneficial to beginning researchers in non-smooth harmonic analysis.

Inside the book, we cover a number of theorems due to a diverse list of authors and co-authors. In some cases, the authors and co-authors are simply identified by name; in some cases, if it isn't clear from the context, also one or more research papers are cited. In the latter case, we use the usual citation codes; for example, the paper [**DJP09**] is co-authored by Dorin Ervin Dutkay, Palle E. T. Jorgensen, and Gabriel Picioroaga, and appeared in 2009. And, of course, full details are included in the Reference list. Yet for other theorems, the co-authors' names are listed in parenthesis, in the statement of the theorem itself. For example: Theorem 3.3.9 (Jo-Pedersen); with my name Jorgensen abbreviated "Jo."

The 10 lectures. The material in the present book corresponds to the areas covered in the 10 lectures. But, for pedagogical reasons, we chose to organize the material a bit differently in the book. Readers may wish to compare the book-form *table of contents* with the title of the 10 lectures. The latter list is included below (also see Figure 1.1.1):

Lecture 1.	Harmonic analysis of measures: Analysis on fractals
Lecture 2.	*Spectra of measures*, tilings, and wandering vectors
Lecture 3.	*The universal tiling conjecture* in dimension one and operator fractals
Lecture 4.	*Representations of Cuntz algebras* associated to quasi-stationary Markov measures
Lecture 5.	The Cuntz relations and *kernel decompositions*
Lecture 6.	Harmonic analysis of *wavelet filters*: input-output and state-space models
Lecture 7.	Spectral theory for *Gaussian processes*: reproducing kernels, boundaries, and L^2-wavelet generators with fractional scales

Lecture 8.	Reproducing kernel Hilbert spaces arising from groups
Lecture 9.	Extensions of positive definite functions
Lecture 10.	*Reflection positive* stochastic processes indexed by Lie groups

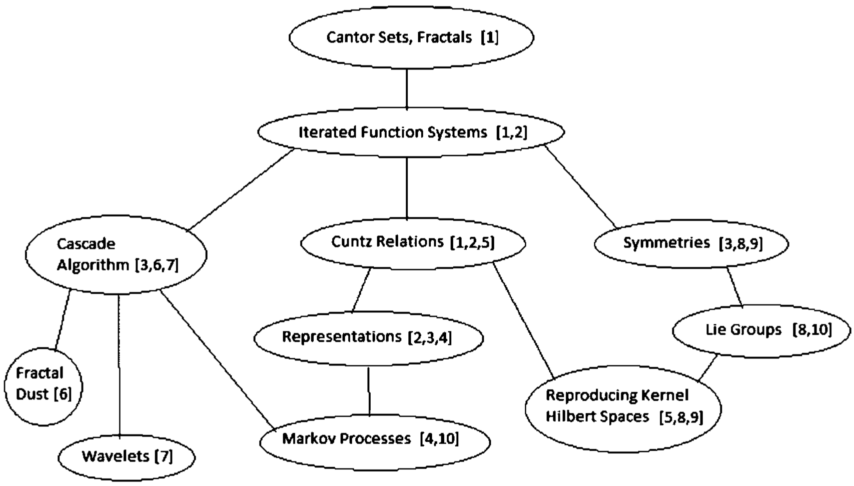


FIGURE 1.1.1. Flow and Connections of Topics: The figure gives a bird's eye view of the main topics in the book, and the lines indicate interconnections. They will be fleshed out in full detail inside the book. The numbers above, inside square brackets, indicate which of the 10 lectures cover the topic in question. The complete title of each of the 10 lectures is listed in the table above.

1.2. Historical context

One of the most fruitful achievements of mathematics in the past two hundred years has been the development of *Fourier series.* Such a series may be thought of as the decomposition of a periodic function into sinusoid waves of varying frequencies. Application of such decompositions are naturally abundant, with waves occurring in all manner of physics, and uses for periodic functions being present in other areas such as *economics* and *signal processing*, just to name a few. The importance of Fourier series is well-known and incontestable.

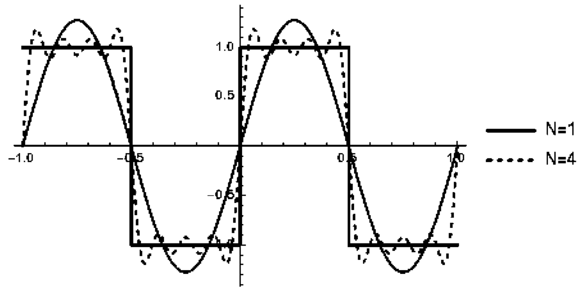
Fourier series. While to many non-mathematicians and undergraduate math majors, a Fourier series is regarded as a breakdown into sine and cosine waves, the experienced analyst will usually think of it (equivalently), as a decomposition into sums of complex exponentials. For instance, in the classical setting of the unit interval $[0,1)$, a Lebesgue integrable function $f:[0,1)\to\mathbb{C}$ will induce a Fourier series

$$f(x)\sim\sum\nolimits_{n\in\mathbb{Z}}\hat{f}(n)\,e^{i2\pi nx} \tag{1.2.1}$$

where

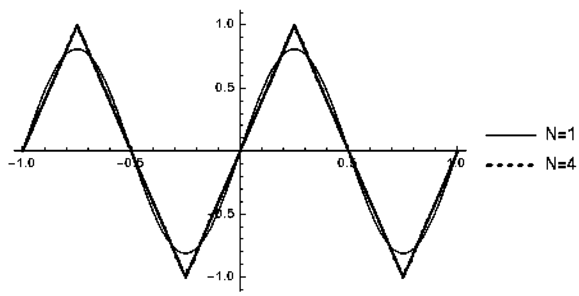
$$\hat{f}(n):=\int_0^1 f(x)\,e^{-i2\pi nx}dx. \tag{1.2.2}$$

See Figures 1.2.1-1.2.2.



$$f\left(x\right) \sim \frac{4}{\pi} \sum_{k=1}^{N} \frac{\sin\left(2\pi\left(2k-1\right)x\right)}{2k-1}$$

FIGURE 1.2.1. Fourier series approximation of square wave. The figure illustrates the known difficulty with Fourier series approximation of step functions. In view of this, it seems even more surprising that some fractals admit convergent Fourier series (see Section 2.1.) Also compare the function in Figure 1.2.1 with the mother function for the Haar wavelet, see Section 4.3.



$$f\left(x\right) \sim \frac{8}{\pi^2} \sum_{k=1}^{N} \frac{(-1)^{k-1}\sin\left(2\pi\left(2k-1\right)x\right)}{\left(2k-1\right)^2}$$

FIGURE 1.2.2. Fourier series approximation of triangle wave. Of course, here Fourier yields a good fit. But it is also clear that it is not good from the point of view of numerical analysis. For example, most wavelet algorithms will do a lot better; see Section 4.3.

Because the Fourier series is intended to represent the function $f\left(x\right)$, it is only natural to ask in what senses, if any, the sum above converges to $f\left(x\right)$. One can ask important questions about *pointwise convergence*, but it is more relevant for our purposes to restrict attention to various *normed spaces* of functions or, as we will be most concerned with hereafter, a Hilbert space consisting of square-integrable functions, and then ask about norm convergence. In our present context, if we let $L^2\left(\left[0,1\right)\right)$ denote the Hilbert space of (equivalence classes of) functions

$f : [0,1) \to \mathbb{C}$ satisfying

$$\|f\|^2 := \int_0^1 |f(x)|^2 \, dx < \infty \tag{1.2.3}$$

and equipped with the inner product

$$\langle f, g \rangle := \int_0^1 f(x) \overline{g(x)} dx, \tag{1.2.4}$$

then if $f \in L^2([0,1))$, the convergence in (1.2.1) will occur in the norm of $L^2([0,1))$. It is also easy to see that in $L^2([0,1))$,

$$\left\langle e^{i2\pi mx}, e^{i2\pi nx} \right\rangle = \int_0^1 e^{i2\pi mx} e^{-i2\pi mx} dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{otherwise.} \end{cases} \tag{1.2.5}$$

That is, the set of complex exponentials $\left\{e^{i2\pi nx}\right\}_{n\in\mathbb{Z}}$ is *orthogonal* in $L^2([0,1))$. Since every function in $L^2([0,1))$ can be written in terms of these exponentials, $\left\{e^{i2\pi nx}\right\}_{n\in\mathbb{Z}}$ is in fact an *orthonormal basis* of $L^2([0,1))$.

Because there exists a countable set of complex exponential functions that form an orthogonal basis of $L^2([0,1))$, we say that the set $[0,1)$ is *spectral*. The set of frequencies of such an orthogonal basis of exponentials, which is this case is $\mathbb{Z}$, is called a *spectrum*.

Like most areas of analysis, the historical and most common contexts for Fourier series are also the most mundane: The functions they decompose are defined on $\mathbb{R}$, the unit interval $[0,1)$, or sometimes a discrete set. The underlying measure used for integration is *Lebesgue measure*. It is thanks to the work of many individuals, including the author, that modern Fourier analysis has been able to aspire beyond these historical paradigms. Table 1 below provides an overview of *generalized Fourier duality*.

The first paradigm break is to consider a wider variety of domains in a wider variety of dimensions. In general, if C is a compact subset of $\mathbb{R}^n$ of nonzero Lebesgue measure, then we say that C is spectral if there exists a countable set $\Lambda \subset \mathbb{R}^n$ such that $\left\{e^{i2\pi\lambda\cdot\vec{x}}\right\}_{n\in\Lambda}$ is an orthogonal basis of $L^2(C)$, where

$$L^2(C) := \left\{ f : C \to \mathbb{C} \mid \int_C |f(\vec{x})|^2 \, d\lambda^n(\vec{x}) < \infty \right\}. \tag{1.2.6}$$

Here λ^n is Lebesgue measure in $\mathbb{R}^n$.

Fuglede's conjecture. The famous Fuglede Conjecture surmised that C would be spectral if and only if it would tessellate by translation to cover $\mathbb{R}^n$. Iosevich, Katz, and Tao proved in 2001 that the conjecture holds for convex planar domains [**IKT03**]. In the same year, they also proved that a smooth, symmetric, convex body with at least one point of nonvanishing Gaussian curvature cannot be spectral [**IKT01**]. However, in 2003 Tao devised counterexamples to the Fuglede Conjecture in $\mathbb{R}^5$ and $\mathbb{R}^{11}$ [**Tao04**]. The conjecture remains open in low dimensions.

The second paradigm break is to substitute a different Borel measure in place of Lebesgue measure. For example, if μ is any Borel measure on $[0,1)$, one can form the Hilbert space

$$L^2(\mu) = \left\{ f : [0,1) \to \mathbb{C} \mid \int_0^1 |f(x)|^2 \, d\mu(x) < \infty \right\} \tag{1.2.7}$$

TABLE 1. Harmonic analysis of measures with the use of *Fourier bases*, *Parseval frames*, or *generalized transforms*: An overview of generalized Fourier duality: Measures vs spectra.

	Measure side	Spectrum side ($e_\lambda(x) = e^{i2\pi\lambda\cdot x}$)
1	$\Omega \subset \mathbb{R}^d$ a Borel set with finite d-dimensional Lebesgue measure λ_d	$\Lambda \subset \mathbb{R}^d$, a subset such that $\{e_\lambda \mid \lambda \in \Lambda\}$ restricts to an orthogonal total system in $L^2(\Omega)$ (w.r.t. λ_d).
2	μ a compactly supported measure in $\mathbb{R}^d$	$\Lambda \subset \mathbb{R}^d$, a subset such that $\{e_\lambda \mid \lambda \in \Lambda\}$ is an orthogonal $L^2(\mu)$ basis.
3	μ as above, but $d = 1$, μ assumed singular	$\{g_n \mid n \in \mathbb{N}_0\} \subset L^2(\mu)$ is a Parseval frame, i.e., $\|f\|^2_{L^2(\mu)} = \sum_0^\infty \lvert \langle f, g_n\rangle_{L^2(\mu)} \rvert^2$ with $L^2(\mu)$ expansion for $f \in L^2(\mu)$: $f(x) = \sum_0^\infty \langle f, g_n\rangle_{L^2(\mu)} e_n(x)$, i.e., summation over $n \in \mathbb{N}_0$.
4	Symmetric case: μ, ν two Borel measures in $\mathbb{R}^d$ such that $(F_\mu f)(\xi) = \int_{\mathbb{R}^d} f(x) e_\xi(x) d\mu(x)$, defines an isometric isomorphism onto $L^2(\nu)$, i.e., $$\int_{\mathbb{R}^d} \lvert F_\mu(f)(\xi)\rvert^2 d\nu(\xi) = \int_{\mathbb{R}^d} \lvert f(x)\rvert^2 d\mu(x), \ \forall f \in L^2(\mu).$$	

with inner product

$$\langle f, g\rangle_\mu = \int_0^1 f(x)\overline{g(x)} d\mu(x). \tag{1.2.8}$$

Comparing equations (1.2.7) and (1.2.8) with equations (1.2.3) and (1.2.4), respectively, we see that we can then regard spectrality as a property of measures rather than of sets: The measure μ is spectral if there exists a countable index set Λ such that the set of complex exponentials $\{e^{i2\pi\lambda x}\}_{\lambda\in\Lambda}$ is an orthogonal basis of $L^2(\mu)$. The index set Λ is then called a spectrum of μ.

Guide to readers. Inside the text in present Introduction, we have been, and will be, using some technical terms which might perhaps not be familiar to all readers. For example, the notion of *iterated function systems* (IFS) are mentioned, and they will be explored in detail in Section 1.3 below, and again later in Chapters 2, 3, and 6. In the present discussion, around the themes of Figure 1.2.4 and the table, we use the concept of *selfadjoint extensions* of densely defined Hermitian symmetric operators in Hilbert space. This is from the theory of unbounded operators in Hilbert space. Again, these tools will be made precise later in the book, for example in Subsection 3.4.1, especially Lemma 3.4.2. In fact these tools will also play an important role in Chapters 3 and 6.

In the Introduction we also refer to *representations of the Cuntz relations* (see eq (1.3.8)), especially in connection with multi-frequency band analysis; see e.g., Remark 1.3.5, and Figure 1.4.1. These notions from representation theory, and their applications, will be taken up in a systematic fashion in Chapter 5 below.

<table>
<tr><th>A: spectrum</th><th>B: translation tiling</th></tr>
<tr><td>$\Omega \subset \mathbb{R}^d$, $|\Omega| < \infty$. $\exists \Lambda$ s.t. $\{e_\lambda\,;\,\lambda \in \Lambda\}$ is an orthogonal basis in $L^2(\Omega)$.</td><td>$\exists T \subset \mathbb{R}^d$ s.t. $\Omega \dotplus T = \mathbb{R}^d$</td></tr>
<tr><th colspan="2">C: operator theory</th></tr>
<tr><td colspan="2">$\exists$ s.a. commuting extension operators $H_j \supset \frac{1}{i}\frac{\partial}{\partial x_j}\big|_{C_c^\infty(\Omega)}$, $1 \le j \le d$;
$H_j = \int_{\mathbb{R}} \lambda\, E_j(d\lambda)$, $E_j(A)\,E_k(B) = E_k(B)\,E_j(A)$, $\forall j,k$, $\forall A,B \in \mathscr{B}(\mathbb{R})$.</td></tr>
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FIGURE 1.2.3. Three related properties for open subsets Ω in $\mathbb{R}^d$: (A) Ω is spectral, (B) Ω admits a translation tiling set, and (C) the minimal symmetric partial derivative operators for Ω admit commuting selfadjoint extensions. Equivalence of (A) and (B) is called the Fuglede conjecture. It is open for $d = 1$, and $d = 2$. But for $d = 3$ and higher, (A) and (B) are known not to be equivalent. In general (A) implies (C); and if Ω is also assumed connected, then (C) implies (A). Also see Table 1.

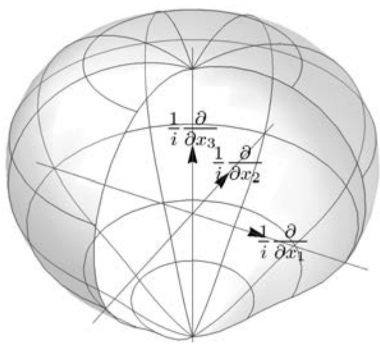


FIGURE 1.2.4. $H_j \supset \frac{1}{i}\frac{\partial}{\partial x_j}\big|_{C_c^\infty(\Omega)}$, $1 \le j \le d$, the partial derivatives.

In our initial discussion and presentation here, we have chosen to start out by first describing a number of applications, and then postpone a more complete treatment of technical issues involved, until later in the book.

1.3. Iterated function systems (IFS)

The title above refers to a class of measures arising naturally in geometric measure theory; they are generated by a prescribed system of functions, and the construction is based on iteration; hence the name Iterated function systems (IFS). The purpose of the brief outline below is to explain a selection of measures. This sample includes measures, and associated maps, the measures with support in an ambient space $\mathbb{R}^d$, and the maps defined in $\mathbb{R}^d$. Examples for all values of d. Here we focus on the case when the initial system of maps are from the class of affine maps in $\mathbb{R}^d$, but the theory of IFSs includes a much wider array of examples and applications; some of which will be taken up later inside the book.

One of the tools we shall employ in our harmonic analysis considerations is as follows: To a particular IFS we shall associate certain systems of operators (S_j); also called systems of *Cuntz isometries*. Even though the initial setting of IFSs is commutative, the consideration of the Cuntz isometries is highly non-commutative. Nonetheless, we wish to demonstrate their use and power in analysis of IFS measures.

There do, of course, exist some measures that are not spectral. Of great interest are measures that arise naturally from affine iterated function systems. An *iterated function system* (IFS) is a finite set of contraction operators $\tau_0, \tau_1, \cdots, \tau_n$ on a complete metric space S. As a consequence of Hutchinson's Theorem [**Hut81**], for an IFS on $\mathbb{R}^n$, there exists a unique compact set $X \subset \mathbb{R}^n$ left invariant by system in the sense that $X = \cup_{j=0}^{n} \tau_j(X)$. There will then exist a unique Borel measure μ on X such that

$$\int_X f(x)\, d\mu(x) = \frac{1}{n+1} \sum\nolimits_{j=0}^{n} \int_X f(\tau_j(x))\, d\mu(x) \tag{1.3.1}$$

for all continuous f.

In many cases of interest, X is a fractal set. In particular, if we take the iterated function system

$$\tau_0(x) = \frac{x}{3}, \quad \tau_1(x) = \frac{x+2}{3}$$

on $\mathbb{R}$, then the attractor is the *ternary Cantor* set C_3. The set C_3 has another construction: One starts with the interval $[0,1]$ and removes the middle third, leaving only the intervals $[0,1/3]$ and $[2/3,1]$, and then successively continues to remove the middle third of each remaining interval. Intersecting the sets remaining at each step yields C_3. The ternary Cantor measure μ_3 is then the measure induced in (1.3.1). Alternatively, μ_3 is the *Hausdorff measure* of dimension $\frac{\ln 2}{\ln 3}$ restricted to C_3.

In [**JP98a**] Jorgensen and Pedersen used the zero set of the Fourier-Stieltjes transform of μ_3 to show that μ_3 is not spectral (see Section 2.2). Equally remarkably, they showed that the quaternary (4-ary) Cantor set, which is the measure induced in (1.3.1) under the IFS

$$\tau_0(x) = \frac{x}{4}, \quad \tau_1(x) = \frac{x+2}{4} \tag{1.3.2}$$

is spectral by using Hadamard matrices and a completeness argument based on the Ruelle transfer operator. The attractor set for this IFS can be described in a manner similar to the ternary Cantor set: The 4-ary set case is as follows,

$$C_4 = \left\{ x \in [0,1] : x = \sum\nolimits_{k=1}^{\infty} \frac{a_k}{4^k},\ a_k \in \{0,2\} \right\},$$

and the invariant measure is denoted by μ_4. Jorgensen and Pedersen prove that

$$\begin{aligned} \Gamma_4 &= \left\{ \sum\nolimits_{n=0}^{N} l_n 4^n : l_n \in \{0,1\},\ N \in \mathbb{N} \right\} \\ &= \{0, 1, 4, 5, 16, 17, 20, 21, 64, 65, \cdots\} \end{aligned} \tag{1.3.3}$$

is a spectrum for μ_4, though there are many spectra [**DHS09, DHL13**]. The proof that this is a spectrum is a two step process: first the orthogonality of the exponentials with frequencies in Γ_4 is verified, and second the completeness of those exponentials is verified.

The orthogonality of the exponentials can be checked in several ways:

(1) checking the zeroes of the Fourier-Stieltjes transform of μ_4;
(2) using the representation of a particular Cuntz algebra on $L^2(\mu_4)$;
(3) generating Γ_4 as the invariant set for a second IFS that is "dual" in a sense to the IFS in (1.3.2) ("fractals in the large").

While these three methods are distinct, they all rely on the fact that a certain matrix associated to the IFSs is a (complex) Hadamard matrix. All three of these methods are, more or less, contained in the original paper [**JP98a**].

As a Borel probability measure, μ_4 is determined uniquely by the following IFS-fixed-point property:

$$\mu_4 = \frac{1}{2}\left(\mu_4 \circ \tau_0^{-1} + \mu_4 \circ \tau_1^{-1}\right), \tag{1.3.4}$$

see (1.3.2) for the affine maps τ_i, $i = 0, 1$; and one checks that the support of μ_4 is *the 4-ary Cantor set* C_4.

The conclusions for the pair (μ_4, Γ_4) from (1.3.3)-(1.3.4) are as follows:

THEOREM 1.3.1 ([**JP98a**]). *Let the pair* (μ_4, Γ_4) *be as described. Then we get a spectral pair; more precisely:*

$$\langle e_\gamma, e_{\gamma'}\rangle_{L^2(\mu_4)} = \widehat{\mu_4}(\gamma - \gamma') = \delta_{\gamma\gamma'} \left(= \begin{cases} 1 & \text{if } \gamma = \gamma' \text{ in } \Gamma_4 \\ 0 & \text{if } \gamma \neq \gamma', \text{ both in } \Gamma_4 \end{cases}\right) \tag{1.3.5}$$

Moreover, if $f \in L^2(\mu_4)$, *and*

$$\widehat{f}(\gamma) = \langle f, e_\gamma\rangle_{L^2(\mu_4)} = \int_{C_4} f(x)\, \overline{e_\gamma(x)}\, d\mu_4(x) \tag{1.3.6}$$

then we have the following $L^2(\mu_4)$ *limit:*

$$\lim_{N\to\infty} \Big\| f - \underbrace{\sum\nolimits_{\Gamma_4(N)} \widehat{f}(\gamma)\, e_\gamma(\cdot)}_{\text{Fractal Fourier series}} \Big\|_{L^2(\mu_4)} = 0$$

where Γ_4 *is as in (*1.3.3*).*

The proof and the ramifications of Theorem 1.3.1 will be discussed in detail inside the book; especially in the following sections below: Sections 2.4, 3.1, 4.1, and 6.1.

REMARK 1.3.2. It is known that, for classical Fourier series, there are continuous functions whose Fourier series may fail to be pointwise convergent.

Now the Fourier expansion from Theorem 1.3.1 turns out not to have this "defect." The reason is that those gap-fractals which have orthogonal frequency expansions, turn out to also possess a *localization* property (which is not present in the classical setting of Fourier series for functions on an interval.) Indeed, Strichartz [**Str93**] proved that, in the setting of Theorem 1.3.1, *every* continuous function on C_4 has its Γ_4 Fourier expansion be pointwise convergent.

By contrast, when this is modified to (μ_3, C_3), the middle-third Cantor, Jorgensen and Pedersen proved that then there cannot be more than two orthogonal Fourier functions $e_\lambda(x) = e^{i2\pi\lambda x}$, for any choices of points λ in $\mathbb{R}$.

The *completeness* of the exponentials (for the cases when the specified Cantor measure is spectral) can be shown in several ways as well, though the completeness is more subtle. The original argument for completeness given in [**JP98a**] uses a delicate analysis of the spectral theory of a *Ruelle transfer operator*. Jorgensen and Pedersen construct an operator on $C(\mathbb{R})$ using filters associated to the IFS in (1.3.2), which they term a Ruelle transfer operator. The argument then is to check that the eigenvalue 1 for this operator is a simple eigenvalue. An alternative argument for completeness given by Strichartz in [**Str98b**] uses the convergence of the cascade algorithm from *wavelet theory* [**Mal89**, **Dau88**, **Law91**]. Later arguments for completeness were developed in [**DJ09c**, **DJ12b**] again using the representation theory of Cuntz algebras.

The *Cuntz algebra* $\mathscr{O}_N$ for $N \geq 2$ is the universal C^*-algebra generated by a family $\{S_0, \cdots, S_{N-1}\}$ of N isometries satisfying the relation

$$\sum_{j=0}^{N-1} S_j S_j^* = I, \quad \text{and} \quad S_i^* S_j = \delta_{ij} I. \tag{1.3.7}$$

When N is fixed, and a system of operators S_j is identified satisfying (1.3.7), we say that $\{S_j\}$ is a system of Cuntz isometries; or that they define a representation of the Cuntz algebra $\mathscr{O}_N$. Equivalently, we say that the operators S_j satisfy the Cuntz relations. In the present book, we shall stress the role of representations of the Cuntz algebras in the study of multi-frequency band signal processing, of wavelet multiresolution generators, as filter-banks, and as generators of an harmonic analysis of iterated function systems (IFSs).

Now the Cuntz algebras $\mathscr{O}_N$ and their representations are of independent interest. And there is a rich literature dealing with them. In fact, it is known that the family of equivalence classes (unitary equivalence) of representations of $\mathscr{O}_N$ (N fixed) *does not admit Borel cross sections*; i.e., it is too big for classification. Nonetheless, we shall show that the class of representations corresponding to solutions to the filter bank systems in Figure 1.4.1 covers an infinite dimensional variety of equivalence classes of representations of $\mathscr{O}_N$.

Solutions to (1.3.7) $\{S_i\}$, realized in Hilbert space $\mathscr{H}$, play an important role in the construction of *multiresolutions*.

1.3.1. $\mathscr{O}_2$ vs $\mathscr{O}_\infty$. We shall discuss sequences $\{F_n\}_{n\in\mathbb{N}}$ of operators in a fixed Hilbert space, say $\mathscr{H}$, so $F_n : \mathscr{H} \to \mathscr{H}$.

Convergence of such sequences will be in the *strong operator topology* (SOT), defined as follows:

If $G : \mathscr{H} \to \mathscr{H}$ is an operator, we say that $F_n \xrightarrow[n\to\infty]{} G$ (SOT) if, for all vectors $h \in \mathscr{H}$, we have:

$$\lim_{n\to\infty} \|F_n h - Gh\|_{\mathscr{H}} = 0.$$

DEFINITION 1.3.3. A system of isometries $\{T_i\}_{i\in\mathbb{N}_0}$ is said to be a solution to the $\mathscr{O}_\infty$-relations iff (Def.)

$$T_i^* T_j = \delta_{ij} I, \quad \text{and} \quad \sum_{i=0}^{\infty} T_i T_i^* = I; \tag{1.3.8}$$

compare with (1.3.7).

LEMMA 1.3.4. *Let $\{S_0, S_1\}$ be a solution to the $\mathscr{O}_2$-relations (*1.3.7*), and set*

$$T_i = S_0^i S_1, \quad i \in \mathbb{N}_0. \tag{1.3.9}$$

Then $\{T_i\}_{i\in\mathbb{N}_0}$ *satisfies the* $\mathscr{O}_\infty$*-relations (*1.3.8*) if and only if*

$$\lim_{n\to\infty} S_0^{*n} = 0.$$

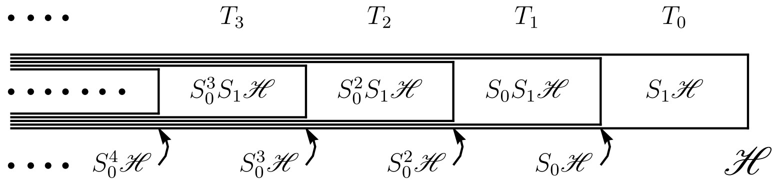
Proof. Let S_0, S_1, and $T_i = S_0^i S_1$ be as stated. For $k \in \mathbb{N}$, we then have

$$\sum_{i=0}^{k-1} T_i T_i^* = \sum_{i=0}^{k-1} S_0^i \left(I - S_0 S_0^*\right) S_0^{*^i} = I - S_0^k S_0^{*k},$$

and the desired conclusion follows immediately. □

Remark 1.3.5. To appreciate the role of the lemma in building *multiresolutions*, consider the following diagram, sketching closed subspaces in $\mathscr{H}$.

Assume $\{S_i\}_{i=0}^1$ is an $\mathscr{O}_2$-system, then



with the system  representing an orthogonal resolution, i.e., a system of orthogonal closed subspaces.

There are many ways to generate such families. For example, consider the isometries S_0, S_1 on $L^2[0,1]$ given by defining their adjoints

$$\left(S_0^* f\right)(x) = \frac{1}{\sqrt{2}} f\left(\frac{x}{2}\right) \quad \text{and} \quad \left(S_1^* f\right)(x) = \frac{1}{\sqrt{2}} f\left(\frac{x+1}{2}\right),$$

$f \in L^2[0,1]$, $x \in [0,1]$. One can check that the range isometries $S_0 S_0^* = \chi_{[0,1/2]}$ and $S_1 S_1^* = \chi_{[1/2,1]}$, so that the Cuntz relations are satisfied.

Developing this example a bit further, we can see a relationship between Cuntz isometries and iterated function systems. Let C be the standard Cantor set in $[0,1]$, consisting of those real numbers whose ternary expansions are of the form $x = \sum_{k=1}^\infty \frac{x_k}{3^k}$ where $x_k \in \{0,2\}$ for all k. Let

$$\varphi : C \to [0,1], \quad \varphi\left(\sum\nolimits_{k=1}^\infty \frac{x_k}{3^k}\right) = \sum\nolimits_{k=1}^\infty \frac{x_k}{2^{k+1}}.$$

Let m be Lebesgue measure on $[0,1]$, and define the Cantor measure μ on C by $\mu\left(\varphi^{-1}(B)\right) = m(B)$ if $B \subset [0,1]$ is Lebesgue measurable. This is well defined since φ is bijective except at countably many points.

Now define isometries R_0, R_1 on $\left(L^2(C), \mu\right)$ by defining their adjoints:

$$R_0^*(f) = S_0^*(f \circ \varphi) \quad \text{and} \quad R_1^*(f) = S_1^*(f \circ \varphi), \quad f \in \left(L^2(C), \mu\right).$$

Then

$$R_0^*(f)(x) = \frac{1}{\sqrt{2}} f\left(\frac{x}{3}\right) \quad \text{and} \quad R_1^*(f)(x) = \frac{1}{\sqrt{2}} f\left(\frac{x+2}{3}\right),$$

$f \in \left(L^2(C), \mu\right)$, $x \in C$. Thus we see the iterated function system for the Cantor set $\tau_0(x) = x/3$, $\tau_1(x) = (x+2)/3$ arising in the definition of Cuntz isometries on the Cantor set.

TABLE 2. Some popular affine IFSs

	Scaling factor	Number of affine maps τ_i	Ambient dimension	Hausdorff dimension
Middle-third C_3	3	2	1	$\log_3 2 = \frac{\ln 2}{\ln 3}$
The 4-ary C_4	4	2	1	$\frac{1}{2}$
Sierpinski triangle	2	3	2	$\log_2 3 = \frac{\ln 3}{\ln 2}$

Multiresolutions as outlined in Remark 1.3.5 are versatile, they are algorithmic. Here their construction is based on representation theory. We shall discuss their wider use in harmonic analysis, both in the case of traditional wavelet expansions, and in their fractal counterparts. This will be developed in detail in the following three later sections, 2.4, 4.3, and 5.2. For their use in Chapter 4, see especially equations (4.3.6)–(4.3.7), and Figure 4.3.2.

The Cuntz relations can be represented in many different ways. In their paper [**DJ15a**], Dutkay and Jorgensen look at finite *Markov processes*, and the *infinite product* of the state space is a compact set on which different measures can be defined, and these form the setting of representations of the Cuntz relations.

To construct a Fourier basis for a spectral measure arising from an iterated function system generated by contractions $\{\tau_0, \cdots, \tau_{N-1}\}$, Jorgensen (and others, [**JP98a**, **DPS14**, **DJ15a**, **PW17**]) choose *filters* $m_0, \cdots, m_{N-1}$ and define Cuntz isometries $S_0, \cdots, S_{N-1}$ on $L^2(\mu)$ by

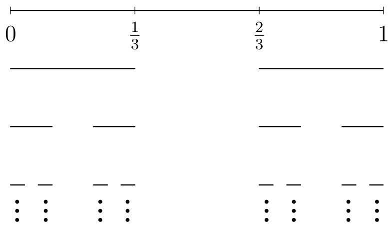
$$S_j f = \sqrt{N} m_j f \circ R,$$

where R is the common left inverse of the τ's. The filters, functions defined on the attractor set of the iterated function system, are typically chosen to be continuous, and are required to satisfy the relation $\sum_{j=0}^{N-1} |m_j|^2 = 1$. The Cuntz relations are satisfied by the S_j's provided the filters satisfy the *orthogonality condition*

$$\mathcal{M}^* \mathcal{M} = I, \quad (M)_{jk} = m_j(\tau_k(\cdot)). \tag{1.3.10}$$

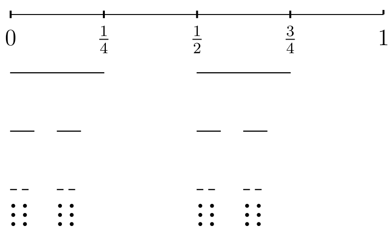
To obtain Fourier bases, the filters m_j are chosen specifically to be exponential functions when possible. This is not possible in general, however, and is not possible in the case of the middle-third Cantor set and its corresponding measure μ_3.

The fact that some measures, such as μ_3, are not spectral leaves us with a conundrum: We still desire Fourier-type expansions of functions in $L^2(\mu)$, that is, a representation as a series of complex exponential functions, but we cannot get such an expansion from an orthogonal basis of exponentials in the case of a non-spectral measure. For this reason, we turn to another type of sequence called a *frame*, which has the same ability to produce series representations that an orthogonal basis does, but has redundancy that orthogonal bases lack and has no orthogonality requirement. Frames for Hilbert spaces were introduced by Dun and Schaeer [**DS52**] in their study of non-harmonic Fourier series. The idea then lay essentially dormant until Daubechies, Grossman, and Meyer reintroduced frames in [**DGM86**]. Frames are now pervasive in mathematics and engineering. For recent applications, we refer the reader to [**Web04**, **ALTW04**, **PW17**].



$$\tau_0(x) = \frac{x}{3}$$
$$\tau_1(x) = \frac{x+2}{3}$$

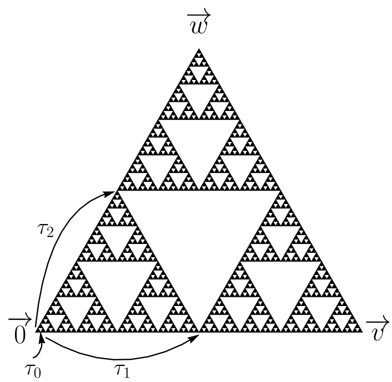
FIGURE 1.3.1. Middle-third Cantor C_3



$$\tau_0(x) = \frac{x}{4}$$
$$\tau_1(x) = \frac{x+2}{4}$$

FIGURE 1.3.2. The 4-ary Cantor C_4

$$\vec{w} = \left(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right)$$
$$\vec{v} = (1, 0)$$



$$\tau_0(\vec{x}) = \frac{1}{2}\vec{x}$$
$$\tau_1(\vec{x}) = \frac{1}{2}(\vec{x} + \vec{v})$$
$$\tau_2(\vec{x}) = \frac{1}{2}(\vec{x} + \vec{w})$$

FIGURE 1.3.3. Sierpinski triangle

1.4. Frequency bands, filters, and representations of the Cuntz-algebras

Our analysis of the Cuntz relations here in the form $\{S_i\}_{i=0}^{N-1}$ turns out to be a modern version of the rule from *signal-processing engineering* (SPEE): When complex frequency response functions are introduced, the (SPEE) version of the Cuntz relations $S_i^* S_j = \delta_{ij} I_{\mathscr{H}}$, $\sum_{i=0}^{N-1} S_i S_i^* = I_{\mathscr{H}}$, where $\mathscr{H}$ is a Hilbert space of time/frequency signals, and where the N isometris S_i are expressed in the following form:

$$(S_i f)(z) = m_i(z) f\left(z^N\right), \ f \in \mathscr{H}, \ z \in \mathbb{C}; \tag{1.4.1}$$

and where $\{m_i\}_{i=0}^{N-1}$ is a system of *bandpass-filters*, m_0 accounting for the low band, and the filters $m_i(z)$, $i > 0$, accounting for the remaining bands in the subdivision into a total of N bands. The diagram form (SPEE) is then as in Figure 1.4.1.

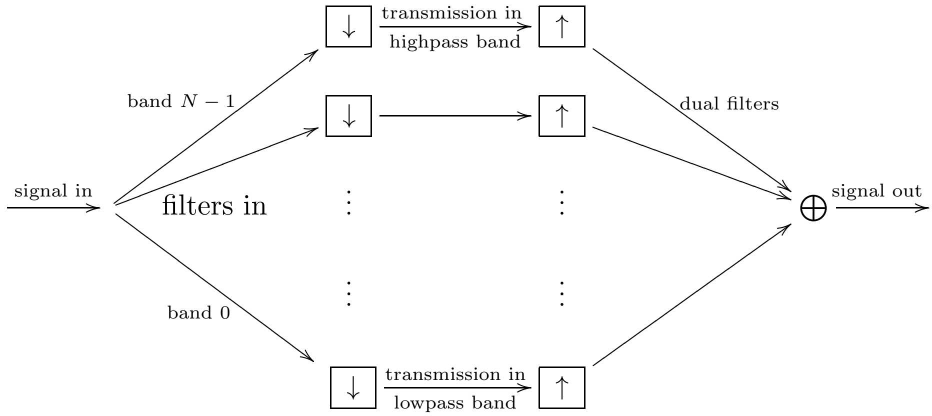


FIGURE 1.4.1. *Down-sampling* $\boxed{\downarrow}$, and *up-sampling* $\boxed{\uparrow}$. The picture is a modern math version of one I (PJ) remember from my early childhood: In our living room, my dad was putting together some of the early versions of low-pass/high-pass frequency band filters for transmitting speech signals over what was then long distance. One of the EE journals had a picture which is much like the one I reproduce here; after hazy memory. Strangely, the same multi-band constructions are still in use for modern wireless transmission, both speech and images. The down/up arrows in the figure stand for down-sampling, up-sampling, respectively. Both operations have easy expressions in the complex frequency domain. For example up-sampling becomes substitution of z^N where N is the fixed total number of bands.

The operators making up the multiband filters in Figure 1.4.1 are expressed in (1.4.1) in the frequency variable z ($\in \mathbb{C}$, or in $\mathbb{T}$). With the usual inner product in the Hilbert space $L^2(\mathbb{T})$, and

$$\int_0^1 \left|\sum\nolimits_{n\in\mathbb{Z}} c_n e_n(x)\right|^2 dx = \sum\nolimits_{n\in\mathbb{Z}} |c_n|^2, \tag{1.4.2}$$

one checks that the adjoint of the above operators (1.4.1) are:

$$\left(S_j^* f\right)(z) = \frac{1}{N} \sum_{w\in\mathbb{T},\, w^N = z} \left(\overline{m}_j f\right)(w), \tag{1.4.3}$$

for $\forall z \in \mathbb{T}$, $0 \le j < N$.

Now, there is a time-frequency duality, and operators in one side of the duality have a counterpart in the other side. As evidenced by (1.4.2), the discrete-time dual version of the frequency function

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e_n(x) \tag{1.4.4}$$

is simply the time-series $(c_n)_{n\in\mathbb{Z}}$:

$$\cdots, c_{-2}, c_{-1}, c_0, c_1, c_2, c_3, \cdots \tag{1.4.5}$$

If $\widetilde{m}(x) = \sum_{n\in\mathbb{Z}} h_n e_n(x)$, then the multiplication operator, $f \longmapsto mf$, is simply:

$$(c_n)_{n\in\mathbb{Z}} \longmapsto (\widetilde{m}[c])_n = \sum_{m\in\mathbb{Z}} h_m c_{n-m}. \tag{1.4.6}$$

The up and down-sampling operations $\boxed{\uparrow}$ vs $\boxed{\downarrow}$ acting on the time-series are:

(1.4.7)

$$\begin{aligned}\left(\boxed{\uparrow}[c]\right)_n &= \begin{cases} c_{n/N} & \text{if } N \mid n \\ 0 & \text{if } n \text{ is not divisible by } N \end{cases} \\ &= (\underset{\text{places}}{\cdots}, 0, \underbrace{c_{-1}}_{-N}, 0, \cdots, 0, \underbrace{c_0}_{0}, 0, \cdots, 0, \underbrace{c_1}_{N}, 0, \cdots, 0, \underbrace{c_2}_{2N}, 0, \cdots)\end{aligned}$$

and

$$\left(\boxed{\downarrow}[c]\right)_n = c_{nN}, \quad \forall n \in \mathbb{Z}. \tag{1.4.8}$$

In many applications, the operators from (1.4.1) and (1.4.3) have matrix realizations. The respective matrices are *slanted* (see Figure 1.4.2), and they are used in algorithms for digitized representations of signals and of images.

To appreciate the matrix point of view, we restrict here to the special case where the functions m_i in (1.4.1)-(1.4.3) are polynomials, so M (= one of the functions m_i) has the form

$$M(z) = h_0 + h_1 z + \cdots + h_d z^d. \tag{1.4.9}$$

Assume a signal is given in the form (1.4.4), i.e., with

$$(c) = (c_0, c_1, c_2, \cdots)$$

representing a time series with discrete time $n \in \mathbb{N}_0$. When realized in this form, one checks that the operators

$$(S_M f)(z) = M(z) f\left(z^N\right),$$

and

$$(S_M^* f)(z) = \frac{1}{N} \sum_{w\in\mathbb{T},\, w^N = z} \left(\overline{M} f\right)(w),$$

yield the respective matrix forms:

$$(S_M c)_k = \sum_{n\in\mathbb{Z}} h_{k-Nn} c_n;$$

and

$$(S_M^* c)_k = \sum_{n\in\mathbb{Z}} \overline{h_{Nk-n}} c_n.$$

The slanted matrices themselves are given in Figure 1.4.2 below.

These are wavelet tools, and they will be revisited in a number of applications, later in the book, starting with Section 4.3.

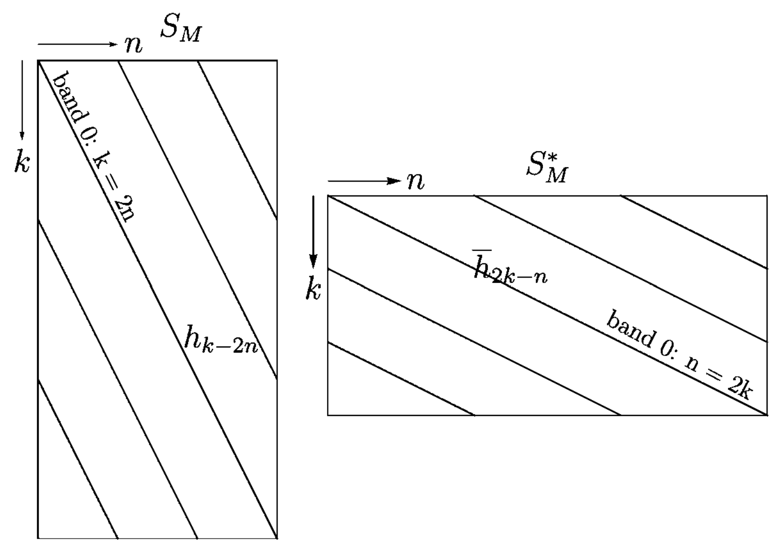


FIGURE 1.4.2. The two slanted matrices in the special case when $N = 2$.

1.5. Frames

Let $\mathscr{H}$ be a separable Hilbert space with *inner product* $\langle \cdot, \cdot \rangle$, and let $\mathbb{J}$ be a *countable index set.* A *frame* for $\mathscr{H}$ is a sequence $\{x_j\}_{j \in \mathbb{J}} \subset \mathscr{H}$ such that there exist constants $0 < C_1 \leq C_2 < \infty$ such that for all $v \in \mathscr{H}$,

$$C_1 \|v\|^2 \leq \sum |\langle v, x_j \rangle|^2 \leq C_2 \|v\|^2 .$$

If C_1 and C_2 can be chosen so that $C_1 = C_2 = 1$, we say that $\{x_j\}$ is a *Parseval frame.*

If $\mathbb{X} \subset \mathscr{H}$ is a frame, then any other frame $\tilde{\mathbb{X}} := \{\tilde{x}_j\} \subset \mathscr{H}$ that satisfies

$$\sum \langle v, \tilde{x}_j \rangle x_j = v, \tag{1.5.1}$$

for all $v \in \mathscr{H}$ is called a *dual frame* for $\mathbb{X}$. Every frame possesses a dual frame, and in general, dual frames are not unique. A Parseval frame is self-dual, that is, $v = \sum \langle v, x_j \rangle x_j$.

Returning to our current interest, we say that a measure μ is *frame-spectral* if there exists a countable set $\Lambda \subset \mathbb{R}$ such that $\left\{e^{i2\pi\lambda x}\right\}_{\lambda \in \Lambda}$ is a frame in $L^2(\mu)$. In general, for a compact subset C of $\mathbb{R}^d$ with nonzero measure, Lebesgue measure restricted to that set is not spectral, but it will always be frame spectral. In general, a singular measure will not be frame-spectral [**DHSW11**, **DL14b**], but many singular measures are frame-spectral [**EKW16**, **PW17**]. It is currently unknown whether or not μ_3 is frame-spectral.

The *redundancy* of frames makes them more immune to error in transmission: Multiple frame elements will capture the same dimensions of information, and so if one series coefficient in the frame expansion of a function is transmitted incorrectly, the adverse effect on the reconstructed function will be minimal. However, expansions in terms of a given frame are in general not unique, and this can be a desirable or undesirable quality depending on the application. If we want the best of both worlds — a frame with redundancy but with a unique expansion for each function, then we must turn to the realm of Riesz bases.

A *Riesz basis* in a Hilbert space $\mathscr{H}$ is a sequence $\{x_j\}_{j=1}^{\infty}$ which has dense span in $\mathscr{H}$ and is such that there exist $0 < A \leq B$ such that for any nite sequence of scalars $c_1, c_2, \cdots, c_N$, we have

$$A \sum_{j=1}^{N} |c_j|^2 \leq \| \sum_{j=1}^{N} c_j x_j \|^2 \leq B \sum_{j=1}^{N} |c_j|^2 . \tag{1.5.2}$$

A Riesz basis is a frame that has only one dual frame. Equivalently, $\{x_j\}_{j=1}^{\infty}$ is a Riesz basis if an only if there is a topological isomorphism $T : \mathscr{H} \to \mathscr{H}$ such that $\{Tx_j\}_{j=1}^{\infty}$ is an orthonormal basis of $\mathscr{H}$.

The unit disk $\mathbb{D}$, for example, as a convex planar body has no orthogonal basis of complex exponential functions, but it does possess a frame of complex exponential functions. However, it is still an open problem whether it possesses a Riesz basis of complex exponential functions.

1.6. Key themes in the book

Beginning with the foundational results in "*Dense analytic subspaces in fractal L^2-spaces*" [**JP98a**], Chapter 2 will cover the construction of spectral measures, the constructions of various spectra, characterizations and invariance of spectra for spectral measures. It will include initial connections to *representation theory* of Cuntz algebras, *spectra* and *tiling* properties in $\mathbb{R}^d$, the Fuglede conjecture, and Reproducing Kernel Hilbert spaces.

The existence of *orthogonal Fourier bases for classes of fractals* came as somewhat of a surprise, referring to the 1998 paper [**JP98a**]. There are several reasons for why existence of orthogonal Fourier bases might have been unexpected: For one, existence of orthogonal Fourier bases, as in the classical case of Fourier, tends to imply a certain amount of "smoothness" which seems inconsistent with fractal geometries, and fractal dimension. Nonetheless, when feasible, such a orthogonal Fourier analysis holds out promise for applications to large chaotic systems, or to analysis of *noisy signals*; areas that had previously resisted analysis by Fourier tools.

When Fourier duality holds, it further yields a *duality of scale*, fractal scales in the small, and for the dual frequency domain, fractals in the large.

While the original framework for the Jorgensen-Pedersen fractals, and associated L^2-spaces, was a rather limited family, this original fractal framework for orthogonal Fourier bases has since been greatly expanded. While the original setting was restricted to that of *affine selfsimilarity*, determined by certain *iterated affine function systems* in one and higher dimension, this has now been broadened to the setting of say conformal selfsimilar IFS systems, and to associated maximal entropy measures. Even when the strict requirements entailed by orthogonal Fourier bases is suitably relaxed, there are computational Fourier expansions (Herr-Jorgensen-Weber) which lend themselves to analysis/synthesis for most *singular measures.*

Inherent in the study of fractal scales is the notion of *multiresolution analyses*, in many ways parallel to the more familiar Daubechies wavelet multiresolutions. Moreover, Strichartz proved that when an orthogonal Fourier expansions exist, they have localization properties which parallel the kind of localization which has made wavelet multiresolutions so useful. The presence of multiresolutions further implies powerful algorithms, and it makes connections to representation theory and to signal/image processing; subjects of the later chapters. Dutkay-Jorgensen proved that all affine IFS fractals have wavelet bases.

Chapter 2 will build on the themes from Chapter 1, detailing the constructions of spectra arising from Cuntz algebras, characterizations of spectra using the spectral theory of *Ruelle operators*, connections between tilings, and *wandering vectors* for unitary groups and unitary systems.

There is an intimate relations between systems of *tiling by translations* on the one hand, and *orthogonal Fourier bases* on the other. Representation theory makes a link between the two, but the tile-spectral question is deep and difficult; so far only partially resolved. One tool of inquiry is that of "wandering vectors" or *wandering subspaces*. The term "wandering" has its origin in the study of systems of isometries in Hilbert space. It has come to refer to certain actions in a Hilbert space which carries representations: When the action generates orthogonal vectors, we refer to them as wandering vectors; similarly for closed subspaces. In the case of representations of groups, this has proved a useful way of generating orthogonal Fourier bases; — when they exist. In the case of representations of the Cuntz algebras, the "wandering" idea has become a tool for generating nested and orthogonal subspaces. The latter includes multiresolution subspaces for wavelet systems and for signal/image processing algorithms.

Chapter 3 will focus on the *tiling properties* arising from the study of spectral measures, specifically in dimension one; advances in the Fuglede conjecture in dimension one, non-commutative fractal analogues in infinite dimensions.

Fuglede (1974) conjectured that a domain Ω admits an operator spectrum (has an orthogonal Fourier basis) if and only if it is possible to tile $\mathbb{R}^d$ by a set of translates of Ω [**Fug74**]. Fuglede proved the conjecture in the special case that the tiling set or the spectrum are lattice subsets of $\mathbb{R}^d$, and Iosevich et al. [**IKT01**] proved that no smooth symmetric convex body Ω with *at least one point of nonvanishing Gaussian curvature* can admit an orthogonal basis of exponentials.

Using complex *Hadamard matrices* of orders 6 and 12, Tao [**Tao04**] constructed counterexamples to the conjecture in some small Abelian groups, and lifted these to counterexamples in $\mathbb{R}^5$ or $\mathbb{R}^{11}$. Tao's results were extended to lower dimensions, down to $d = 3$, but the problem is still open for $d = 1$ and $d = 2$.

Summary of some affirmative recent results: The conjecture has been proved in a great number of special cases (e.g., all convex planar bodies) and remains an open problem in small dimensions. For example, it has been shown in dimension 1 that a nice algebraic characterization of finite sets tiling $\mathbb{Z}$ indeed implies one side of Fuglede's conjecture [**CM99**]. Furthermore, it is sufficient to prove these conditions when the tiling gives a factorization of a non-Hajós cyclic group [**Ami05**].

Ironically, despite a large number of great advances in the area, Fuglede's original question is still unsolved in the planar case. In the planar case, the question is: *Let Ω be a bounded open and connected subset of $\mathbb{R}^2$. Does it follow that $L^2(\Omega)$ with respect to planar Lebesgue measure has an orthogonal Fourier basis if and only if Ω tiles $\mathbb{R}^2$ with translations by some set of vectors from $\mathbb{R}^2$?* Of course, if Ω is a fundamental domain for some rank-2 lattice, the answer is affirmative on account of early work.

Another direction is to restrict the class of sets Ω in $\mathbb{R}^3$ to be studied. One such recent direction is the following affirmative theorem for the case when Ω is assumed to be a *convex polytope*: Nir Lev et al [**GL17**] proved that a spectral *convex polytope* (i.e., having a Fourier basis) must tile by translations. This implies in particular that Fuglede's conjecture holds true for convex polytopes in $\mathbb{R}^3$.

Chapter 4 is devoted to the RKHSs that appear in the study of spectral measures. Spectral measures give rise to positive definite functions via the Fourier transform. Reversing this process, the chapter will set the stage by discussing RKHSs that appear in the context of positive definite functions, and the associated harmonic analysis in such spaces.

Since the measures are spectral, the corresponding *positive definite functions* have special properties in terms of their zero sets. This correspondence leads to the natural question of whether this process can be reversed. Bochner's theorem implies that positive definite functions are the Fourier transform of measures, but whether those measures are spectral becomes a subtle problem. Thus, by considering certain functions on appropriate subsets, the question of spectrality can be formulated as whether the function can be extended to a positive definite function. The answer is sometimes yes, using the harmonic analysis of RKHSs.

Chapter 5 concerns *representations of Cuntz algebras* that arise from the action of stochastic matrices on sequences from $\mathbb{Z}_n$. This action gives rise to an invariant measure, which depending on the choice of stochastic matrices, may satisfy a finite tracial condition. If so, the measure is ergodic under the action of the shift on the sequence space, and thus yields a representation of a Cuntz algebra. The measure provides spectral information about the representation in that equivalent representations of the Cuntz algebras for different choices of stochastic matrices occur precisely when the measures satisfy a certain equivalence condition.

Recursive multiresolutions and basis constructions in Hilbert spaces are key tools in analysis of fractals and of iterated function systems in dynamics: Use of multiresolutions, selfsimilarity, and locality, yield much better pointwise approximations than is possible with traditional Fourier bases. The approach here will be via representations of the Cuntz algebras. It is motivated by applications to an analysis of frequency sub-bands in signal or image-processing, and associated multi-band filters: With the representations, one builds recursive subdivisions of signals into frequency bands.

Concrete realizations are presented of a class of *explicit representations*. Starting with Hilbert spaces $\mathscr{H}$, the representations produce recursive families of closed subspaces (projections) in $\mathscr{H}$, in such a way that "non-overlapping, or uncorrelated, *frequency bands*" correspond to orthogonal subspaces in $\mathscr{H}$. Since different frequency bands must exhaust the range for signals in the entire system, one looks for orthogonal projections which add to the identity operator in $\mathscr{H}$. Representations of Cuntz algebras (see Figure 1.4.1) achieve precisely this: From representations we obtain classification of families of *multi-band filters*; and representations allow us to deal with non-commutativity as it appears in both time/frequency analysis, and in scale-similarity. The representations further offer canonical selections of special families of commuting orthogonal projections.

The chapter will focus on the connections between harmonic analysis on fractals and the *cascade algorithm* from wavelet theory. Wavelets have a dual existence between the discrete and continuous realms manifested in the discrete and continuous wavelet transforms. Wavelet filters give another bridge between the smooth and non-smooth domains in that the convergence of the cascade algorithm yields wavelets and wavelet transforms in a smooth setting, i.e. $\mathbb{R}^d$, and also the non-smooth setting such as the Cantor dust, depending on the parameters embedded in the choice of wavelet filters.

Chapter 6 concerns *Gaussian processes* for whose spectral (meaning generating) measure is spectral (meaning possesses orthogonal Fourier bases). These Gaussian processes admit an Itô-like stochastic integration as well as harmonic and wavelet analyses of related *Reproducing Kernel Hilbert Spaces* (RKHSs).

Chapter 7 will focus on stochastic processes that appear in the representation theory of *Lie groups*. Motivated by reflection symmetries in Lie groups, we will consider representation theoretic aspects of reflection positivity by discussing reflection positive Markov processes indexed by Lie groups, measures on path spaces, and invariant Gaussian measures in spaces of distribution vectors. This provides new constructions of reflection positive unitary representations.

Since early work in mathematical physics, starting in the 1970ties, and initiated by A. Jaffe, and by K. Osterwalder and R. Schrader, the subject of *reflection positivity* has had an increasing influence on both non-commutative harmonic analysis, and on duality theories for spectrum and geometry. In its original form, the Osterwalder-Schrader idea served to link Euclidean field theory to relativistic quantum field theory. It has been remarkably successful; especially in view of the abelian property of the Euclidean setting, contrasted with the non-commutativity of quantum fields. Osterwalder-Schrader and reflection positivity have also become a powerful tool in the theory of unitary representations of Lie groups. Co-authors in this subject include G. Olafsson, and K.-H. Neeb.

Below we list suggested papers readers might wish to consult on four central themes:

(1) Fourier analysis on affine fractals [**JP87**, **JP92**, **JP93a**, **JP93b**, **JP94**, **JP95**, **JP96**, **JP98a**, **JP98b**, **JP98c**, **JP98d**, **JP99**, **JP00**, **JPT12**, **JPT14**, **JPT15a**, **JPT15b**]

(2) Multiresolution analyst, fractals, and representations of the Cuntz relations [**DJ05b**, **DJ05a**, **DJ06b**, **DJ06d**, **DJ06c**, **DJ06a**, **DJ07c**, **DJ07a**, **DJ07b**, **DJ07d**, **DJ07e**, **DJ07f**, **DJ08a**, **DJ08b**, **DJ09c**, **DHJS09**, **DJ09b**, **DHJ09**, **DJP09**, **DJ09a**, **DJ11b**, **DJ11a**, **DJS12**, **DJ12a**, **DJ12b**, **DHJP13**, **DJ13b**, **DJ13a**, **DJ14a**, **DJ14b**, **DHJ15**, **DJ15c**, **DJ15b**, **DJ15a**]

(3) Frame analysis of singular measures [**HJW18b**]

(4) Reflection positivity [**JO98**, **JO00**, **JNO16**, **JNO18**, **JT18c**]

The past two decades has seen a rich and diverse flourishing of research in the areas of analysis on fractals, and their applications. While the present lectures have stressed a certain harmonic analysis approaches, and their associated applications, there are others.

And in fact, it will be nearly impossible to cover all directions, even by way of citations, and we apologize for omissions. Nonetheless, we believe that the following supplementary references will help readers broaden their perspective: First, the book [**BP17**] stresses connections to probability and Markov processes. And there is the work by Poltoratski et al with a different perspective on harmonic analysis on fractals; see e.g., [**dRP99**, **dRFP02**, **Pol15**, **Pol13**].

There are many standard textbooks dealing with harmonic analysis and applications. Two of these books might perhaps be more helpful for students; filling in prerequisites. They are [**DM72**, **DM76**] by Dym and McKean. (While they are extremely useful for background material, they do not get into the fractal variants of Fourier series.)

Our present focus as far as the fractals go is harmonic analysis. Many of our tools apply to large and varied classes of fractals, but we have chosen to illustrate most of our results with the fractals called affine iterated function systems (IFS). Fractals are studied in a variety of areas both in mathematics and in diverse applications. And the literature is vast. Readers who want to get started are referred to the book [**Fed88**].