CHAPTER 1

Introduction

1.0.1. Some historical references. These lecture notes concern the study of the asymptotics of large systems of particles in very strong mean field interaction and in particular the study of their fluctuations. Examples are given by the distributions of eigenvalues of Gaussian random matrices, β-ensembles, random tilings and discrete β-ensembles, or several random matrices. These models display a much stronger interaction between the particles than the underlying randomness so that classical tools from probability theory fail. Fortunately, these models have in common that their correlators (basically moments of a large class of test functions) obey an infinite system of equations that we will call the Dyson-Schwinger equations. They are also called loop equations, Master equations or Ward identities. Dyson-Schwinger equations are usually derived from some invariance or some symmetry of the model, for instance by some integration by parts formula. We shall argue in these notes that even though these equations are not closed, they are often asymptotically closed (in the limit where the dimension goes to infinity) so that we can asymptotically solve them and deduce asymptotic expansions for the correlators. This in turn allows to retrieve the global fluctuations of the system, and eventually even more local information such as rigidity.

This strategy has been developed at the formal level in physics [**2**] for a long time. In particular in the work of Eynard and collaborators [**17**, **49**–**51**], it was shown that if one assumes that correlators expand formally in the dimension N, then the coefficients of these expansions obey the so-called topological recursion. For instance, in [**29**, **30**], it was shown that assuming a formal expansion holds, Dyson-Schwinger equations induce recurrence relations on the terms in the expansion which can be solved by algebraic geometry means. These recurrence relations can even be interpreted as topological recursion, so that the coefficients of these expansions can be given combinatorial interpretations. In fact, it was realized in the seminal works of t'Hooft [**88**] and Brézin-Parisi-Itzykson-Zuber [**43**] that moments of Gaussian matrices and matrix models can be interpreted as the generating functions for maps. One way to retrieve this result is by using Dyson-Schwinger equations and checking that asymptotically they are similar to the topological recursion formulas obeyed by the enumeration of maps, as found by Tutte [**92**]. In this case, one first needs to analyze the limiting behavior of the system, given by the so-called equilibrium measure or spectral curve, and then the Dyson-Schwinger equations, that is the topological recursion, will provide the large dimension expansion of the observables.

The study of the asymptotics of our large system of particles also starts with the analysis of its limiting behaviour. I usually derive this limiting behaviour as the minimizer of an energy functional appearing as a large deviation rate functional [**7**], or in concentration of measure estimates [**73**], but, according to fields, people can prefer to see it as the optimizer of Fekete points [**81**], or as the solution of a Riemann-Hilbert problem [**37**]. This study often amounts to the analysis of some equation. The same type of analysis appears in combinatorics when one counts for example triangulations of the sphere. Indeed, it can be seen, thanks to Tutte surgery [**92**], that the generating function for this enumeration satisfies some equation. Sometimes, one can solve explicitly this equation, for instance thanks to the quadratic method and catalytic variables [**24, 27**] or [**56**, Section 2.9]. In our models, we will also be able to derive equations for our equilibrium measure thanks to Dyson-Schwinger equations. But sometimes, these equations may have several solutions, for instance in the setting of a double well potential in β-models. The absence of uniqueness of solutions to these equations prevents the analysis of many interesting models, such as several matrix models at low temperature. In good cases such as the β-models, we may still get uniqueness for instance if we add the information that the equilibrium measure minimizes a strictly convex energy. Dyson-Schwinger equation can then be regarded as the equations satisfied by the critical points of this energy.

The Dyson-Schwinger equations will be our key to get precise informations on the convergence to equilibrium, such as large dimension expansion of the free energy or fluctuations. These types of questions were attacked also in the Riemann Hilbert problems community based on a fine study of the asymptotics of orthogonal polynomials [**11, 26, 35, 36, 44, 54**]. It seems to me however that such an approach is more rigid as it requires more technical steps and assumptions and can not apply in such a great generality than loop equations. Yet, when it can be used, it provides eventually more detailed information. Moreover, in certain cases, such as the case of potentials with Fisher Hartwig singularities, Riemann Hilbert techniques could be used but not yet loop equations [**38, 65**].

To study the asymptotic properties of our models we need to get one step further than the formal approach developped in the physics litterature. In other words, we need to show that indeed correlators can expand in the dimension up to some error which is quantified in the large N limit and shown to go to zero. To do so, one needs in general a priori concentration bounds in order to expand the equations around their limits. For β-models, such a priori concentration of measure estimates can be derived thanks to a result of Boutet de Monvel, Pastur and Shcherbina [**25**] or Maida and Maurel-Segala [**73**]. It is roughly based on the fact that minus the logarithm of the density of such models is very close to a distance of the empirical measure to its equilibrium measure, hence implying a priori estimates on this distance. In more general situations where densities are unknown, for instance when one considers the distributions of the traces of polynomials in several matrices, one can rely on abstract concentration of measures estimates for instance in the case where the density is strictly log-concave or the underlying space has a positive Ricci curvature (e.g $SU(N)$) [**60**]. Dyson-Schwinger equations are then crucial to obtain optimal concentration bounds and asymptotics.

This strategy was introduced by Johansson [**66**] to derive central limit theorems for β-ensembles with convex potentials. It was further developped by Shcherbina and collaborators [**1, 84**] and myself, together with Borot [**55**], to study global fluctuations for β-ensembles when the potential is off-critical in the sense that the equilibrium has a connected support and its density vanishes like a square root at its boundary. These assumptions allow to linearize the Dyson-Schwinger equations around their limit and solve these linearizations by inverting the so-called Master operator. The case where the support of the density has finitely many connected component but the potential is off-critical was adressed in [**13**, **82**]. It displays the additional tunneling effect where eigenvalues may jump from one connected support to the other, inducing discrete fluctuations. However, it can also be solved asymptotically after a detailed analysis of the case where the number of particles in each connected components is fixed, in which case Dyson-Schwinger equations can be asymptotically solved. These articles assumed that the potentials are real analytic in order to use Dyson-Schwinger equations for the Stieltjes functions. We will see that these techniques generalize to sufficiently smooth potentials by using more general Dyson-Schwinger equations. Global fluctuations, together with estimates of the Wasserstein distance, were obtained in [**70**] for off-critical, one-cut smooth potentials. One can obtain by such considerations much more precise estimates such as the expansion of the partition function up to any order for general off-critical cases with fixed filling fractions, see [**13**]. Such expansion can also be derived by using Riemann-Hilbert techniques, see [**44**] in a perturbative setting and [**31**] in two cut cases and polynomial potential.

But β-models on the real line serve also as toy models for many other models. Borot, Kozlowski and myself considered more general potentials depending on the empirical measure in [**18**]. We studied also the case of more complicated interactions (in particular sinh interactions) in [**19**] : the main problems are then due to the non-linearity of the interaction which induces multi-scale phenomenon. The case of critical potentials was tackled recently in [**40**]. Also Dyson-Schwinger (often called Ward identities) equations are instrumental to study Coulomb gas systems in higher dimension. One however has to deal with the fact that Ward identities are not nice functions of the empirical measure anymore, so that an additional term, the anisotropic term, has to be controlled. This could very nicely be done by Leblé and Serfaty [**71**] by using local large deviations estimates. Recently we also generalized this approach to study discrete β-ensembles and random tilings [**14**] by analyzing the so-called Nekrasov's equations in the spirit of Dyson-Schwinger equations.

The same approach can be developed to study multi-matrix questions. Originally, I developed this approach to study fluctuations and large dimension expansion of the free energy with E. Maurel Segala [**57**, **58**] in the context of several random matrices. In this case we restrict ourselves to perturbations of the quadratic potential to insure convergence and stability of our equations. We could extend this study to the case of unitary or orthogonal matrices following the Haar measure (or perturbation of this case) in [**32**, **59**]. This strategy was then applied in a closely related setting by Chatterjee [**28**], see also [**34**].

Dyson-Schwinger equations are also central to derive more local results such as rigidity and universality, showing that the eigenvalues are very close to their deterministic locus and that their local fluctuations does not depend much on the model. For instance, in the case of Wigner matrices with non Gaussian entries, a key tool to prove rigidity is to show that the Stieltjes transform approximately satisfies the same quadratic equation than in the Gaussian case up to the optimal scale [**4**, **45**, **46**]. Recently, it was also realized that closely connected ideas could lead to universality of local fluctuations, on one hand by using the local relaxation flow [**22**, **46**, **47**], by using Lindenberg strategy [**86**, **87**] or by constructing approximate transport maps [**5**, **53**, **84**]. Such ideas could be generalized in the discrete Beta ensembles [**64**] where universality could be derived thanks to optimal rigidity (based on the study of Nekrasov's equations) and comparisons to the continuous setting.

1.0.2. A toy model. Let us give some heuristics for the type of analysis we will do in these lectures thanks to a toy model. We will consider the distribution of N real-valued variables $\lambda_1, \dots, \lambda_N$ and denote by

$$\hat{\mu}^N = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}$$

their empirical measure : for a test function f, $\hat{\mu}^N(f) = \frac{1}{N}\sum f(\lambda_i)$. Then, the correlators are moments of the type

$$M(f_1, \dots, f_p) = \mathbb{E}[\prod_{i=1}^p \hat{\mu}^N(f_i)]$$

where f_i are test functions, which can be chosen to be polynomials, Stieltjes functionals or some more general set of smooth test functions. Dyson-Schwinger equations are usually retrieved from some underlying invariance or symmetries of the model. Let us consider the continuous case where the law of the λ_i's is absolutely continuous with respect to $\prod d\lambda_i$ and given by

$$dP_N^V(\lambda_1, \dots, \lambda_N) = \frac{1}{Z_N^V}\exp\{-\sum_{i_1=1}^N\sum_{i_2=1}^N V(\lambda_{i_1}, \lambda_{i_2})\}\prod d\lambda_i$$

where V is some symmetric smooth function. Then a way to get equations for the correlators is simply by integration by parts (which is a consequence of the invariance of Lebesgue measure under translation) : Let $f_0, f_1, \dots, f_\ell$ be continuously differentiable functions. Then

$$\begin{aligned}
\mathbb{E}[\hat{\mu}^N(f_0')\prod_{i=1}^\ell \hat{\mu}^N(f_i)] &= \mathbb{E}[\left(\frac{1}{N}\sum_k \partial_{\lambda_k} f_0(\lambda_k))\right)\prod_{i=1}^\ell \hat{\mu}^N(f_i)] \\
&= -\frac{1}{N}\mathbb{E}\left[(\frac{dP_N^V}{d\lambda})^{-1}\sum_k f_0(\lambda_k)\partial_{\lambda_k}\left(\prod_{i=1}^\ell \hat{\mu}^N(f_i)(\frac{dP_N^V}{d\lambda})\right)\right] \\
&= 2N\mathbb{E}[\left(\int f_0(x_1)\partial_{x_1}V(x_1,x_2)d\hat{\mu}^N(x_1)d\hat{\mu}^N(x_2)\right)\prod_{i=1}^\ell \hat{\mu}^N(f_i)] \\
&\quad -\frac{1}{N}\sum_{j=1}^\ell \mathbb{E}[\left(\hat{\mu}^N(f_0 f_j')\right)\prod_{i\neq j}\hat{\mu}^N(f_i)]
\end{aligned}$$

where we noticed that since V is symmetric $\partial_x V(x,x) = 2\partial_x V(x,y)|_{y=x}$. The case $\ell = 0$ refers to the case $f_1 = \dots = f_\ell = 1$. We will call the above equations Dyson -Schwinger equations. One would like to analyze the asymptotics of the correlators. The idea is that if we can prove that $\hat{\mu}^N$ converges, then we can linearize the above equations around this limit, and hopefully solve them asymptotically by showing that only few terms are relevant on some scale, solving these simplified equations and then considering the equations at the next order of correction. Typically in the case above, we see that if $\hat{\mu}^N$ converges towards μ^* almost surely (or in L^p) then by the previous equation (with $\ell = 0$) we must have

$$\int f_0(x_1)\partial_{x_1}V(x_1,x_2)d\mu^*(x_1)d\mu^*(x_2) = 0\,. \tag{1.1}$$

We can then linearize the equations around μ^* and we find that if we set $\Delta_N = \hat{\mu}^N - \mu^*$, we can rewrite the above equation with $\ell = 0$ as

$$\mathbb{E}[\Delta_N(\Xi f_0)] = \frac{1}{N}\mathbb{E}[\hat{\mu}^N(f_0')] - 2\mathbb{E}[\int f_0(x_1)\partial_{x_1}V(x_1,x_2)d\Delta_N(x_1)d\Delta_N(x_2)] \tag{1.2}$$

where Ξ is the Master operator given by

$$\Xi f_0(x) = 2f_0(x)\int \partial_{x_1}V(x,x_1)d\mu^*(x_1) + 2\int f_0(x_1)\partial_{x_1}V(x_1,x)d\mu^*(x_1)\,.$$

Let us show heuristically how such an equation can be solved asymptotically. Let us assume that we have some a priori estimates that tell us that Δ_N is of order δ_N almost surely (or in all L^k's)[that is that for sufficiently smooth functions g, $\Delta_N(g) = (\hat{\mu}^N - \mu^*)(g)$ is with high probability (i.e with probability greater than $1 - N^{-D}$ for all D and N large enough) at most of order $\delta_N C_g$ for some finite constant C_g]. Then, the right hand side of (1.2) should be smaller than $\max\{\delta_N^2, N^{-1}\}$ for sufficiently smooth test functions. Hence, if we can invert the master operator Ξ, we see that the expectation of Δ_N is of order at most $\max\{\delta_N^2, N^{-1}\}$. We would like to bootstrap this estimate to show that δ_N is at most of order N^{-1}. This requires to estimate higher moments of Δ_N. Let us do a similar derivation from the Dyson-Schwinger equations when $\ell = 1$ to find that if $\bar{\Delta}_N(f) = \Delta_N(f) - \mathbb{E}[\Delta_N(f)]$,

$$\begin{aligned}\mathbb{E}[\Delta_N(\Xi f_0)\bar{\Delta}_N(f_1)] &= -2\mathbb{E}[\int f_0(x_1)\partial_{x_1}V(x_1,x_2)d\Delta_N(x_1)d\Delta_N(x_2)\bar{\Delta}_N(f_{0)}] \\ &\quad + \frac{1}{N}\mathbb{E}[\Delta_N(f_0')\bar{\Delta}_N(f_1)] + \frac{1}{N^2}\mathbb{E}[\hat{\mu}^N(f_0 f_1')]\,.\end{aligned} \tag{1.3}$$

Again if Ξ is invertible, this allows to bound the covariance $\mathbb{E}[\Delta_N(f_0)\bar{\Delta}_N(f_1)]$ by $\max\{\delta_N^3, \delta_N^2/N, N^{-2}\}$, which is a priori better than δ_N^2 unless δ_N is of order $1/N$. Since $\Delta_N(f) - \bar{\Delta}_N(f)$ is at most of order δ_N^2 by (1.2), we deduce that also $\mathbb{E}[\Delta_N(f_0)\Delta_N(f_1)]$ is at most of order δ_N^3. We can plug back this estimate into the previous bound and show recursively (by considering higher moments) that δ_N can be taken to be of order $1/N$ up to small corrections. We then deduce that

$$C(f_0,f_1) = \lim_{N\to\infty} N^2\mathbb{E}[(\Delta_N - \mathbb{E}[\Delta_N])(f_0)(\Delta_N - \mathbb{E}[\Delta_N])(f_1)] = \mu^*(\Xi^{-1}f_0 f_1')$$

and

$$m(f_0) = \lim_{N\to\infty} N\mathbb{E}[\Delta_N(f_0)] = \mu^*((\Xi^{-1}f_0)')\,.$$

We can consider higher order equations (with $\ell \geq 1$) to deduce higher orders of corrections, and the convergence of higher moments.

1.0.3. Rough plan of the lecture notes. We will apply these ideas in several cases where V has a logarithmic singularity in which case the self-interaction term in the potential has to be treated with more care. More precisely we will examine the following models.

(1) *The law of the GUE.* We consider the case where the λ_i are the eigenvalues of the GUE and we take polynomial test functions. In this case the operator Ξ is triangular and easy to invert. Convergence towards μ^* and a priori estimates on Δ_N can also be proven from the Dyson-Schwinger equations.
(2) *The Beta ensembles.* We take smooth test functions. Convergence of $\hat{\mu}^N$ is proven by large deviation principle and quantitative estimates on δ_N are obtained by concentration of measure. The operator Ξ is invertible if μ^* has a single cut, with a smooth density which vanishes like a square root at the boundary of the support. We then obtain full expansion of the correlators. In the case where the equilibrium measure has p connected components in its support, we can still follow the previous strategy if we fix the number of eigenvalues in a small neighborhood of each connected pieces (the so-called filing fractions). Summing over all possible choices of filing fractions allows to estimate the partition functions as well as prove a form of central limit theorem depending on the fluctuations of the filling fractions.
(3) *Discrete Beta ensembles.* These distributions include the law of random tilings and the λ_i's are now discrete. Integration by parts does not give nice equations a priori but Nekrasov found a way to write new equations by showing that some observables are analytic. These equations can in turn be analyzed in a spirit very similar to Dyson-Schwinger equations.
(4) *Several matrix models.* In this case, large deviations results are not yet known despite candidates for the rate function were proposed by Voiculescu [**96**] and a large deviation upper bound was derived [**10**]. However, we can still write the Dyson-Schwinger equations and prove that limits exist provided we are in a perturbative setting (with respect to independent GUE matrices). Again in perturbative settings we can derive the expansion of the correlators by showing that the Master operator is invertible.

We will discuss also one idea related with our approach based on Dyson-Schwinger to study more local questions, in particular universality of local fluctuations. It is based on the construction of approximate transport maps as introduced in [**5**]. The point is that the construction of these transport maps goes through solving a Poisson equation $Lf = g$ where L is the generator of the Langevin dynamics associated with our invariant measure. It is symmetric with respect to this invariant measure and therefore closely related with integration by parts. In fact, solving this Poisson equation is at the large N limit closely related with inverting the master operator Ξ above, and in general follows the strategy we developed to analyze Dyson-Schwinger equations. Another strategy to show universality of local fluctuations is by analyzing the Dyson-Schwinger equations but for less smooth test functions, that is prove local laws [**46**]. We will not developp this approach here. These ideas were developed in [**64**] for discrete beta-ensembles, based on a strategy initiated in [**23**]. The argument is to show that optimal bounds on Stieltjes functionals can be derived from Dyson-Schwinger equation away from the support of the equilibrium measure, but at some distance. It is easy to get it at distance of order $1/\sqrt{N}$, by straightforward concentration inequalities. To get estimates up to distance of order $1/N$, the idea is to localize the measure. Rigidity follows from this approach, as well as universality eventually.