Preface

Polynomials have been around for a long time. They play a prominent role in many applications and are basic objects in commutative algebra and algebraic geometry. In the 20th century, the foundations of algebraic geometry required a level of abstraction that made these tools hard to use for many people interested in applications. But with the advent of powerful algorithms, personal computers, and the theory of sparse polynomials, we now have *computational commutative algebra* and *computational algebraic geometry*. Coupled with accessible introductory texts, recent years have witnessed a remarkable range of applications.

The goal of this volume is to introduce some of these applications. The five chapters are based on the NSF-CBMS Regional Research Conference Applications of Polynomial Systems held at Texas Christian University (TCU), June 4–8, 2018. The format of the conference, replicated in this book, is that each day David Cox gave two lectures on a topic, followed by a third lecture by an expert on the topic.

The book is not a complete introduction to the topics covered. Many proofs are sketched or omitted, and the emphasis is on the examples. The hope is that the brief presentation provided here inspires you to learn more.

The first two chapters set the background for the rest of the book. Chapter 1 focuses on the rich history of elimination theory from the work of Newton and Bézout in the 17th and 18th centuries through the decline of elimination theory in the middle of the 20th century, followed by the emergence of symbolic computation in the latter part of the century. In the final section of the chapter, Carlos D'Andrea brings elimination theory into the 21st century.

Chapter 2 begins with the tension between the perfect information needed for symbolic computation and the messiness of the information coming from the real world. The chapter explores linear algebra and homotopy continuation, two commonly used methods for solving systems of polynomial equations numerically. This is where we meet our first substantial applications. Jonathan Hauenstein concludes the chapter with a discussion of sampling in numerical algebraic geometry.

The remaining chapters of the book focus on three substantial applications. Chapter 3 explores the geometry and algebra of geometric modeling, with some unexpected connections to toric varieties, algebraic statistics, and Rees algebras. At the end of the chapter, Hal Schenck surveys a variety of algebraic tools which are used in geometric modeling.

Chapter 4 is devoted to the geometry and combinatorics of rigidity theory. For bar-and-joint frameworks, basic objects include graphs and rigidity matrices, though polytopes and matroids also have an important role to play. A fun result due to Gross and Sullivant uses the rigidity matroid to study maximum likelihood estimates of Gaussian graphical models coming from planar graphs. Jessica Sidman ends the chapter with a discussion of body-and-bar and body-and-cad frameworks and the study of non-generic frameworks via the Grassmann-Cayley algebra.

Chapter 5 shifts the scene to chemical reaction networks, where the Law of Mass Action leads to a lovely combination of graph theory and dynamical systems. This framework includes examples from chemistry, biology, epidemiology, and population genetics. Algebraic geometry enters the picture because the steady states are defined by a polynomial system. Toric varieties also have a role to play by the pioneering work of Karin Gatermann. Alicia Dickenstein finishes the chapter with an exposition of the interesting things that can happen in biochemical reaction networks.

Each chapter starts at a relatively elementary level, with more advanced topics introduced as needed. The algebraic geometry background can be found in the book [**104**]. In addition to the main applications presented in Chapters 3, 4, and 5, the twin themes of toric varieties and algebraic statistics play a prominent role in the book. Readers unfamiliar with this material should consult the expository papers [**98**] and [**214**].

We hope you enjoy the book! We had fun writing it.

Acknowledgments

The authors first want to thank the Conference Board of the Mathematical Sciences and the National Science Foundation for creating and supporting the NSF-CBMS Regional Research Conference Series. This book is a direct result of the NSF-CBMS conference held in June 2018, funded by NSF grant DMS-1741730 with additional support provided by the CBMS, the TCU Department of Mathematics, and the TCU College of Science and Engineering. Thank you all!

We are also tremendously grateful to the local organizers Greg Friedman, José Carrión, Eric Hanson, Scott Nollett, and Efton Park for their work in making the conference possible. They helped create a wonderful atmosphere for an intense week of mathematics. Thank you!

Final thanks go to the participants. The conference and the book benefited tremendously from your enthusiasm and excellent questions.

David Cox
I first want to thank the local organizers for inviting me to be principal lecturer for the CBMS conference they hoped to host. I have equal thanks for my wonderful coauthors, Carlos D'Andrea, Jonathan Hauenstein, Hal Schenck, Jessica Sidman, and Alicia Dickenstein, whose contributions enriched the conference and the book immensely.

I am also very grateful to Doug Arnold, Dan Bates, Alicia Dickenstein, Ron Goldman, Jonathan Hauenstein, Martin Kreuzer, Kaie Kubjas, Kuei-Nuan Lin, Nathan Pflueger, Henk Poulisse, Jan Peter Schäfermeyer, Gabriel Sosa, Tom Sederberg, Jessica Sidman, Frank Sottile, Simon Telen, Caroline Uhler, and Charles Wampler for their suggestions, comments, and assistance.

Finally, I want to acknowledge my indebtedness to Bernd Sturmfels for writing *Solving Systems of Polynomial Equations* [**334**]. This wonderful book, the result of a CBMS conference in 2002, is a model of vivid, concise writing that conveys the amazing ability of algebraic geometry to shed light on systems of polynomial equations.

Carlos D'Andrea
I am very grateful to Laurent Busé, Alicia Dickenstein, Emiliano Gomez, Martín Sombra, Teresa Krick, and Josephine Yu for their careful reading of an earlier version of this manuscript and several suggestions for improving the presentation.

Jonathan Hauenstein
I am very appreciative of discussions with Danielle Brake, Maggie Regan, and Sam Sherman to help formulate the presentation. I am also grateful for many conversations regarding sampling with Dan Bates, Heather Harrington, Chris Peterson, Andrew Sommese, Frank Sottile, Bernd Sturmfels, and Charles Wampler.

Hal Schenck
I thank David Cox—while I was a postdoc, David asked me a question on vanishing of syzygies, which led to a joint paper followed by a 20-year collaboration and lasting friendship. I'm also grateful to my coauthors Alicia Dickenstein, Alexandra Seceleanu, Javid Validashti, Eliana Duarte, and Nicolas Botbol.

Jessica Sidman
I am deeply indebted to Audrey St. John for introducing me to rigidity theory and am grateful to collaborators Zvi Rosen, Louis Theran, and Cynthia Vinzant, from whom I have learned so much. I would also like to thank all of the undergraduate contributors to projects mentioned here, including James Farre, Helena Kleinschmidt, Stephanie Stark, and Xilin Yu. Thanks also go to Bernd Sturmfels for suggesting computations in distance coordinates.

Alicia Dickenstein
I am very grateful to David Cox for including me in this project. I am also indebted to Bernd Sturmfels for introducing me to the subject of Chapter 5 while I was walking along a corridor at MSRI during the Commutative Algebra Program in the spring of 2003. I deeply thank Mercedes Pérez Millán, Nicolás Botbol, Magalí Giaroli, Frédéric Bihan, Anne Shiu, Elisenda Feliu, and David Cox for all I learned from them while working together.

David A. Cox, Amherst
Carlos D'Andrea, Barcelona
Jonathan Hauenstein, South Bend
Hal Schenck, Auburn
Jessica Sidman, South Hadley
Alicia Dickenstein, Buenos Aires