Preface

The zeta and L-functions play a central role in number theory. They provide important information of arithmetic nature. For example, the analytic behavior of the Riemann zeta function $\zeta(s)$ on the closed right half-plane $\Re(s) \geq 1$ leads to the estimate of the number of prime numbers up to x by the logarithmic integral $\mathrm{Li}(x)$, and the error of this estimate is controlled by the location of the zeros of $\zeta(s)$ in the critical strip $0 < \Re(s) < 1$. The celebrated Riemann Hypothesis asserts that these zeros should all lie on the line of symmetry $\Re(s) = 1/2$. Under this assumption one can bound the error by $O(x^{1/2} \log x)$. This is the Prime Number Theorem. Furthermore, given a finite Galois extension of $\mathbb{Q}$ with Galois group G, we can partition the prime numbers according to their associated Frobenius conjugacy classes in G. For a conjugacy class C of G, denote by S_C the collection of prime numbers whose associated Frobenius conjugacy classes equal C. The distribution of prime numbers in S_C is described by the Chebotarev Density Theorem, which says that S_C has natural density $|C|/|G|$. In particular, the prime numbers are equidistributed among the sets S_C when the Galois group G is abelian. Specialized to cyclotomic extensions of $\mathbb{Q}$, this is Dirichlet's Theorem on primes in arithmetic progressions. The Chebotarev Density Theorem follows from the holomorphy and nonvanishing of the Artin L-functions attached to nontrivial irreducible representations of G on the half-plane $\Re(s) \geq 1$.

On the geometric side, to a d-dimensional projective smooth irreducible variety V defined over a finite field $\mathbb{F}_q$, Artin and Smith attached the zeta function $Z(V, u)$, which counts points on V with coordinates in finite extensions of $\mathbb{F}_q$. As shown by Grothendieck, $Z(V, u)$ is an alternating product of polynomials $P_i(u)$, $0 \leq i \leq 2d$, arising geometrically; and Deligne proved the Riemann Hypothesis as conjectured by Weil; that is, the zeros of a nonconstant $P_i(u)$ have absolute value $q^{-i/2}$. Like the Riemann zeta function, $Z(V, u)$ has an Euler product over closed points, which play the role of primes of V. When V is a curve, the analytic behavior of $Z(V, u)$ and similarly defined Artin L-functions give rise to analogues of the Prime Number Theorem and the Chebotarev Density Theorem.

These remarkable achievements in number theory are reviewed in Lecture 1. The purpose of this monograph is to provide a systematic and comprehensive account of the developments of these topics in geometry and combinatorics for graduate students and researchers. We shall highlight interactions between number theory and other fields and compare similarities and dissimilarities under different settings. This is done in chronological order. Lecture 2 introduces the first instance, considered by Selberg in the 1950s, on compact Riemann surfaces arising as quotients of the upper half-plane $\mathfrak{H}$ by discrete torsion-free cocompact subgroups Γ of $SL_2(\mathbb{R})$. In this setting the primes of $\Gamma\backslash\mathfrak{H}$ are primitive closed geodesic cycles not counting the starting points, and the fundamental group Γ plays the role of absolute Galois group. Selberg introduced zeta functions attached to finite-dimensional representations of Γ as suitable products over such primes. These functions have nice analytic properties like $\zeta(s)$, and they satisfy the Riemann Hypothesis for all except possibly finitely many real zeros in the critical strip. The Prime Geodesic Theorem and the Chebotarev Density Theorem for $\Gamma\backslash\mathfrak{H}$ are established by Huber and Sarnak.

Lecture 3 is a digression to finite-dimensional compact Riemannian manifolds and the connection to dynamical systems. We shall see that the distribution of primes is related to the analytic behavior of the associated L-functions of Artin type, which are variations of the Ruelle zeta function in dynamical systems. While this is a much explored topic in dynamical systems, our exposition will stay close to the theme of this monograph.

The remainder of this monograph is devoted to the combinatorial setting. In the 1960s, by interpreting the upper half-plane $\mathfrak{H}$ as the homogeneous space $PGL_2(\mathbb{R})/PO_2(\mathbb{R})$, Ihara extended Selberg's results from the real field $\mathbb{R}$ to a nonarchimedean local field F with q elements in its residue field. The upper half-plane is replaced by a $(q+1)$-regular tree, known as the building of $PGL_2(F)$, on which $PGL_2(F)$ acts. The quotient of the tree by a discrete torsion-free cocompact subgroup of $PGL_2(F)$ is hence a finite $(q+1)$-regular graph X, whose primes are primitive closed geodesic cycles up to starting points, similar to the Riemann surfaces discussed in Lecture 2. The Ihara zeta function $Z(X,u)$ of X counts closed geodesic cycles on X; as such, it can be expressed as a product over the primes of X, like the zeta function of a curve reviewed in Lecture 1. Serre observed that the same definition applies to all finite graphs. In Lecture 4 we study different closed-form expressions of the Ihara zeta function as a rational function in u. Lecture 5 is devoted to the spectral theory for regular graphs. We shall see that, for a $(k+1)$-regular graph X, its zeta function satisfies the Riemann Hypothesis in the sense that its nontrivial poles have the same absolute value (which is $k^{-1/2}$) if and only if X is a Ramanujan graph; that is, its eigenvalues other than $\pm(k+1)$ fall in the spectrum of its universal cover. Ramanujan graphs are extremal expanders with nice properties and wide applications. Explicit constructions of infinite families of $(k+1)$-regular Ramanujan graphs are introduced in Lecture 6. For k equal to a prime power, the construction by Margulis and independently by Lubotzky-Phillips-Sarnak in the 1980s is number-theoretic. The existence of such families for general $k \geq 3$ was established by Marcus-Spielman-Srivastava in 2015 for bipartite graphs using combinatorial and analytical means. It is still an open question to find infinite families of nonbipartite Ramanujan graphs for general k. Lecture 7 deals with Artin L-functions on graphs. It is shown that, as before, good analytic behavior of zeta and L-functions on X leads to the Prime Geodesic Theorem and the Chebotarev Density Theorem on the distribution of primes of graphs.

Graphs are 1-dimensional simplicial complexes. Lecture 8 concerns extensions to higher dimensional simplicial complexes obtained in this century. This is a budding and rapidly evolving research area. Two main themes are considered in this lecture. The first one is the generalization from Ramanujan graphs based on $PGL_2(F)$ to Ramanujan complexes based on $PGL_n(F)$ developed in the early 2000s. There have been several explicit number-theoretic constructions of infinite families of Ramanujan complexes as finite quotients of the building of $PGL_n(F)$.

We shall present the one given by Sarveniazi in 2007. To understand the combinatorial properties and to find applications of these Ramanujan complexes are currently active research areas in mathematics and computer science. The second theme is on zeta and L-functions and prime distributions for 2-dimensional simplicial complexes X_Γ arising as finite quotients by discrete torsion-free cocompact subgroups Γ of the building of $G = PGL_3(F)$ and $PGSp_4(F)$, with the results obtained in the past decade. In this situation for each $i \in \{1, 2\}$, there are two types of i-dimensional simplices and hence two zeta functions counting i-dimensional closed geodesics of X_Γ using simplices of a given type. These zeta functions as well as L-functions of Artin type attached to finite-dimensional irreducible representations of Γ have similar analytic behavior as their counterparts for graphs, and they imply similar and more refined Prime Geodesic Theorems and Chebotarev Density Theorems for primes of X_Γ. In the case that X_Γ is a Ramanujan complex for $PGL_3(F)$, a good error term in each estimate is also obtained. It should be pointed out that a suitable alternating product of these four zeta functions gives rise to the Langlands L-function for $L^2(\Gamma\backslash G)$, which is reminiscent of the zeta function for a surface over a finite field. It is natural to seek similar results for other p-adic groups of Lie type as well as real Lie groups of rank at least 2.

This monograph grew out of the lectures and courses I gave during the years 2014–2017 on various occasions. More materials are supplemented at each stage to broaden the scope. It started with the 10 lectures on combinatorial zeta and L-functions delivered at the NSF-CBMS Regional Research Conference in the Mathematical Sciences, May 12–16, 2014, at the Sundance Resort, Utah. I am indebted to the organizers Jasbir Chahal and Michael Barrus from Brigham Young University for their invitation and hard work, the National Science Foundation, the Conference Board of the Mathematical Sciences, and Brigham Young University for their financial support, and the participants for their enthusiasm and feedback. Special thanks are due to Steve Butler and Alia Hamieh for leading discussion groups and making daily lecture notes available to the participants the next day and to Steve Butler for typing the lecture notes. This manuscript is the skeleton of the Distinguished Lecture Series at the National Tsing Hua University, Taiwan, from May to July of 2015, the one semester graduate course at the Pennsylvania State University in the fall of 2016, and finally a two-month short course at the University of Hong Kong from May to July of 2017. I would like to express my gratitude to the Mathematics Department of the National Tsing Hua University in Taiwan, the Institute of Mathematical Research at the University of Hong Kong, and the Simons Foundation in the USA, which supported the research and writing of this work.

August 6, 2018
The Pennsylvania State University

Wen-Ching W. Li