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PREFACE

A substantial portion of this volume is based on my lectures at the NSF-CBMS Regional Research Conference held at the Texas Christian University, May 19-24, 1996. When I was asked to give lectures at the conference and to write up eventually the contents of the lectures in monograph form, my idea was rather different from, if not unrelated to, what I am presenting now. At that time I thought I would include the results I had published in a series of papers, which concerned Euler products and Eisenstein series on symplectic and metaplectic groups, and I would also discuss the arithmeticity problems of the special values of the Euler products.

After thinking about this project for a few weeks, I found the idea unexciting. Though the question of arithmeticity had never been fully explored in those cases and I still intend to treat it on a future occasion, the whole program lacked the allure of making me brave the burden of writing a book of fair length. Therefore I decided to take up something new and more challenging which had been occupying my mind for some time, and on which I had only incomplete results but felt that I had enough technical ideas to complete them. However, in addition to the obvious question of whether those ideas were enough, there was another problem, namely, whether the proposed book could be accessible to many readers. After a few more months of experimenting, I convinced myself that I would be able to accomplish my aims satisfactorily, and began the work of which the outcome is the present volume.

What are then the main features of the book? Leaving the details to the Introduction, let us merely say that there are three chief objectives: (i) the determination of local Euler factors on classical groups, in an explicit rational form; (ii) Euler products and Eisenstein series on a unitary group of an arbitrary signature; (iii) a class number formula for a totally definite hermitian form.

Though these form the principal new results obtained in the book, we start with quite a general setting, and include many topics of expository nature so that the book can be viewed as an introduction to the theory of automorphic forms of several variables. We eventually specialize our exposition to unitary groups, but we treat them as a model case so that the reader can easily formulate the corresponding facts in other cases. For that purpose we find unitary groups better than symplectic groups as will be explained in the Introduction.

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INTRODUCTION

The first main theme of this book is to associate an Euler product to an automorphic form that is a Hecke eigenform on a classical group in a suitable sense. As the name indicates, Hecke treated the case of holomorphic modular forms with respect to a congruence subgroup Γ of $SL_2(\mathbf{Z})$. Each Hecke operator is given by a double coset $\Gamma\alpha\Gamma$ with α belonging to a semigroup Ξ of matrices, containing Γ , whose entries are integers and determinants are positive. Taking an eigenform f in the sense that $f|\Gamma\alpha\Gamma = \lambda(\alpha)f$ with a complex number $\lambda(\alpha)$ for every $\alpha \in \Xi$, one considers a Dirichlet series of the form

$$(1) \quad \mathfrak{Z}(s) = \sum_{\alpha \in \Gamma \backslash \Xi / \Gamma} \lambda(\alpha) \det(\alpha)^{-s}.$$

With a suitable choice of Ξ one can show that \mathfrak{Z} has an Euler product whose Euler factors have degree 2; moreover $\Gamma(s)\mathfrak{Z}(s)$ can be continued to a meromorphic function on the whole s -plane with at most two simple poles; $\Gamma(s)\mathfrak{Z}(s)$ is entire if f is a cusp form.

Various kinds of generalization of this theory of Hecke have been attempted and carried out, but any attempt must be preceded by a choice of formulation that is practicable. Though one might be able to present a framework including every imaginable Euler product, it is of little use if one cannot indeed prove the desired results. Therefore in this book we choose one type of Euler product which is somewhat different from Hecke's type. To be explicit, we take an algebraic number field K with an automorphism ρ of order 1 or 2; we then put $F = \{x \in K \mid x^\rho = x\}$ and

$$(2) \quad G(\varphi) = G^\varphi = \{ \alpha \in GL_n(K) \mid \alpha\varphi \cdot {}^t\alpha^\rho = \varphi \}$$

with a fixed $\varphi \in GL_n(K)$ such that ${}^t\varphi^\rho = \varepsilon\varphi$, $\varepsilon = \pm 1$. Thus our group is either symplectic, orthogonal, or unitary. Suppose that we can speak of an automorphic form with respect to a congruence subgroup Γ of G^φ on a space on which G^φ acts. Taking a certain subgroup Ξ of G^φ which contains Γ and is dense in almost all nonarchimedean localizations of G^φ and an eigenform f of all $\Gamma\alpha\Gamma$ with $\alpha \in \Xi$, we consider a series

$$(3) \quad \mathfrak{Z}(s) = \sum_{\alpha \in \Gamma \backslash \Xi / \Gamma} \lambda(\alpha)\nu(\alpha)^{-s}$$

with a suitable function ν on G^φ and $\lambda(\alpha)$ defined as above in the present case. Strictly speaking we have to formulate everything on the adelicization $G_{\mathbf{A}}^\varphi$ of G^φ , and the series of (3) is the right one only if "the class number is one" in a certain

sense. But to avoid excessive details, we do not give here the precise definition in the general case. Though this may not be the best way to produce a zeta function in other cases, it has the advantage that we can connect the Hecke eigenvalues naturally and directly to the desired Euler product for a large class of classical groups. Also, in certain cases this definition allows us to express our Euler product in terms of the Fourier coefficients of f , though we do not touch on that aspect in this book.

For the purpose of illustration, let $n = 2r$, $G^\varphi = Sp(r, \mathbf{Q})$, and $\Gamma = Sp(r, \mathbf{Z})$; then we take $\Xi = G^\varphi$ and $\nu(\alpha)$ to be the product of the denominators of the elementary divisors of α . If $r = 1$, then the group is $SL_2(\mathbf{Q})$ and the series has the form

$$(4) \quad \mathfrak{Z}(s) = \sum_{m=1}^{\infty} \lambda(\text{diag}[m^{-1}, m]) m^{-s},$$

which produces an Euler product of degree 3, that is usually called the symmetric square zeta function associated to a series of type (1), but it is also natural to call it an Euler product on $SL_2(\mathbf{Q})$, with no reference to $GL_2(\mathbf{Q})$.

Now our first main task is to show that this type of series with ν defined in a similar manner has an expression

$$(5) \quad \Lambda(s)\mathfrak{Z}(s) = \prod_{\mathfrak{p}} W_{\mathfrak{p}}(N(\mathfrak{p})^{-s})^{-1},$$

where Λ is a product of L -functions of F , \mathfrak{p} runs over all the prime ideals in F , and $W_{\mathfrak{p}}$ is a polynomial with constant term 1 whose degree is $n[K : F]$ for almost all \mathfrak{p} , except when G^φ is a symplectic group or an orthogonal group of odd degree, in which case the degree of $W_{\mathfrak{p}}$ is $n + 1$ or $n - 1$ accordingly. This fact is purely algebraic, or rather local, and can be formulated without the notion of automorphic forms. Therefore we obtain the result for an arbitrary G^φ of type (2) (Theorem 16.16).

Now, assuming that $K \neq F$, take an arbitrary Hecke character χ of K and denote by χ_1 its restriction to F ; denote also by $\mathfrak{Z}(s, \chi)$ and $\mathcal{Z}(s, \chi)$ the twists of \mathfrak{Z} and the right-hand side of (5) by χ by viewing them as Euler products over K ; further denote by $\Lambda(s, \chi_1)$ the twist of Λ by χ_1 . Then

$$(6) \quad \Lambda(s, \chi_1)\mathfrak{Z}(s, \chi) = \mathcal{Z}(s, \chi).$$

Our next problem is to prove that if f is a holomorphic cusp form in the unitary case, then $\mathcal{Z}(s, \chi)$ times suitable gamma factors can be continued to a meromorphic function on the whole plane with finitely many possible simple poles. In principle our methods are applicable to nonholomorphic forms, but the incomplete state of knowledge of such forms makes it difficult to find explicit forms of the gamma factors. In the holomorphic case, however, it is relatively easy to calculate the gamma factors, which is the main reason why we restrict our exposition in later sections to holomorphic forms.

We prove the desired result on \mathcal{Z} for G^φ when K is a totally imaginary quadratic extension of totally real F , in which case G^φ is a unitary group acting on a hermitian symmetric space \mathfrak{Z}^φ . There are several reasons why we consider unitary groups

instead of symplectic or orthogonal groups: (i) Unitary groups can be split or nonsplit, and therefore they have the characteristics of general reductive algebraic groups, while symplectic groups are split and special in that sense. (ii) Besides, the symplectic case has been treated in detail in a series of recent papers by the author, and so it seems desirable to present the “nonsplit aspect” of the theory. (iii) Holomorphic forms can be considered also on orthogonal groups of a restricted type. A uniform treatment of all these classical groups is not impossible, but at some point the task will become cumbersome. For example, it requires a careful analysis of certain Eisenstein series in the orthogonal case, which would have made the book much longer. For this reason, we discuss only the arithmetic aspect of the orthogonal case, but not its analytic aspect.

Now the function \mathcal{Z} is closely connected with an Eisenstein series of the following type. Given G^φ as above with $\varphi = {}^t\varphi^\rho$, we consider G^{ψ} and G^η with

$$\psi = \begin{bmatrix} \varphi & 0 \\ 0 & \eta \end{bmatrix} \quad \text{and} \quad \eta = \eta_m = \begin{bmatrix} 0 & 1_m \\ 1_m & 0 \end{bmatrix},$$

where m is a positive integer. Let P be the parabolic subgroup of G^ψ consisting of all the matrices whose lower left $m \times (m+n)$ -block is 0. Given a congruence subgroup Δ of G^ψ , a holomorphic cusp form f with respect to a congruence subgroup of G^φ defined as a function on \mathfrak{Z}^φ , and a Hecke character χ of K , we can define an infinite series $E(z, s; f, \chi)$ for $(z, s) \in \mathfrak{Z}^\psi \times \mathbf{C}$ under a natural consistency condition on Δ , f , and χ . In the simplest case in which $F = \mathbf{Q}$, it can be given in the form

$$E(z, s; f, \chi) = \sum_{\alpha \in R} \text{sgn}(\lambda_0(\alpha))^k \chi^*(\lambda_0(\alpha)\mathbf{Z}) \delta(z, s; f) \|_k \alpha,$$

$$R = P \backslash G^\psi, \quad \delta(z, s; f) = f(\wp(z)) [\delta_\psi(z) / \delta_\varphi(\wp(z))]^{s-k/2}.$$

Here $\lambda_0(\alpha)$ is the determinant of the lower right $m \times m$ -block of α , \wp is a certain projection map of \mathfrak{Z}^ψ into \mathfrak{Z}^φ , δ_φ is the function on \mathfrak{Z}^φ such that $\delta_\varphi(\gamma w) = |j_\gamma(w)|^{-2} \delta_\varphi(w)$ for every $\gamma \in G^\varphi$ with the standard scalar factor of automorphy $j_\gamma(w)$, δ_ψ is similarly defined on \mathfrak{Z}^ψ , k is the weight of f , and $(g \|_k \alpha)(z) = j_\gamma(z)^{-k} g(\alpha z)$ for a function g on \mathfrak{Z}^ψ .

In order to study the analytic nature of this series, we consider another group G^ω with $\omega = \text{diag}[\psi, -\varphi]$. Now it can easily be seen that G^ω is isomorphic to $G(\eta_{m+n})$, and we can define an Eisenstein series \mathcal{E} on $G(\eta_{m+n})$ with respect to its standard parabolic subgroup. Since $G^\psi \times G^\varphi$ can be embedded in G^ω in an obvious fashion, we obtain a function $H(z, w; s)$ of $(z, w; s) \in \mathfrak{Z}^\psi \times \mathfrak{Z}^\varphi \times \mathbf{C}$ by pulling back a suitable transform \mathcal{E}' of \mathcal{E} to $\mathfrak{Z}^\psi \times \mathfrak{Z}^\varphi$. Then we prove a formula which in the simplest case can be written

$$(7) \quad \int_{\Gamma \backslash \mathfrak{Z}^\varphi} H(z, w; s) f(w) \delta_\varphi(w)^k dw = c(s) \mathfrak{I}(s, \chi) E(z, s; f, \chi),$$

where c is a product of explicitly given gamma factors.

To define $E(z, s; f, \chi)$, we assumed $m > 0$. However, we can make H and the integral meaningful even when $m = 0$ by taking $\psi = \varphi$. Then we obtain

$$(8) \quad \int_{\Gamma \backslash \mathfrak{Z}^\varphi} H(z, w; s) f(w) \delta_\varphi(w)^k dw = c'(s) \mathfrak{I}(s, \chi) f(z),$$

where c' is the function c in the present case. Now we can find a product Λ' of L -functions of F and a product \mathcal{G} of gamma factors such that $\Lambda'\mathcal{G}\mathcal{E}$ can be continued to a meromorphic function of s on the whole plane with finitely many poles which are all simple. Clearly we can say the same for $\Lambda'\mathcal{G}H$. In the setting of (8), this Λ' coincides with $\Lambda(s, \chi_1)$, and hence from (8) we obtain the desired meromorphic continuation of $c'(s)\mathcal{G}(s)\mathcal{Z}(s, \chi)$ (Theorem 20.5). In a similar way, multiplying by a factor of the type $\Lambda'\mathcal{G}$, we can show that $\mathcal{Z}(s, \chi)E(z, s; f, \chi)$ times suitable gamma factors can be continued to a meromorphic function on the whole plane with finitely many poles which are all simple (Theorem 20.7).

Strictly speaking, (8) is valid only for the character χ whose archimedean factor is consistent with the weight of f in a certain sense. To obtain $\mathfrak{T}(s, \chi)$ for an arbitrary χ , we have to replace \mathcal{E}' by $A\mathcal{E}'$ with a differential operator A which is not so simple.

It should be noted that Garrett gave in [Ga] a formula for the pullback of the standard Eisenstein series on $Sp(r, \mathbf{Z})$, from which one could derive an equality of type (7) in that case. However, he did not carry out the calculation, which was later done by Böcherer in [Bö]. It may be noted also that equalities of type (8) were employed in a few earlier papers of the author when the group in question is obtained from a quaternion algebra.

The final main theorem of the book concerns a generalization of the class number of a hermitian form, which we call *the mass of G^φ* relative to a specified open subgroup of $G_{\mathbf{A}}^\varphi$. To explain the concept, denote by V the vector space of all n -dimensional row vectors with components in K on which G^φ acts by right multiplication, and by \mathfrak{r} the maximal order of K . Then we can find a finitely generated \mathfrak{r} -submodule M of V with the property that $x\varphi \cdot {}^t x^\rho \in \mathfrak{r}$ for every $x \in M$ and M is maximal among such submodules of V . To make a transparent formulation of the problem, we now have to consider the adelization $G_{\mathbf{A}}^\varphi$ of G^φ , which we have avoided so far. Taking an arbitrary integral ideal \mathfrak{c} in F , we define an open subgroup D of $G_{\mathbf{A}}^\varphi$ containing the archimedean factor of $G_{\mathbf{A}}^\varphi$ such that its v -factor D_v for each nonarchimedean prime v of F is defined by

$$D_v = \{ \alpha \in G_v^\varphi \mid M_v \alpha = M_v, M_v(\alpha - 1) \subset \mathfrak{c}_v M_v \}.$$

Then we can find a finite set \mathcal{B} so that $G_{\mathbf{A}}^\varphi = \bigsqcup_{a \in \mathcal{B}} G^\varphi a D$. Let T be the set of elements of G^φ that act trivially on \mathfrak{Z}^φ ; let $\Gamma^a = G^\varphi \cap a D a^{-1}$ for each $a \in \mathcal{B}$. We then put

$$(9) \quad \mathfrak{m}(\varphi, \mathfrak{c}) = \sum_{a \in \mathcal{B}} [\Gamma^a \cap T : 1]^{-1} \text{vol}(\Gamma^a \backslash \mathfrak{Z}^\varphi),$$

where we understand that $T = G^\varphi$ and $\text{vol}(\Gamma^a \backslash \mathfrak{Z}^\varphi) = 1$ if φ is totally definite, so that we have

$$(10) \quad \mathfrak{m}(\varphi, \mathfrak{c}) = \sum_{a \in \mathcal{B}} [\Gamma^a : 1]^{-1}$$

for such a φ . In this special case clearly $\mathfrak{m}(\varphi, \mathfrak{c}) = \#(\mathcal{B})$ if Γ^a is trivial for every a , which can happen under a suitable condition on \mathfrak{c} . Thus $\mathfrak{m}(\varphi, \mathfrak{c})$ is similar to the class number. The point of our formulation is that this quantity is computable

if φ is anisotropic. Namely, for such a φ we shall prove, as the last main result of this book, a formula which takes the following form if n is odd, the discriminant of φ is represented by a unit, and $\mathfrak{c} = (1)$ (Theorem 24.4):

$$(11) \quad \mathfrak{m}(\varphi, (1)) = 2^{1-t} b_\varphi \prod_{k=1}^{n-1} (k!)^d \cdot D_F^{(n^2-n)/2} \\ \cdot \prod_{k=1}^n \left\{ N(\mathfrak{d})^{k/2} D_F^{1/2} (2\pi)^{-kd} L(k, \tau^k) \right\}.$$

Here t is the number of prime ideals in F ramified in K , b_φ is an explicitly given constant depending only on the type of \mathfrak{F}^φ , $d = [F : \mathbf{Q}]$, D_F is the discriminant of F , \mathfrak{d} is the different of K relative to F , τ is the Hecke character of F corresponding to K/F , and $L(s, \tau^k)$ is the L -function of τ^k . Similar and somewhat more complicated formulas can be given for an arbitrary \mathfrak{c} and also for even n . (Strictly speaking, we prove the formula for a group D which is somewhat different from the above one.) We have $b_\varphi = 1$ if φ is totally definite.

If $\mathfrak{c} = (1)$, the quantity of (10) in the orthogonal case is what Siegel called, following Eisenstein and Minkowski, *the mass of a genus* in his celebrated theory of quadratic forms. Therefore one should be able to deduce (11) from his formula stated in the hermitian case, and vice versa, except that the deduction of one formula from the other is highly nontrivial. Also, the formulation of Siegel's formula in terms of the Tamagawa number has popularized the subject, but at the same time it has obscured the significance of other classical problems in this area well worthy of further investigations. Indeed, one important aspect of Siegel's formula is that the mass is theoretically computable as he showed by some examples, but later researchers completely neglected that aspect, apparently thinking that the computation is impracticable in general.

Without relying on the formula of Siegel's type, we derive (11) as an easy consequence of our methods. To be exact, we first prove an equality of type (8) with the constant 1 as f for each class belonging to a fixed genus, and compare the residues of its both sides. Adding the results for all such classes, we obtain (11) with no ambiguous factors, thus fulfilling one of Siegel's wishes at least in the unitary case. Though this by no means supersedes his method, it offers a new insight into the nature of the quantity \mathfrak{m} . Indeed, our proof shows that \mathfrak{m} is, up to some factors, the residue of our Euler product for $f = 1$, which may be called *the zeta function of G^φ* . We believe also that the consideration of $\mathfrak{m}(\varphi, \mathfrak{c})$ with an arbitrary \mathfrak{c} is natural and at least technically advantageous. Our methods are applicable to the orthogonal case, which the author intends to treat in a separate article.

Let us now briefly describe the contents of each section. We first develop in Section 1 a purely algebraic theory of quadratic or hermitian forms over an involutorial division algebra K which is not necessarily commutative. We consider G^φ as in (2) with such a K , and define in Section 2 its parabolic subgroup P_J^φ with respect to a totally isotropic subspace J of V . We then prove in Propositions 2.4 and 2.7 two basic facts on $P^\omega \backslash G^\omega / (G^\psi \times G^\varphi)$ for $\omega = \text{diag}[\psi, -\varphi]$ as above, which hold in a general setting and which play crucial roles in the proof of (7). Here P^ω is a parabolic subgroup of G^ω with respect to a maximal totally isotropic subspace.

In Section 3 we introduce the notion of the denominator ideal of a matrix with entries in the field of quotients of a principal ideal domain, which is essentially the

quantity $\nu(\alpha)$ of (3). Most noteworthy among several facts proved in this section are Propositions 3.9 and 3.10 concerning $\nu(\alpha)$ for α belonging to a parabolic subgroup of a general linear group and also for α of “degenerate type.” Sections 4 and 5 concern quadratic and hermitian forms over the field of quotients of a ring which is first a Dedekind domain, later a principal ideal domain, and finally a discrete valuation ring. We introduce the notion of maximal lattices, and describe them in terms of a refined form of Witt’s decomposition. We prove a product expression of the type $G^\varphi = P_J^\varphi C$ with the stabilizer C of a maximal lattice.

In Section 6 we define the space \mathfrak{Z}^φ and also basic factors of automorphy in the unitary case. Various elementary facts concerning the archimedean version of $G^\varphi = P_J^\varphi C$ and the projection map $\varphi : \mathfrak{Z}^\psi \rightarrow \mathfrak{Z}^\varphi$ are collected in this section. The symplectic case, not included in these three sections, is treated in Section 7.

The adélization $G_{\mathbf{A}}$ of an algebraic group G and some related concepts are introduced in Section 8. For our purposes it is essential to examine the coset decompositions of $G_{\mathbf{A}}$ relative to an open subgroup and a parabolic subgroup. This will be done in Section 9. We introduce the notion of automorphic forms in Section 10, prove easy facts on Hecke operators in Section 11, and define Eisenstein series in Section 12. We treat these as functions and operators on \mathfrak{Z}^φ , and also as objects on $G_{\mathbf{A}}^\varphi$. Our exposition in these sections is restricted to the unitary case, though we add some comments in the symplectic case.

Sections 13 through 15 are devoted to the investigation of a type of local Dirichlet series that appears as an Euler factor of a Fourier coefficient of an Eisenstein series on a split group. This local series plays also a crucial role in the computation of the Euler factors of our zeta functions. In Section 16 we determine the explicit rational expression for $W_{\mathfrak{p}}$ of (5). The key fact in this is Proposition 16.10, which gives $\nu(\alpha)$ for $\alpha \in P_J^\varphi$. Section 17 concerns several formulas for the group indices which are necessary for the proof of (11). We investigate in Sections 18 and 19 the Eisenstein series on G^η which we denoted by \mathcal{E} in the above. We first give an explicit form for each Fourier coefficient and determine a product Λ' of L -functions and another product \mathcal{G} of gamma factors such that $\Lambda' \mathcal{G} \mathcal{E}$ has only finitely many poles on the whole plane. We then give an explicit formula for the residue at a special pole when the weight is 0.

We state our main theorems on $\mathcal{Z}(s, \chi)$ and $E(z, s, f, \chi)$ in Section 20, and prove them in the next three sections. One of the main technical difficulties arises in the analysis of the pullback denoted by $H(z, w; s)$ in the above. Though the description of $P^\omega \backslash G^\omega / (G^\psi \times G^\varphi)$ given in Section 2 is not complicated, we have to describe it in connection with various open subgroups of the adélized groups. It should also be mentioned that in order to obtain (7) and (8), we must choose \mathcal{E}' carefully, because an arbitrary or a seemingly natural choice of \mathcal{E}' often produces a vanishing integral or ambiguous factors. It is one of the main points of our treatment to give such formulas in nonvanishing exact forms with all factors explicitly determined.

As we said earlier, to establish the meromorphic continuation of $\mathcal{Z}(s, \chi)$ in the most general case, it is necessary to apply a certain differential operator to \mathcal{E} . In Section 23 we define such an operator and prove a formula on its effect on each term of \mathcal{E} , which eventually leads to the proof of the desired fact. Finally in Section 24 we prove a formula for $\mathfrak{m}(\varphi, \mathfrak{c})$ when φ is anisotropic. If φ is not totally definite, we can derive from it another formula concerning $\text{vol}(\Gamma^a \backslash \mathfrak{Z}^\varphi)$ (Theorem 24.7).

The Appendix at the end of the book consisting of eight sections contains various facts which could have been included in the main text, but would have interrupted

the smooth flow of the principal ideas. Some of these sections are quite elementary and contain only the results which are either well known or essentially known. They are intended for the reader who is not familiar with such standard facts. However, it seems that some results in less elementary sections, A4 and A7 for example, have never been stated in the forms we present them.