LECTURE 8

Zeta and L-functions of complexes

8.1. The building attached to $PGL_n(F)$

Throughout this lecture F denotes a nonarchimedean local field with q elements in its residue field. Such a field is a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((x))$, where q is a power of p. Denote by $\mathcal{O}_F$ the ring of integers of F and by π a uniformizer of $\mathcal{O}_F$, thus the residue field $\mathcal{O}_F/\pi\mathcal{O}_F$ is isomorphic to $\mathbb{F}_q$. Similar to $PGL_2(\mathbb{Q}_p)/PGL_2(\mathbb{Z}_p)$ discussed in §4, Lecture 6, each coset $gPGL_n(\mathcal{O}_F)$ in $PGL_n(F)/PGL_n(\mathcal{O}_F)$ can be interpreted as an equivalence class of rank-n lattices (i.e, $\mathcal{O}_F$-modules) $[L_g]$ in F^n. They are the vertices of the building $\mathcal{B}_n(F)$ attached to $PGL_n(F)$. Two vertices v_1 and v_2 are adjacent if they can be represented by lattices L_1 and L_2 such that $L_1 \supsetneq L_2 \supsetneq \pi L_1 \supsetneq \pi L_2$, as before. Note that $L_1/\pi L_1$ is isomorphic to $(\mathbb{F}_q)^n$ and $L_2/\pi L_1$ is a proper nonzero subspace of $\mathbb{F}_q^n$. Thus a maximal set of mutually adjacent vertices has cardinality n; they form an $(n-1)$-dimensional simplicial complex, called a *chamber*. In other words, vertices v_1,..., v_n form a chamber if they can be represented by lattices $L_1, ..., L_n$ such that

$$L_1 \supsetneq L_2 \supsetneq \cdots \supsetneq L_n \supsetneq \pi L_1,$$

or equivalently,

$$0 \subsetneq L_n/\pi L_1 \subsetneq L_{n-1}/\pi L_1 \subsetneq \cdots \subsetneq L_2/\pi L_1 \subsetneq L_1/\pi L_1 = (\mathbb{F}_q)^n$$

is a complete flag of $\mathbb{F}_q$-subspaces in $(\mathbb{F}_q)^n$. For example, the standard chamber has vertices $[L_{g_i}]$ for $0 \le i \le n-1$, where g_i is the diagonal matrix $\mathrm{diag}(1, ..., 1, \pi, ..., \pi)$ of $\det g_i = \pi^i$. The chambers sharing a fixed vertex correspond bijectively to the complete flags in $(\mathbb{F}_q)^n$. The building $\mathcal{B}_n(F)$, as the union of the chambers, is a contractible $(n-1)$-dimensional simplicial complex. For $0 \le j \le n-1$, the j-dimensional facets of the building $\mathcal{B}_n$ consist of the j-dimensional facets of chambers. The group $PGL_n(F)$ acts on $\mathcal{B}_n(F)$ by left translation, preserving j-dimensional facets for all $0 \le j \le n-1$. Hence we may take discrete torsion-free cocompact subgroups Γ of $PGL_n(F)$ with $\mathrm{ord}_\pi \det \Gamma \subset n\mathbb{Z}$ and obtain finite $(n-1)$-dimensional complexes $X_\Gamma = \Gamma\backslash\mathcal{B}_n(F)$ as before.

Each vertex $[L_g]$ has a type, given by $\mathrm{ord}_\pi \det g \mod n$. Adjacent vertices have different types. In particular, the vertices in a chamber exhaust all possible types exactly once. The 1-skeleton of $\mathcal{B}_n(F)$ is the graph consisting of the vertices and edges in the building; the vertices are partitioned according to their types and the adjacency relation makes it an n-partite graph. There are $n-1$ Hecke operators A_i, $i = 1, ..., n-1$, based on the double cosets $PGL_n(\mathcal{O}_F)g_iPGL_n(\mathcal{O}_F)$; the action of A_i on $f \in L^2(PGL_n(F)/PGL_n(\mathcal{O}_F))$ is given by

$$A_i f(x) = \sum_{\substack{y \text{ adjacent to } x,\\ \textit{type } y = (\textit{type } x)+i \mod n}} f(y).$$

Equivalently, we may express the double coset $PGL_n(\mathcal{O}_F)g_iPGL_n(\mathcal{O}_F)$ as a disjoint union $\cup_{j\in I_i} g_{i,j}PGL_n(\mathcal{O}_F)$ of right $PGL_n(\mathcal{O}_F)$-cosets indexed by a finite set I_i, then

$$A_i f(gPGL_n(\mathcal{O}_F)) = \sum_{j\in I_i} f(gg_{i,j}PGL_n(\mathcal{O}_F)),$$

similar to what happened for the case $n = 2$ discussed in §4, Lecture 6. The operators A_i and A_{n-i} are transpose of each other, and the A_i's commute. Consequently the induced operators A_i on a finite quotient X_Γ are simultaneously diagonalizable.

Each $(n-2)$-dimensional simplex is contained in $q+1$ chambers. Each vertex x of $\mathcal{B}_n(F)$ has

$$q_{n,i} = \frac{(q^n-1)\cdots(q-1)}{(q^i-1)\cdots(q-1)(q^{n-i}-1)\cdots(q-1)}$$

neighbors of type equal to $(type\ x) + i \mod n$. We say that $\mathcal{B}_n(F)$ is $(q+1)$-regular. Topologically the building $\mathcal{B}_n(F)$ is simply connected, hence it serves as the universal cover of its finite left quotients, which are finite simplicial complexes of dimension $n-1$.

8.2. Spectral theory of regular complexes from $\mathcal{B}_n(F)$

MacDonald [**70, 71**] proved that, for $1 \le \ell \le n-1$, the spectrum of A_ℓ on $\mathcal{B}_n(F)$ is $q^{\ell(n-\ell)/2}\Omega_{n,\ell}$, where

$$\Omega_{n,\ell} = \{\sigma_\ell(z_1, ..., z_n) : z_1, ..., z_n \in S^1, z_1\cdots z_n = 1\}$$

and $\sigma_\ell(z_1, ..., z_n) = \sum_{1\le i_1<\cdots<i_\ell\le n} z_{i_1}\cdots z_{i_\ell}$ is the ℓ^{th} elementary symmetric polynomial in n-variables. Note that $\Omega_{2,1}$ is the interval $[-2, 2]$. For $n \ge 3$, the spectrum $\Omega_{n,\ell}$ of A_ℓ is no longer real; it is a region in the complex plane invariant under multiplication by the n^{th} roots of unity.

The following Alon-Boppana type theorem holds for finite (left) quotients of the bulding $\mathcal{B}_n(F)$. Let $\{X_j\}$ be a family of $(q+1)$-regular finite quotients of $\mathcal{B}_n(F)$ with $|X_j| \to \infty$ as $j \to \infty$. The operators A_ℓ for $1 \le \ell \le n-1$ defined above act on functions on vertices of X_j. The trivial eigenvalues of A_ℓ are $q_{n,\ell}e^{2\pi ir/n}$ for $r = 1, ..., n$.

THEOREM 8.1 (Li [**62**]). *Assume that each X_j contains a ball isomorphic to a ball in $\mathcal{B}_n(F)$ with radius approaching ∞ as $j \to \infty$. Then, for each $1 \le \ell \le n-1$, the closure of the collection of eigenvalues of $A_\ell(X_j)$ for all $j \ge 1$ contains the region $q^{\ell(n-\ell)/2}\Omega_{n,\ell}$.*

When $n = 2$, this reduces to the Alon-Boppana Theorem 5.1 and its counterpart Theorem 5.4 for graphs.

8.3. Ramanujan complexes as finite quotients of $\mathcal{B}_n(F)$

In view of the definition of Ramanujan graphs for regular and bi-regular bipartite graphs as discussed in Lectures 5 and 6, a finite quotient X of $\mathcal{B}_n(F)$ is called a Ramanujan complex if, for $1 \le \ell \le n-1$, all nontrivial eigenvalues of A_ℓ on X fall in $q^{\ell(n-\ell)/2}\Omega_{n,\ell}$, the spectrum of A_ℓ on $\mathcal{B}_n(F)$. Theorem 8.1 says that a Ramanujan complex is spectrally extremal. Like graphs, there are explicitly constructed families of Ramanujan complexes.

THEOREM 8.2. *For q equal to a prime power and $n \geq 2$, there exist explicitly constructed infinite families of $(q+1)$-regular $(n-1)$-dimensional Ramanujan complexes arising as quotients of $\mathcal{B}_n(F)$.*

The explicit constructions rely on the Ramanujan conjecture established for certain representations of GL_n or the multiplicative group of division algebras over function fields. We sketch the basic approach, which is similar to the explicit construction for Ramanujan graphs by Lubotzky-Phillips-Sarnak explained in §5, Lecture 6. Let $K = \mathbb{F}_q(x)$ be a rational function field such that $F = K_v$ is the completion of K at a degree-1 place v not equal to the place at infinity ∞. Let H be a central simple division algebra of dimension n^2 over K, totally ramified at ∞ and unramified at v. Put $D = H^\times/\text{center}$. Then $D(K_v) \cong PGL_n(F)$ so that the building $\mathcal{B}_n(F)$ is isomorphic to $D(K_v)/D(\mathcal{O}_v)$. By strong approximation theorem, for a compact open subgroup $\mathcal{K}$ of $\prod_{w\neq v,\infty} D(\mathcal{O}_w)$, the double coset space

$$X_{\mathcal{K}} = D(K)\backslash D(\mathbb{A}_K)/D(K_\infty)D(\mathcal{O}_v)\mathcal{K}$$

can be expressed locally at v as

$$X_{\mathcal{K}} = \Gamma_{\mathcal{K}}\backslash D(K_v)/D(\mathcal{O}_v) = \Gamma_{\mathcal{K}}\backslash \mathcal{B}_n(F).$$

Here $\Gamma_{\mathcal{K}} = D(K)\cap\mathcal{K}$ is a discrete cocompact subgroup of $\mathcal{B}_n(F)$. Shrinking $\mathcal{K}$ if necessary we may also assume $\Gamma_{\mathcal{K}}$ is torsion-free and $\operatorname{ord}_\pi \det \Gamma_{\mathcal{K}} \equiv 0 \mod n$ so that it preserves the type of each vertex. As such, $X_{\mathcal{K}}$ is a finite $(q+1)$-regular $(n-1)$-dimensional simplicial complex.

The space of functions on vertices of $X_{\mathcal{K}}$ are automorphic forms on $D(\mathbb{A}_K)$ right invariant by $D(K_\infty)D(\mathcal{O}_v)\mathcal{K}$. It contains the constant functions and their twists by n^{th} roots of unity. They are common eigenfunctions of A_ℓ for $1 \leq \ell \leq n-1$ with trivial eigenvalues $q_{n,\ell}e^{2\pi i r/n}$ for $r = 1, ..., n$. By Jacquet-Langlands correspondence the orthogonal complement $V_{\mathcal{K}}$ of these functions can be interpreted as certain automorphic forms on $PGL_n(\mathbb{A}_K)$ with the same eigenvalues for the Hecke operators A_ℓ for $1 \leq \ell \leq n-1$. As Lafforgue [**55**] proved the Ramanujan conjecture for cuspidal representations for $GL_n(\mathbb{A}_K)$, so we just have to choose appropriate $\mathcal{K}$ so that only cuspidal automorphic forms occur in $V_{\mathcal{K}}$ in order that $X_{\mathcal{K}}$ is Ramanujan. By varying the congruence subgroups $\mathcal{K}$, an infinite family of $(n-1)$-dimensional Ramanujan complexes is obtained.

It should be pointed out that, when $n = 2$, the building $\mathcal{B}_2(F)$ is a $(q+1)$-regular tree and any choice of a compact open subgroup $\mathcal{K}$ of $\prod_{w\neq v,\infty} D(\mathcal{O}_w)$ such that $\Gamma_{\mathcal{K}}$ is torsion-free gives rise to a Ramanujan graph. For $n \geq 3$, Lubotzky, Samuels and Vishne [**69**] showed that the same method yields Ramanujan complexes if n is a prime. When n is a composite, more care is required in order to avoid the residual spectrum of PGL_n to occur in the space $V_{\mathcal{K}}$ since such representations do not satisfy the Ramanujan conjecture. They showed that there are indeed infinitely many $\mathcal{K}$ giving rise to complexes that are not Ramanujan.

There are three explicit constructions of Ramanujan complexes in the literature, by Li [**62**], Lubotzky-Samuels-Vishne [**68**], and Sarveniazi [**86**] from 2004 to 2007. At the time when their works were done, the Jacquet-Langlands correspondence for the multiplicative groups of division algebras of dimension n^2 over function fields was established only for primes n. The proof of the correspondence for all n appeared only in 2016 by Badulescu and Roche [**7**]. Li's construction [**62**] avoided this problem by using the results of Laumon-Rapoport-Stuhler [**59**], where the Ramanujan conjecture was shown to hold for automorphic representations of $D(\mathbb{A}_K)$ having a local component at a place $w \neq v, \infty$ being Steinberg. Her construction did not use Lafforgue's result [**55**] either, but required the division algebras to ramify at least at 4 places. The other two constructions assumed the validity of the Jacquet-Langlands correspondence and used Lafforgue's result, but could be applied to division algebras totally ramified only at two places. The method in [**68**] closely parallels that in [**67**] by finding a group to represent the vertices of $\mathcal{B}_n(F)$. So did [**86**], which established this fact in a different and elegant way. To show variations of viewpoints, we sketch below Sarveniazi's approach [**86**].

Denote by H the simple central division algebra over $K = \mathbb{F}_q(t)$ of dimension n^2 so that

$$H(K) = \mathbb{F}_{q^n}(t) + \mathbb{F}_{q^n}(t)\tau + \cdots + \mathbb{F}_{q^n}(t)\tau^{n-1},$$

where $\tau^n = t$ and $\tau\alpha = \alpha^q\tau$ for $\alpha \in \mathbb{F}_{q^n}$. Then H is totally ramified at $t = \infty$ with invariant $-1/n$ and at $t = 0$ with invariant $1/n$, and unramified elsewhere.

The group of norm 1 elements in $\mathbb{F}_{q^n}$ is

$$N_1 = \{\alpha \in \mathbb{F}_{q^n} : N_{\mathbb{F}_{q^n}/\mathbb{F}_q}(\alpha) = 1\} = \{\beta^{q-1} : \beta \in \mathbb{F}_{q^n}^\times\}.$$

Here the second expression follows from the Hilbert Theorem 90. It has cardinality

$$|N_1| = \frac{q^n - 1}{q - 1} = q^{n-1} + q^{n-2} + \cdots + q + 1.$$

Observe that, up to multiplication by elements in N_1, $1 - \alpha\tau$ with $\alpha \in N_1$ are the elements in $H(K)$ of reduced norm $1 - t$.

Think of τ as the Frobenius automorphism on $\mathbb{F}_{q^n}$ sending x to x^q and regard a polynomial $\sum_{0 \le i \le n-1} a_i\tau^i$ with $a_i \in \mathbb{F}_{q^n}$ as an $\mathbb{F}_q$-linear endomorphism on $\mathbb{F}_{q^n}$ which sends x to $\sum_{0 \le i \le n-1} a_i x^{q^i}$. View $\mathbb{F}_{q^n}$ as an n-dimensional vector space over $\mathbb{F}_q$. Given a complete flag of $\mathbb{F}_q$-vector spaces

$$0 \subset < v_1 > \subset < v_1, v_2 > \subset \cdots \subset < v_1, v_2, ..., v_n > = \mathbb{F}_{q^n},$$

we shall find unique elements $1 - \alpha_1\tau$, $1 - \alpha_2\tau$,..., $1 - \alpha_n\tau$ with $\alpha_i \in N_1$ such that for $1 \le i \le n$,

$$< v_1, ..., v_i > \text{ is the kernel of } (1 - \alpha_i\tau) \cdots (1 - \alpha_1\tau) \text{ on } \mathbb{F}_{q^n}.$$

First we note that for any $\beta^{1-q} \in N_1$, the kernel of $1-\beta^{1-q}\tau$ is 1-dimensional, equal to $< \beta >$. This is because $x \in \mathbb{F}_{q^n}^\times$ is such that $(1 - \beta^{1-q}\tau)(x) = x - \beta^{1-q}x^q = 0$ if and only if $x^{q-1} = \beta^{q-1}$, which means that x and β differ by a multiple in $\mathbb{F}_q^\times$. Hence we choose $\alpha_1 = v_1^{1-q}$ so that $< v_1 >$ is the kernel of $1 - \alpha_1\tau$. The image of v_2 under $1 - \alpha_1\tau$ is $v_2 - \alpha_1 v_2^q =: \beta_2$ is a nonzero element in $\mathbb{F}_{q^n}$. Letting $\alpha_2 = \beta_2^{1-q}$, we see that v_2 is annihilated by $(1 - \alpha_2\tau)(1 - \alpha_1\tau)$ so that the kernel of $(1 - \alpha_2\tau)(1 - \alpha_1\tau)$ is precisely $< v_1, v_2 >$. Inductively, suppose we have found $\alpha_1, ..., \alpha_i$ in N_1 so that the kernel of $(1 - \alpha_i\tau) \cdots (1 - \alpha_1\tau)$ is $< v_1, ..., v_i >$ for $1 \le i < N$, then $\alpha_{i+1} = \beta_{i+1}^{1-q}$ with $\beta_{i+1} = (1 - \alpha_i\tau) \cdots (1 - \alpha_1\tau)(v_{i+1})$ has the desired property that the kernel of $(1-\alpha_{i+1}\tau) \cdots (1-\alpha_1\tau)$ is $< v_1, ..., v_{i+1} >$. Note that the α_i's are uniquely determined by the complete flag.

Since the product $(1 - \alpha_n\tau) \cdots (1 - \alpha_1\tau) = 1 + \cdots + a_n\tau^n$ is a polynomial in τ of degree n which is the zero map on $\mathbb{F}_{q^n}$, and so is $1 - \tau^n = 1 - t$. The difference of the two, if nonzero, would yield a nontrivial polynomial relation of degree $\le n - 1$

in the Frobenius of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$, contradicting the linear independence of 1, τ,..., τ^{n-1}. Hence we have

$$(1-\alpha_n\tau)\cdots(1-\alpha_1\tau) = 1-t.$$

This shows that a complete flag of $\mathbb{F}_{q^n}$ gives rise to a unique factorization of $1-t$ as a product of n terms of the form $1-\alpha\tau$ with $\alpha \in N_1$. Conversely, any such factorization yields a complete flag with $< v_1, ..., v_i >$ being the kernel of $(1-\alpha_i\tau)\cdots(1-\alpha_1\tau)$. Hence there is a bijection between the complete flags of $\mathbb{F}_{q^n}$ and the factorizations of $1-t$ as a product of n terms of the form $1-\alpha\tau$ with $\alpha \in N_1$.

Now let F be the completion of $K = \mathbb{F}_q(t)$ at the place $t = 1$. Put $D = H^\times/\text{center}$. As H is unramified at $t = 1$, we have $D(F) \cong PGL_n(F)$ and the building $\mathcal{B}_n(F) = D(F)/D(\mathcal{O}_F)$. Let S consist of $1-\alpha_1\tau$, $(1-\alpha_2\tau)(1-\alpha_1\tau)$,..., $(1-\alpha_{n-1}\tau)\cdots(1-\alpha_1\tau)$, where all α_i are in N_1, and each element of S divides $1-t$. To show that S is symmetric, it suffices to prove that given $u = (1-\alpha_i\tau)\cdots(1-\alpha_1\tau)$ in S there is an element $v = (1-\alpha_n\tau)\cdots(1-\alpha_{i+1}\tau)$ in S such that the product vu is $1-t$. Since $1-t$ lies in the center of H, this would imply v and u are inverse to each other in $D(F)$. To find v, write $< v_1 >= ker(1-\alpha_1\tau)$, $< v_1, v_2 >= ker(1-\alpha_2\tau)(1-\alpha_1\tau)$, ..., $< v_1, ..., v_i >= ker\ u$ as we did before. This gives a partial flag $0 \subset< v_1 >\subset< v_1, v_2 >\subset \cdots \subset< v_1, ..., v_i >$ in $\mathbb{F}_{q^n}$, which we complete by adding a choice of subspaces

$$< v_1, ..., v_i, v_{i+1} >\subset< v_1, ..., v_{i+1}, v_{i+2} >\subset \cdots \subset< v_1, ..., v_n >= \mathbb{F}_{q^n}.$$

As shown before, there are unique $\alpha_{i+1}, ..., \alpha_n$ in N_1 such that this complete flag corresponds to the factorization

$$(1-\alpha_n\tau)\cdots(1-\alpha_{i+1}\tau)(1-\alpha_i\tau)\cdots(1-\alpha_1\tau) = 1-t.$$

Then $v = (1-\alpha_n\tau)\cdots(1-\alpha_{i+1}\tau)$ lies in S and $vu = 1-t$, as desired.

Let Λ be the group generated by S. Sarveniazi showed that the 1-skeleton of $\mathcal{B}_n(F)$ is the Cayley graph $Cay(\Lambda, S)$. To see this, recall that all chambers containing a fixed vertex x correspond bijectively to the complete flags in $(\mathbb{F}_q)^n$. By viewing $\mathbb{F}_{q^n}$ as an n-dimensional vector space over $\mathbb{F}_q$, these chambers correspond bijectively to the factorizations of $1-t$ as a product of n factors of the form $1-\alpha\tau$ with $\alpha \in N_1$, as observed before. In other words, given a factorization

$$(1-\alpha_n\tau)(1-\alpha_{n-1}\tau)\cdots(1-\alpha_1\tau) = 1-t,$$

we obtain a chamber with vertices $x_1 = x$, $x_2 = (1-\alpha_1\tau)x_1$, $x_3 = (1-\alpha_2\tau)(1-\alpha_1\tau)x_1 = (1-\alpha_2\tau)x_2$, ..., $x_n = (1-\alpha_{n-1}\tau)x_{n-1}$, and $x_1 = (1-\alpha_n\tau)x_n$.

In this setting the Hecke operators A_ℓ, $1 \le \ell \le n-1$, can be neatly expressed as sending $f \in L^2(\Lambda)$ to

$$A_\ell f(x) = \sum f((1-\alpha_\ell\tau)\cdots(1-\alpha_1\tau)x),$$

where the sum is over all elements $(1-\alpha_\ell\tau)\cdots(1-\alpha_1\tau)$ in S which have reduced norm $(1-t)^\ell$.

Sarveniazi constructed infinite families of finite $(q+1)$-regular Ramanujan complexes by taking the building modulo varying monic irreducible polynomials $f(t) \in \mathbb{F}_q[t]$ of degree d and coprime to t and $t-1$. To simplify our exposition, assume n divides d so that the residue field of $f(t)$ contains the field $\mathbb{F}_{q^n}$. (This resembles the choice of $q \equiv 1 \mod 4$ in the construction of Ramanujan graphs by

Lubotzky-Phillips-Sarnak discussed in §5, Lecture 6.) In other words, the 1-skeleton of the finite $(n-1)$-dimensional Ramanujan simplicial complex is

$$Cay(\Lambda \mod f(t),\ S \mod f(t)).$$

Like in [**67**], he further identified the vertex set as $PGL_n(\mathbb{F}_{q^d})$ if $1-t$ is not an r^{th} power modulo $f(t)$ for any $r|n$ and $r>1$, and $PSL_n(\mathbb{F}_{q^d})$ if $1-t$ is an n^{th} power modulo $f(t)$. If $1-t$ is congruent to an r^{th} power mod $f(t)$ for some $r|n$ and $r>1$, then the group is an index-r subgroup of $PGL_n(\mathbb{F}_{q^d})$ containing $PSL_n(\mathbb{F}_{q^d})$. The element $1-\alpha\tau$ in S is identified with the following $n\times n$ matrix:

$$\begin{pmatrix} 1 & -\alpha & & & & \\ & 1 & -\alpha^q & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & 1 & -\alpha^{q^{n-2}} \\ -\alpha^{q^{n-1}}t & & & & & 1 \end{pmatrix} \in PGL_n(\mathbb{F}_{q^d}).$$

The combinatorial meaning and possible applications of Ramanujan complexes have been active research topics in recent years. For instance, it was discovered that explicit constructions of Ramanujan complexes also lead to optimal logical gates for quantum computers on a single qubit [**85**].

8.4. Zeta and L-functions of finite quotients of $\mathcal{B}_3(F)$

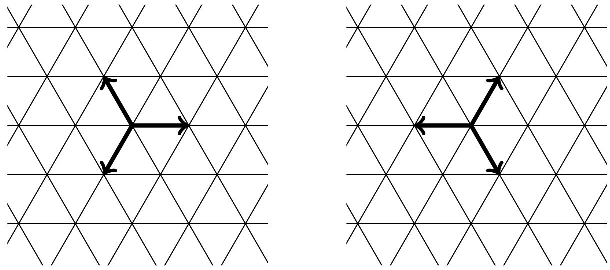
Let Γ be a discrete torsion-free cocompact subgroup of $PGL_3(F)$ such that $\text{ord}_\pi \det\Gamma \equiv 0 \mod 3$. Then left translation by Γ on the building $\mathcal{B}_3(F)$ preserves the types of vertices. The quotient

$$X_\Gamma = \Gamma\backslash\mathcal{B}_3(F) = \Gamma\backslash PGL_3(F)/PGL_3(\mathcal{O}_F)$$

is a finite $(q+1)$-regular 2-dimensional simplicial complex. Our goal is to define the zeta function $Z(X_\Gamma,u)$ of X_Γ such that it has the following nice properties possessed by zeta functions of graphs:

(1) It is a rational function with a closed form expression;
(2) It provides topological and spectral information of X_Γ;
(3) It satisfies the Riemann Hypothesis if and only if X_Γ is Ramanujan.

The first task is to decide what kind of "closed geodesics"to count. The 2-dimensional building $\mathcal{B}_3(F)$ is obtained by gluing together its "apartments "along certain chambers. Each apartment is a Euclidean plane tiled by equilateral triangles, which are chambers. The Euclidean metric on each apartment in turn yields the metric on the building $\mathcal{B}_3(F)$ and thus on the quotient X_Γ. Recall that adjacent vertices have different types. The type of a directed edge from vertex x to vertex y is defined to be $type(y)-type(x)$. A directed path using edges of the same type i is called a path of type i; the same path traveled in opposite direction has type $3-i$. Out of a vertex in an apartment there are three directed edges of type 1 (bold in the left figure) and three of type 2 (bold in the right figure) as depicted below:



edges of type 1 edges of type 2

A geodesic cycle C of X_Γ is a closed 1-dimensional geodesic path in X_Γ. It has a starting vertex and orientation. It lifts to a geodesic path C' starting at a vertex in $\mathcal{B}_3(F)$, which can be shown to lie in an apartment $\mathcal{A}$ of $\mathcal{B}_3(F)$. Hence it is a straight line in $\mathcal{A}$ starting at a vertex v and ending at a vertex γv for some non-identity element $\gamma \in \Gamma$. To achieve our goals, it turns out that we should only consider those C contained in the 1-skeleton of X_Γ so that C' is a straight line in $\mathcal{A}$ using edges of the same type $i \in \{1,2\}$. Furthermore, a geodesic cycle C is required to remain a geodesic cycle after changing the starting vertex. This is equivalent to v, γv and $\gamma^2 v$ being collinear in $\mathcal{A}$. Such a C is called a geodesic cycle of X_Γ of type i; the same cycle traveled in opposite direction is a geodesic cycle of type $3-i$. As before, C is primitive if it is not obtained by repeating a shorter geodesic cycle more than once. Two geodesic cycles are equivalent if they differ by starting points. Denote by $[C]$ the equivalence class of C. As before, the primes of X_Γ are equivalence classes of primitive geodesic cycles. The geometric length $\ell(C)$ of C is the number of edges in C.

For $n \geq 1$, let $N_n(X_\Gamma)$ denote the number of geodesic cycles in X_Γ of type 1 and geometric length n; it is also equal to that of type 2. For $i = 1, 2$ the type i edge zeta function of X_Γ is defined as

$$Z_{1,i}(X_\Gamma, u) = \exp\Big(\sum_{n=1}^{\infty} \frac{N_n(X_\Gamma)}{n} u^n\Big) = \prod_{[C] \text{ prime of type } i} \frac{1}{1-u^{\ell(C)}}.$$

Combined, they define the (edge) zeta function for X_Γ:

$$Z(X_\Gamma, u) = Z_{1,1}(X_\Gamma, u) Z_{1,2}(X_\Gamma, u^2) = \prod_{i=1}^{2} \prod_{[C] \text{ prime of type } i} \frac{1}{1-u^{\ell(C)i}}.$$

Given a type 1 edge $v_1 \to v_2$, its type 1 neighbors are defined to be the type 1 edges $v_2 \to v_3$ in $\mathcal{B}_3(F)$ such that the vertices v_1, v_2, v_3 do not form a chamber. Similar to Theorem 4.3, $Z_{1,1}(X_\Gamma, u)^{-1}$ is a polynomial in u which can be expressed in terms of the type 1 edge adjacency matrix T_E of X_Γ:

$$Z_{1,1}(X_\Gamma, u) = \frac{1}{\det(I - T_E u)}.$$

The stabilizer of the type 1 edge from $[L_{I_3}]$ to $[L_{\mathrm{diag}(1,1,\pi)}]$ in the standard chamber is the parahoric subgroup

$$E = \{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in PGL_3(\mathcal{O}_F) : \pi | g, \pi | h \}.$$

Thus the type 1 edges in $\mathcal{B}_3(F)$ are parameterized by $PGL_3(F)/E$. The operator T_E can be expressed as a parahoric Hecke operator based on the E-double coset $E\,\mathrm{diag}(1,1,\pi)E$. As type 2 edges are the opposite of type 1 edges, the type 2 adjacency matrix is the transpose T_E^t of T_E, and

$$Z_{1,2}(X_\Gamma, u) = \frac{1}{\det(I - T_E^t u)}.$$

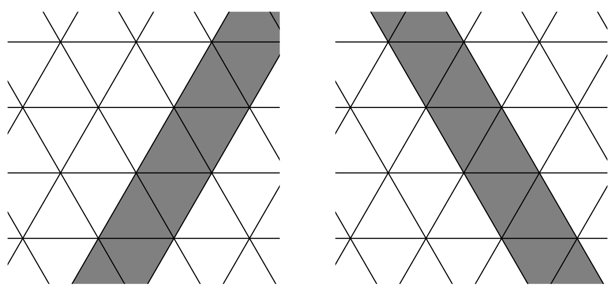
Therefore

$$Z(X_\Gamma, u) = \frac{1}{\det(I - T_E u)\det(I - T_E^t u^2)}.$$

It is much more challenging to find an expression for $Z(X_\Gamma, u)$ analogous to Ihara's theorem for graphs, providing topological information involving the Euler characteristic

$$\chi(X_\Gamma) = \#\text{vertices} - \#\text{edges} + \#\text{chambers}$$

of X_Γ and the spectral information involving the eigenvalues of the Hecke operators A_1 and A_2 acting on functions on vertices of X_Γ. Since X_Γ is 2-dimensional, this will also involve backtrackless sequences of adjacent chambers (i.e. sharing edges) of X_Γ, called *galleries*. We shall only count closed geodesic galleries in X_Γ whose boundaries are geodesic cycles in X_Γ of type 2, as shown in the two examples below.



two geodesic galleries in $\mathcal{B}_3(F)$

Define primitive and equivalent closed geodesic galleries similar to those of cycles. Call the equivalence classes of such primitive closed geodesic galleries 2-dimensional primes of type 2. Note that the same gallery traveled in reverse direction has boundary of type 1, hence the 2-dimensional primes of type 1 are the opposite of those of type 2. The length of a gallery is the number of chambers in the sequence. Denote by $M_n(X_\Gamma)$ the number of closed geodesic galleries in X_Γ of length n. Define the gallery zeta function of X_Γ to be

$$Z_{2,2}(X_\Gamma, u) = \exp\Big(\sum_{n=1}^{\infty} \frac{M_n(X_\Gamma)}{n} u^n\Big) = \prod_{\text{2-dim'l prime } [C] \text{ of type 2}} \frac{1}{1 - u^{\ell(C)}}.$$

The Iwahori subgroup

$$B = \{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in PGL_3(\mathcal{O}_F) : \pi|d, \pi|g, \pi|h\}$$

is the stabilizer of the three vertices of the standard chamber, hence it stabilizes a chamber together with an orientation, called a *directed chamber*, which is expressed as an ordered triple of the three vertices of the chamber up to cyclic permutation. So $PGL_3(F)/B$ parameterizes all directed chambers in the building $\mathcal{B}_3(F)$. Define the out-neighbors of a directed chamber (v_1, v_2, v_3) to be the directed chambers (v_2, v_3, v_4) in $\mathcal{B}_3(F)$ with $v_4 \neq v_1$. Denote by T_B the adjacency matrix of directed chambers of X_Γ. It is an Iwahori Hecke operator based on the B-double coset $B \begin{pmatrix} 0 & 1 & 0 \\ \pi & 0 & 0 \\ 0 & 0 & \pi \end{pmatrix} B$. Then, expression like Theorem 4.3 holds for the gallery zeta function as well:

$$Z_{2,2}(X_\Gamma, u) = \frac{1}{\det(I - T_B u)}.$$

Now we are ready to state our main theorem for the zeta function of a finite quotient of $\mathcal{B}_3(F)$.

THEOREM 8.3 (Kang-Li [**46**], Kang-Li-Wang [**48**], Kang-Li [**47**]). *The edge zeta function for X_Γ is a rational function in u which can be expressed in two ways:*

$$Z(X_\Gamma, u) = \frac{1}{\det(I - T_E u)\det(I - T_E^t u^2)} \tag{2}$$

$$= \frac{(1 - u^3)^{\chi(X_\Gamma)}}{\det(I - A_1 u + A_2 q u^2 - q^3 u^3)\det(I + T_B u)}. \tag{3}$$

Here $\chi(X_\Gamma)$ is the Euler characteristic of X_Γ, A_1 and A_2 are Hecke operators, T_E is a parahoric Hecke operator, and T_B is an Iwahori Hecke operator introduced above.

This theorem has three proofs, revealing different interpretations of the identity. In [**46**] the numbers $N_n(X_\Gamma)$ and $M_n(X_\Gamma)$ were computed and compared in terms of contributions from elements in Γ up to conjugation. The identity was first discovered from these computations. The second proof [**48**] is representation-theoretic. The authors computed the eigenvalues of the operators T_E, T_B, A_1 and A_2 and compared them. The third proof [**47**] is cohomological, analogous to that for graphs but more involved.

The zeta function $Z(X_\Gamma, u)$ is said to satisfy the Riemann Hypothesis if all nontrivial zeros of $\det(I - A_1 u + A_2 q u^2 - q^3 u^3)$ from the nontrivial eigenvalues of A_1 and A_2 have the same absolute value q^{-1}. This happens if and only if X_Γ is Ramanujan. It was shown in [**48**, Theorem 2] that the Ramanujan condition has two more equivalent statements in terms of the operators T_E and T_B, respectively.

THEOREM 8.4 (Kang-Li-Wang [**48**]). *The following statements are equivalent:*

(1) X_Γ is Ramanujan;

(2) The nontrivial zeros of $\det(I - A_1 u + A_2 q u^2 - q^3 u^3)$ *have the same absolute value q^{-1};*

(3) The nontrivial zeros of $\det(I - T_E u)$ *have absolute values q^{-1} and $q^{-1/2}$;*

(4) The nontrivial zeros of $\det(I - T_B u)$ *have absolute values* 1, $q^{-1/2}$, *and* $q^{-1/4}$.

This is reminiscent of the Riemann Hypothesis for the zeta function $Z(V,u)$ attached to a smooth irreducible projective surface V defined over a finite field $\mathbb{F}_q$ discussed in §5 of Lecture 1. Recall that the numerator and denominator of $Z(V,u)$ are decomposed into products $P_1(V,u)P_3(V,u)$ and $P_0(V,u)P_2(V,u)P_4(V,u)$, respectively, where $P_i(V,u) = \det(I - \Phi^{(i)}u)$ is the determinant of the induced Frobenius operator $\Phi^{(i)}$ acting on the finite-dimensional vector space, the i^{th} étale cohomology $H^i(V)$ of V, and the roots of $P_i(V,u) \neq 1$ all have the same absolute value $q^{-i/2}$. The above theorem is extended to $PGL_n(F)$ by First in [**24**].

Let $K = \mathbb{F}_q(T)$ be a rational function field and let v be a degree-1 place of K so that the completion K_v of K at v is isomorphic to $F = \mathbb{F}_q((T))$. Similar to what we saw for curves, in [**59**] Laumon, Rapoport and Stuhler proved the existence of certain moduli surfaces V over K which have good reduction at v and the existence of suitably chosen discrete torsion-free cocompact subgroups $\Gamma(V)$ of $PGL_3(F)$ arising from certain division algebras of dimension 9 over K unramified at v so that in the zeta function $Z(X_{\Gamma(V)},u)$ of the complex $X_{\Gamma(V)} = \Gamma(V)\backslash PGL_3(F)/PGL_3(\mathcal{O}_F)$, the factor

$$\frac{\det(I - A_1(X_{\Gamma(V)})u + A_2(X_{\Gamma(V)})qu^2 - q^3u^3)}{\prod_{j=1}^{3}(1-\mu_3^j u)(1-\mu_3^j qu)(1-\mu_3^j q^2 u)}$$

is equal to the factor $P_2(V,u)$ occurring in the zeta function of V mod v. Here μ_3 is a primitive cubic root of unity and the factor in the denominator comes from trivial eigenvalues of A_1 and A_2.

Another connection with number theory is that the factor

$$\frac{1}{\det(I - A_1(X_\Gamma)u + A_2(X_\Gamma)qu^2 - q^3u^3)}$$

occurring in the zeta function $Z(X_\Gamma,u)$ of X_Γ is a Langlands L-function. To explain this, we begin with the definition of Langlands L-functions. A reductive group G over F has a Langlands dual group $G^\vee$ over $\mathbb{C}$. Each irreducible unramified representation σ of $G(F)$ is induced from an unramified character of the maximal torus of $G(F)$. To σ we associate a conjugacy class $\{c(\sigma)\}$ of $G^\vee(\mathbb{C})$. The Langlands L-function of σ with respect to a chosen representation r of $G^\vee(\mathbb{C})$ is defined as

$$L(\sigma,r,u) = \frac{1}{\det(I - r(c(\sigma))u)}.$$

Given a discrete torsion-free cocompact subgroup Γ of $G(F)$, the Langlands L-function of $\Gamma\backslash G(F)$ with respect to r is

$$L(\Gamma\backslash G(F),r,u) = \prod_\sigma L(\sigma,r,u),$$

where σ runs through all unitary irreducible unramified representations of $G(F)$ occurring in the space $L^2(\Gamma\backslash G(F))$, counting multiplicity.

The Langlands dual group of $PGL_3(F)$ is $SL_3(\mathbb{C})$. We choose r to be the standard representation r_{std} of $SL_3(\mathbb{C})$, namely the identity map from $SL_3(\mathbb{C})$ to itself, viewed as a 3-dimensional representation of $SL_3(\mathbb{C})$. A unitary unramified irreducible representation σ occurring in $L^2(\Gamma\backslash PGL_3(F))$ is a principal series representation induced from three unramified characters χ_1, χ_2, and χ_3

of $F^\times$ with $\chi_1\chi_2\chi_3 = 1$. The conjugacy class attached to such a σ is the set $\{\chi_1(\pi), \chi_2(\pi), \chi_3(\pi)\}$, called the Satake parameter of σ. Thus

$$L(\sigma, r_{\rm std}, u) = \frac{1}{(1-\chi_1(\pi)u)(1-\chi_2(\pi)u)(1-\chi_3(\pi)u)}.$$

One verifies that the Langlands L-function for $\Gamma\backslash PGL_3(F)$ is

$$L(\Gamma\backslash PGL_3(F), r_{\rm std}, qu) = \frac{1}{\det(I - A_1(X_\Gamma)u + A_2(X_\Gamma)qu^2 - q^3u^3)}.$$

The group Γ is the fundamental group of X_Γ. Let ρ be a d-dimensional unitary representation of Γ. The Artin L-function for 1-dimensional complexes of X_Γ is defined as

$$L_1(X_\Gamma, \rho, u) = \prod_{[C] \text{ prime of type } 1} \frac{1}{\det(I_d - \rho(\mathrm{Frob}_{[C]})u^{\ell(C)})}.$$

Here, given a prime $[C]$, lift C to a path C' in $\mathcal{B}_3(F)$ starting at a vertex P, then the end point of C' is γP for some $\gamma \in \Gamma$. The conjugacy class of γ is denoted $\mathrm{Frob}_{[C]}$, called the Frobenius conjugacy class at the prime $[C]$. Unlike the case of graphs, there is no good algebraic description of such γ. More precisely, conjugacy classes of primitive elements in Γ do give rise to $\mathrm{Frob}_{[C]}$ at primes $[C]$; on the other hand, there are $\mathrm{Frob}_{[C]}$ which are not conjugacy classes of primitive elements. See [**46**] for more detail. Similarly one defines the Artin L-function $L_2(X_\Gamma, \rho, u)$ for 2-dimensional complexes of X_Γ as a product over equivalence classes of primitive closed geodesic galleries whose boundaries are geodesic cycles of type 2. Then each Artin L-function is a rational function in u and expressions similar to the zeta functions attached to X_Γ hold for Artin L-functions. More precisely, we have

THEOREM 8.5 (Kang-Li [**47**]). *Let ρ be a d-dimensional unitary representation of Γ. Then*

(1) *For $i = 1, 2$, the Artin L-function $L_i(X_\Gamma, \rho, u)$ is invariant under induction.*
(2) *There exist operators $T_E(\rho)$ and $T_B(\rho)$ such that*

$$L_1(X_\Gamma, \rho, u) = \frac{1}{\det(I - T_E(\rho)u)}, \quad L_2(X_\Gamma, \rho, u) = \frac{1}{\det(I - T_B(\rho)u)},$$

and

$$(1-u^3)^{\chi(X_\Gamma)d} L(\mathrm{Ind}_\Gamma^{PGL_3(F)} \rho, r_{\rm std}, qu) = \frac{L_1(X_\Gamma, \rho, u)L_1(X_\Gamma, \rho, u^2)}{L_2(X_\Gamma, \rho, -u)}.$$

Here the Langlands L-function is the product of the Langlands L-functions of the irreducible representations, counting multiplicity, occurring in the induced representation $\mathrm{Ind}_\Gamma^{PGL_3(F)} \rho$ of $PGL_3(F)$. This theorem is proved using cohomological method, similar in spirit to the argument in §4, Lecture 4 but a lot more complicated.

8.5. Zeta functions of finite quotients of the building $\Delta(F)$ of $Sp_4(F)$

Let V be a 4-dimensional vector space over F equipped with the nondegenerate alternating bilinear form $\langle\ ,\rangle$, whose values on the standard basis $\{e_1, e_2, f_1, f_2\}$ of V are given by

$$\langle e_1, f_2\rangle = \langle e_2, f_1\rangle = 1$$

and

$$\langle e_i, f_i\rangle = \langle e_i, e_i\rangle = \langle f_i, f_i\rangle = 0 \qquad \text{for } i \in \{1, 2\}.$$

The linear transformations on V preserving the bilinear form $\langle\ ,\rangle$ constitute the symplectic group $Sp_4(F)$. Its associated building $\Delta(F)$ is a contractible 2-dimensional simplicial complex. Each apartment of $\Delta(F)$ is an Euclidean plane, tiled by isosceles right triangles. Each isosceles right triangle is a chamber, the two vertices on its hypotenuse are special vertices, and the third vertex is nonspecial. Depicted in Figure 4 is the standard apartment $\mathcal{A}$ in which the vertices are described using the lattice model.[3] More precisely, a vertex marked by $[a, b, c, d]$ represents the equivalence class of the rank-4 lattice with a basis $\pi^a e_1$, $\pi^b e_2$, $\pi^c f_1$, and $\pi^d f_2$. The special vertices are at the crossing of solid lines and the remaining vertices are nonspecial.

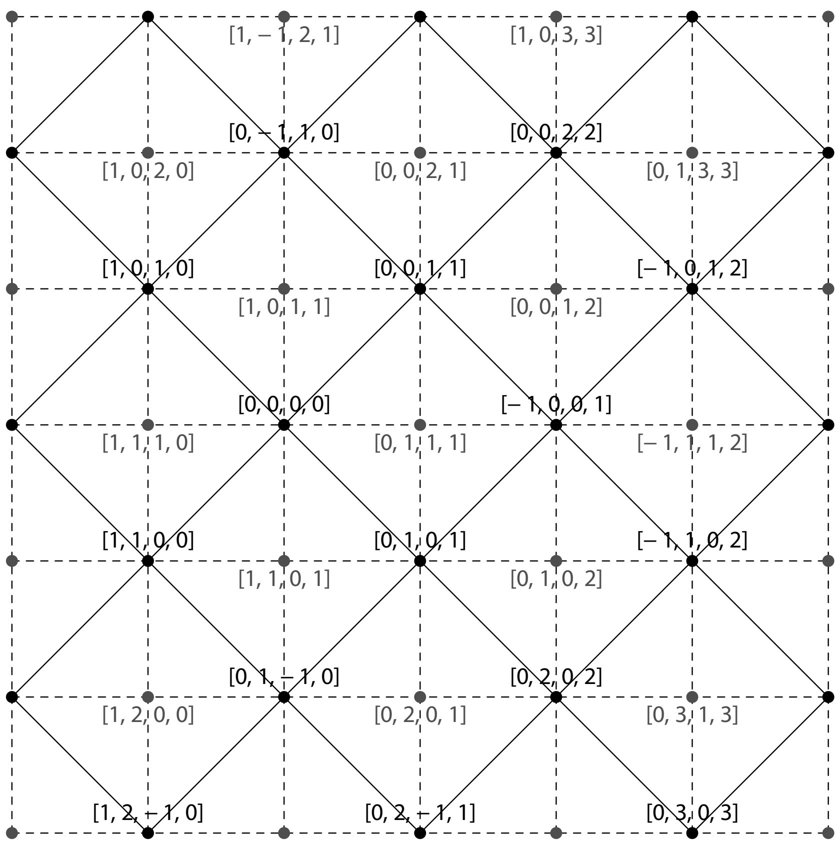


FIGURE 4. The standard apartment $\mathcal{A}$ of $\Delta(F)$

[3]It is Fig. 1 in [**22**], used by permission of Oxford University Press.

The group of symplectic similitudes $GSp_4(F)$ preserves the bilinear form $\langle\ ,\rangle$ up to scalar multiples. Modulo its center, the quotient group $PGSp_4(F)$ acts transitively on the special vertices of $\Delta(F)$ and preserves nonspecial vertices. As the stabilizer of the special vertex $[0,0,0,0]$ is $PGSp_4(\mathcal{O}_F)$, the cosets in $PGSp_4(F)/PGSp_4(\mathcal{O}_F)$ parameterize the special vertices on the building $\Delta(F)$. Out of a special vertex, there are two kinds of directed edges: those between two special vertices are of type 1 and those between a special and a nonspecial vertices are of type 2. The type 1 edges are parameterized by the cosets of the Siegel congruence subgroup P_1 of $PGSp_4(\mathcal{O}_F)$ and the type 2 ones by the cosets of the Klingen congruence subgroup P_2 of $PGSp_4(\mathcal{O}_F)$. The cosets of the Iwahori subgroup $I = P_1 \cap P_2$ parameterize the directed chambers of $\Delta(F)$. For example, on the standard apartment $\mathcal{A}$, the 4 vectors out of $v = [0,0,0,0]$ ending at $[0,0,1,1]$, $[1,0,1,0]$, $[1,1,0,0]$, $[0,1,0,1]$ are type 1 edges, and the 4 vectors ending at $[0,1,1,1]$, $[1,0,1,1]$, $[1,1,1,0]$, $[1,1,0,1]$ are type 2 edges.

Two vertex adjacency operators A_1 and A_2 act on special vertices: A_1 is based on the double coset $PGSp_4(\mathcal{O}_F)\,\mathrm{diag}(1,1,\pi,\pi)PGSp_4(\mathcal{O}_F)$ and A_2 is q^2+1 times the identity operator plus the operator based on the double coset $PGSp_4(\mathcal{O}_F)\,\mathrm{diag}(1,\pi,\pi,\pi^2)PGSp_4(\mathcal{O}_F)$. For $i=1,2$, the type i edge adjacency operator L_{P_i} are based on the double cosets $P_1\,\mathrm{diag}(1,1,\pi,\pi)P_1$ and $P_2\,\mathrm{diag}(1,\pi,\pi,\pi^2)P_2$, respectively. We shall be concerned with two directed chamber adjacency operators L_I based on the double coset ItI with $t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\pi & 0 & 0 \\ 0 & 0 & 0 & \pi \end{pmatrix}$ and L_I' based on the double coset $It'I$ with $t' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \pi^{-1} \\ \pi & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$. See [**22**, **49**] for more details.

Let Γ be a discrete torsion-free cocompact subgroup of $PGSp_4(F)$ such that $\mathrm{ord}_\pi \det \Gamma \equiv 0 \mod 4$. Then Γ preserves the types of vertices and $X_\Gamma = \Gamma\backslash\Delta(F)$ is a finite 2-dimensional simplicial complex. Define 1-dimensional primes of type i and the zeta function of X_Γ counting the closed geodesic cycles contained in the 1-skeleton of X_Γ using edges of the same type as in the previous section:

$$Z(X_\Gamma, u) = \prod_{i=1}^{2} \prod_{1-\text{dim'l prime } [C] \text{ of type } i} \frac{1}{1 - u^{\ell(C)i}}.$$

Similar to Theorem 8.3 for finite quotients of the building $\mathcal{B}_3(F)$, we have two closed form expressions for the zeta function of X_Γ in terms of the various operators introduced above.

THEOREM 8.6 (Fang-Li-Wang [**22**]). *The zeta function $Z(X_\Gamma, u)$ defined above is a rational function in u with two expressions:*

$$\begin{aligned} Z(X_\Gamma, u) &= \frac{1}{\det(I - L_{P_1}u)} \frac{1}{\det(I - L_{P_2}u^2)} \\ &= \frac{(1-u^2)^{\chi(X_\Gamma)}(1-q^2u^2)^{-(q^2-1)N(\Gamma)}}{\det(I - A_1u + qA_2u^2 - q^3A_1u^3 + q^6Iu^4)\det(I - L_Iu)}. \end{aligned}$$

Here $\chi(X_\Gamma) = \#vertices - \#edges + \#chambers$ in X_Γ is the Euler characteristic of X_Γ, and $2N(\Gamma)$ is the number of special vertices in X_Γ.

The dual group of $PGSp_4(F)$ is $\mathrm{Spin}_5(\mathbb{C}) \cong Sp_4(\mathbb{C})$, which has a degree 4 spin representation r_{spin} and a degree 5 standard representation r_{std}. A unitary irreducible representation σ of $PGSp_4(F)$ occurring in $L^2(\Gamma\backslash PGSp_4(F))$ is induced from $\chi_1 \times \chi_2 \rtimes \tau$, where χ_1, χ_2, and τ are unramified characters of $F^\times$ satisfying $\chi_1\chi_2\tau^2 = 1$. The Langlands L-function of σ with respect to r_{spin} is

$$L(\sigma, r_{\mathrm{spin}}, u) = \frac{1}{(1-\chi_1\tau(\pi)u)(1-\chi_2\tau(\pi)u)(1-\chi_1\chi_2\tau(\pi)u)(1-\tau(\pi)u)}$$

and that with respect to r_{std} is

$$\begin{aligned} &L(\sigma, r_{\mathrm{std}}, u) \\ = \quad &\frac{1}{(1-\chi_1(\pi)u)(1-\chi_2(\pi)u)(1-\chi_1^{-1}(\pi)u)(1-\chi_2^{-1}(\pi)u)(1-u)}. \end{aligned}$$

Analogous to the case of $PGL_3(F)$, one verifies that the zeta function $Z(X_\Gamma, u)$ is related to the Langlands L-function of $\Gamma\backslash PGSp_4(F)$ with respect to r_{spin} as follows:

$$L(\Gamma\backslash PGSp_4(F), r_{\mathrm{spin}}, q^{\frac{3}{2}}u) = \frac{1}{\det(I - A_1u + qA_2u^2 - q^3A_1u^3 + q^6Iu^4)}.$$

It is natural to ask whether there is a combinatorial zeta function attached to X_Γ which is related to the Langlands L-function with respect to the standard representation r_{std} of the dual group $\mathrm{Spin}_5(\mathbb{C})$. To answer this, introduce another zeta function

$$Z'(X_\Gamma, u) = \prod_{i=1}^{2} \prod_{1-\dim'\text{l prime } [C] \text{ of type } i} \frac{1}{(1-u^{\ell(C)})^{2/i}}.$$

In [**49**] Kang-Li-Wang obtained a closed form expression of this zeta function, which is more complicated than the theorem above. To describe it, let $A_2' = A_2 - 2q^2I$ and $A_2'' = A_2 - q^2I$.

THEOREM 8.7 (Kang-Li-Wang [**49**]).

$$\begin{aligned} Z'(X_\Gamma, u) &= \frac{1}{\det(I-L_{P_1}u)^2}\frac{1}{\det(I-L_{P_2}u)} \\ = \quad &\frac{(1-u)^2(1+u)^{2\chi(X_\Gamma)-2}(1-qu)^{t_1}(1+qu)^{t_2}}{\det(I - A_2'u + (qA_1^2 - 2q^2A_2'')u^2 - q^4A_2'u^3 + q^8Iu^4)\det(I + L_I'u)}. \end{aligned}$$

Here t_1 and t_2 are integers determined by representations of certain types occurring in $L^2(\Gamma\backslash PGSp_4(F)/I)$. They sum to $-2(q^2-1)N(\Gamma)$, where $2N(\Gamma)$ is the number of special vertices in X_Γ. For detailed expression, see [**49**, Theorem 6.3].

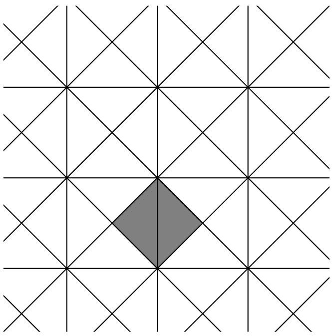
It is related to the Langlands L-function by

$$\begin{aligned} &(1-q^2u)^{2N(\Gamma)}L(\Gamma\backslash PGSp_4(F), r_{\mathrm{std}}, q^2u) \\ = \quad &\frac{1}{\det(I - A_2'u + (qA_1^2 - 2q^2A_2'')u^2 - q^4A_2'u^3 + q^8Iu^4)}. \end{aligned}$$

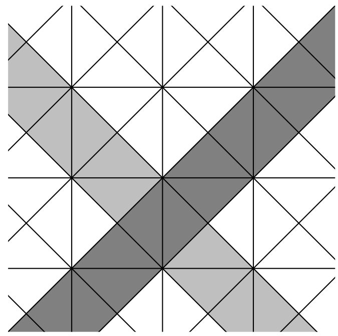
It should be pointed out that, like what we saw in §8.4 for the zeta functions attached to finite quotients of $\mathcal{B}_3$, the factors $\frac{1}{\det(I-L_Iu)}$ occurring in $Z(X_\Gamma, u)$ in

Theorem 8.5 and $\frac{1}{\det(I-L'_I u)}$ in $Z'(X_\Gamma, u)$ in Theorem 8.6 each arise from counting closed geodesic galleries of spin type and of standard type, respectively. More precisely, a chamber of spin type is a "square" in an apartment obtained by gluing together two chambers which share a type 1 edge. Note that the boundaries of this square are four type 2 edges. A sequence of adjacent chambers of spin type with boundry a geodesic path of type 2 edges is called a geodesic gallery of chambers of spin type. Then $\frac{1}{\det(I-L_I u)}$ counts closed geodesic galleries of spin type in X_Γ. A chamber of standard type is a "square" in an apartment obtained by gluing together four chambers along type 2 edges which they share so that the boundaries of the square consist of four type 1 edges. Similarly $\frac{1}{\det(I-L'_I u)}$ counts closed geodesic galleries of standard type in X_Γ.

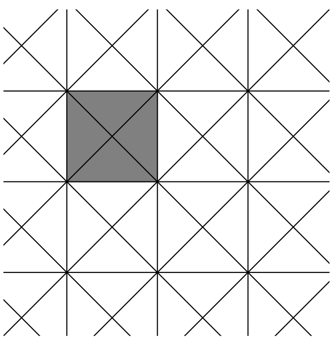
The figures below depict a chamber of spin/standard type and 2 geodesic galleries of chambers of the same type containing it:



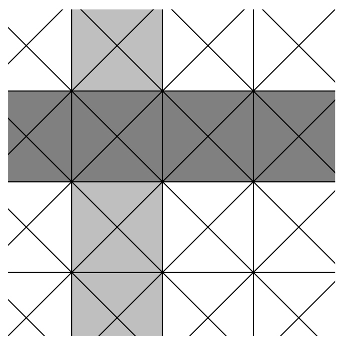
a spin type chamber



2 spin type geodesic galleries



a standard type chamber



2 standard type geodesic galleries

As before, call a 2-dimensional prime of spin/standard type an equivalence class of primitive closed geodesic galleries of spin/standard type. Then $\frac{1}{\det(I-L_I u)}$ (resp. $\frac{1}{\det(I-L'_I u)}$) is a product of $\frac{1}{1-u^{\ell(C)}}$ over 2-dimensional primes $[C]$ of spin (resp. standard) type. Here the length $\ell(C)$ of $[C]$ is the number of chambers of spin (resp. standard) type contained in C. The reader is referred to [**49**] for details.

8.6. Distribution of primes in finite quotients of $\mathcal{B}_3(F)$ and $\Delta(F)$

For a 2-dimensional complex X obtained as a finite quotient of the building $\mathcal{B}_3(F)$ or $\Delta(F)$, we defined 1-dimensional (resp. 2-dimensional) primes which are equivalence classes of primitive closed geodesic cycles (resp. galleries) in X of a given type. In each case there is a zeta function defined as a product over the primes in question, and this zeta function is the reciprocal of a polynomial, of the form $\frac{1}{\det(I-Tu)}$ for a suitable adjacency operator T. Parallel to the case of graphs, very recently Li and Matias [**64**] obtain the "Prime Geodesic Theorem" for 1- and 2-dimensional primes of the given type by studying the analytic behavior of the associated zeta function, or equivalently, the eigenvalues of T. The key is to prove that given two directed edges/chambers of given type, there is a geodesic path of the same type and dimension in X connecting them. This result leads to the determination of the greatest common divisor δ of the length of primes of given type and dimension. Then using the Perron-Frobenius theorem, they determine the largest eigenvalue in absolute value, λ, of T, show that there are δ distinct eigenvalues of T with absolute value λ, and each occurs with multiplicity κ. Their result is summarized below.

THEOREM 8.8 (Prime geodesic theorem for 2-dimensional complexes [**64**]). *Let X be a finite quotient of the building $\mathcal{B}_3(F)$ of $PGL_3(F)$ or $\Delta(F)$ of $PGSp_4(F)$ by a discrete torsion-free cocompact subgroup as before. Given $i \in \{1,2\}$ and a type, let δ, λ and κ be as described above. Then for integers n large, we have*

$$\#\{i - dim'l\ prime\ [C]\ of\ X\ of\ given\ type : \ell(C) = n\delta\} \sim \frac{\kappa\lambda^{n\delta}}{n},$$

and

$$\#\{i - dim'l\ prime\ [C]\ of\ X\ of\ given\ type : \ell(C) < n\delta\} \sim \frac{\kappa\lambda^{n\delta}}{(\lambda^{\delta}-1)n},$$

where the values of λ, δ and κ are as follows:

building	$\mathcal{B}_3(F)$	$\mathcal{B}_3(F)$	$\Delta(F)$	$\Delta(F)$	$\Delta(F)$	$\Delta(F)$
dim i	1	2	1	1	2	2
type	1 *or* 2	1 *or* 2	1	2	*spin*	*standard*
λ	q^2	q	q^3	q^4	q^2	q^3
δ	3	3	2	2	2	1
κ	1	1	1	2	1	2

The Prime Geodesic Theorem provides an estimate of the number of primes of X of given dimension and type with given length. In the case that X is a Ramanujan complex as discussed in §3, this estimate has the best possible error term $O(\frac{\lambda^{n\delta/2}}{n})$, following [**48**, Theorem 2] reviewed in §4.

Given a finite unramified Galois cover $Y \to X$ with Galois group G, to each i-dimensional prime $[C]$ of X of a given type, where $i = 1$ or 2, we associate a conjugacy class in G, called $\mathrm{Frob}_{[C]}$, the same way as we did for graphs. Define, for each conjugacy class $\mathcal{C}$ of G, the set

$$S_{\mathcal{C}} = \{i - \text{dimensional prime } [C] \text{ of } X \text{ of given type} : \mathrm{Frob}_{[C]} = \mathcal{C}\}.$$

In other words, using the conjugacy classes of G we partition the i-dimensional primes of X of a given type, and we may ask a finer distribution question about these primes, namely whether each set $S_{\mathcal{C}}$ has a density. By studying the analytic behavior of the Artin L-functions $L(X,\rho,u)$ attached to finite-dimensional unitary irreducible representations ρ of G, Li and Matias proved in [**64**] that $L(X,\rho,u)$ is holomorphic on the closed disk $|u| \le 1/\lambda$ for all nontrivial ρ. Here λ is as in

Theorem 8.8. From this they conclude that the Chebotarev density theorem in natural density holds for the cover $Y \to X$.

THEOREM 8.9 (Chebotarev density theorem for 2-dimensional complexes [**64**]). *Let X be as in Theorem 8.8. Let $Y \to X$ be a finite unramified Galois cover with Galois group G. Then the Chebotarev density theorem holds in natural density for i-dimensional primes ($i \in \{1,2\}$) of X of a given type. Specifically, for each conjugacy class $\mathcal{C}$ of G, the set $S_{\mathcal{C}}$ defined above has natural density $\frac{|\mathcal{C}|}{|G|}$.*