PREFACE

In each of the six semesters from the Fall Semester of 1921 through the Spring Semester of 1924, I gave a four-hour course on Number Theory at the University of Göttingen. The content of the courses was so very different from the material presented in the standard textbooks that I decided to follow numerous suggestions to publish the material of my lectures. [See Publisher's Preface.]

Among the topics from the theory of integral rational numbers that I have included over and above the basic classical results are the beautiful results of modern number theory that we owe to such leading scholars as van der Corput, Hardy, Littlewood, and Siegel.

As to the theory of algebraic number fields, my one and only fixed goal was to present, after an exposition of the elements, the two most important results that have been obtained concerning the so-called Last Theorem of Fermat, as well as the proofs of those results. Thus, I first give an exposition of Kummer's Theorem to the effect that for every so-called regular prime number $p>2$, the equation $\xi^p+\eta^p+\zeta^p=0$ cannot be solved in algebraic (non-zero) integers of the field of the p-th roots of unity, and thus in particular cannot be solved in rational integers none of which is zero. The proof of this theorem of Kummer's is so difficult that even, for example, in Bachmann's book on the Fermat problem the main lemma (which contains the whole difficulty and which itself can be proved only as the last link of a long chain) is stated without proof. Thus, the only exposition of the matter in the entire literature is Hilbert's *Zahlbericht*; however, everything is presented there in the framework of more general ideas. From this source, I extracted, simplifying as far as possible, the arrangement of the proof in the present work. The second result that I prove concerning Fermat's Theorem is the Theorem of Wieferich, which is as follows: If $x^p+y^p+z^p=0$ has a solution in integers none of which is divisible by p, then $2^{p-1}\equiv 1 \pmod{p^2}$; I also prove Mirimanoff's supplementary result to the effect that $3^{p-1}\equiv 1 \pmod{p^2}$ as well.

As for the rest, my lectures are concerned primarily with the theory of the rational integers. I attach great value throughout to a presentation of the new theories concerning the—for the most part—old complexes of problems which does not stress the greatest generality obtainable but, rather, goes just so far that the most characteristic form of a result applies, but applies to problems with as few parameters as possible. Since each section of the work is prefaced by a more-or-less lengthy introduction, I will mention here only a few details.

1. I do not speak of the Waring-Kamke problem but of the Waring-Hilbert problem and I prove, following Hardy and Littlewood, that all large numbers can be written as a sum of 19 fourth powers, and almost all of them (this notion will be defined) as a sum of 15 fourth powers; similarly, for all exponents k other than 4.

2. I prove Thue's theorem, and Siegel's sharpening of it, only for ordinary diophantine equations, not for equations with algebraic coefficients and unknowns. Thus, the reader can find in this book Thue's famous theorem as a special case, to wit: Every diophantine equation $g(x,y)=a$, where $g(x, y)$ is an irreducible homogeneous polynomial of degree higher than second with integer coefficients, has only a finite number of solutions.

3. In the theory of lattice points, I include, to be sure, van der Corput's main theorem from his Thesis, but otherwise I treat in the main the very special problem of the number $A(x)$ of lattice points in the circular disk $u^2+v^2\leqq x$, because all the essential ideas of the theory show up in this problem. The reader will be astonished to see how little depth is involved in the theorem by Hardy and myself (1915) and in the theorem of van der Corput (1923) to the effect that the (as yet unknown) lower bound ϑ of all a for which $A(x)-\pi x=O(x^a)$ is on the one hand $\geqq\frac{1}{4}$, and on the other hand $<\frac{1}{3}$. True, it was a labor of some years to reduce things to this simplicity. I did not find my new proof of the theorem $\vartheta\geqq\frac{1}{4}$ until toward the end of the sixth semester. Also, the proof given later in this book for $\vartheta\leqq\frac{37}{112}<\frac{1}{3}$ is the union of three new proof arrangements by Littlewood, Walfisz, and myself.

Needless to say, in working out the lectures for publication, I have changed the order of the topics somewhat. On the other hand, I have made hardly any additions or omissions.

I will mention a few more details concerning the content. First, this work is not intended to compete with my two published volumes on Prime Numbers nor with my book on the Theory of Ideals. There are relatively few points of contact. Nor was it my intention to give a comprehensive treatment of all of number theory. A comparison with the content of the relevant encyclopedia articles or with Dickson's *History of the Theory of Numbers* shows how big is the world of number theory and with how small a part of this world the reader will become acquainted in these many pages. But I shall lead him to the classical regions and to the most beautiful of the hitherto hardly accessible regions; some preference has been shown for places the roads to which it was my privilege to help build, a preference, that is, for the analytic part of the theory of rational integers.

For the convenience of the reader, there are no footnotes and only few bibliographical references. From the above-mentioned excellently arranged historical sources (the encyclopedia and Dickson) the reader can ascertain with little trouble where the original papers are to be found.

Following the wishes of my friends, I have retained the lively and sometimes jocular style of my lectures as far as possible and have not completely replaced it with a dry textbook style.

My thanks go first of all to the authors of the beautiful works (especially those from the most recent decades) whose fruits I was able to harvest. Most especially, I thank my assistant of many years past, and present colleague, Privatdozent Dr. K. Grandjot, who knows the entire field thoroughly, gave me the most valuable help during the preparations of the lecture, and finally, checked through the manuscript. In reading proofs, I also enjoyed, in addition to Dr. Grandjot's help, the collaboration of an outstanding expert in the field of analytic and geometric number theory, my student Dr. A. Walfisz. Furthermore, Miss L. Kirchhoff (stud. math.) not only rendered valuable and understanding assistance with the proofs but also went to the great trouble of producing a fair copy of my entire manuscript. I most heartily thank these three faithful collaborators.

I also wish to express my thanks to the firm of S. Hirzel, who undertook the publication of this work and thus made possible its appearance in book form.

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EDMUND LANDAU