

PREFACE

THE MATRIX CALCULUS is widely applied nowadays in various branches of mathematics, mechanics, theoretical physics, theoretical electrical engineering, etc. However, neither in the Soviet nor the foreign literature is there a book that gives a sufficiently complete account of the problems of matrix theory and of its diverse applications. The present book is an attempt to fill this gap in the mathematical literature.

The book is based on lecture courses on the theory of matrices and its applications that the author has given several times in the course of the last seventeen years at the Universities of Moscow and Tiflis and at the Moscow Institute of Physical Technology.

The book is meant not only for mathematicians (undergraduates and research students) but also for specialists in allied fields (physics, engineering) who are interested in mathematics and its applications. Therefore the author has endeavoured to make his account of the material as accessible as possible, assuming only that the reader is acquainted with the theory of determinants and with the usual course of higher mathematics within the programme of higher technical education. Only a few isolated sections in the last chapters of the book require additional mathematical knowledge on the part of the reader. Moreover, the author has tried to keep the individual chapters as far as possible independent of each other. For example, Chapter V, *Functions of Matrices*, does not depend on the material contained in Chapters II and III. At those places of Chapter V where fundamental concepts introduced in Chapter IV are being used for the first time, the corresponding references are given. Thus, a reader who is acquainted with the rudiments of the theory of matrices can immediately begin with reading the chapters that interest him.

The book consists of two parts, containing fifteen chapters.

In Chapters I and III, information about matrices and linear operators is developed *ab initio* and the connection between operators and matrices is introduced.

Chapter II expounds the theoretical basis of Gauss's elimination method and certain associated effective methods of solving a system of n linear equations, for large n . In this chapter the reader also becomes acquainted with the technique of operating with matrices that are divided into rectangular 'blocks.'

In Chapter IV we introduce the extremely important ‘characteristic’ and ‘minimal’ polynomials of a square matrix, and the ‘adjoint’ and ‘reduced adjoint’ matrices.

In Chapter V, which is devoted to functions of matrices, we give the general definition of $f(A)$ as well as concrete methods of computing it—where $f(\lambda)$ is a function of a scalar argument λ and A is a square matrix. The concept of a function of a matrix is used in §§ 5 and 6 of this chapter for a complete investigation of the solutions of a system of linear differential equations of the first order with constant coefficients. Both the concept of a function of a matrix and this latter investigation of differential equations are based entirely on the concept of the minimal polynomial of a matrix and—in contrast to the usual exposition—do not use the so-called theory of elementary divisors, which is treated in Chapters VI and VII.

These five chapters constitute a first course on matrices and their applications. Very important problems in the theory of matrices arise in connection with the reduction of matrices to a normal form. This reduction is carried out on the basis of Weierstrass’ theory of elementary divisors. In view of the importance of this theory we give two expositions in this book: an analytic one in Chapter VI and a geometric one in Chapter VII. We draw the reader’s attention to §§ 7 and 8 of Chapter VI, where we study effective methods of finding a matrix that transforms a given matrix to normal form. In § 8 of Chapter VII we investigate in detail the method of A. N. Krylov for the practical computation of the coefficients of the characteristic polynomial.

In Chapter VIII certain types of matrix equations are solved. We also consider here the problem of determining all the matrices that are permutable with a given matrix and we study in detail the many-valued functions of matrices $\sqrt[m]{A}$ and $\ln A$.

Chapters IX and X deal with the theory of linear operators in a unitary space and the theory of quadratic and hermitian forms. These chapters do not depend on Weierstrass’ theory of elementary divisors and use, of the preceding material, only the basic information on matrices and linear operators contained in the first three chapters of the book. In § 9 of Chapter X we apply the theory of forms to the study of the principal oscillations of a system with n degrees of freedom. In § 11 of this chapter we give an account of Frobenius’ deep results on the theory of Hankel forms. These results are used later, in Chapter XV, to study special cases of the Routh-Hurwitz problem.

The last five chapters form the second part of the book [the second volume, in the present English translation]. In Chapter XI we determine normal forms for complex symmetric, skew-symmetric, and orthogonal mat-

rices and establish interesting connections of these matrices with real matrices of the same classes and with unitary matrices.

In Chapter XII we expound the general theory of pencils of matrices of the form $A + \lambda B$, where A and B are arbitrary rectangular matrices of the same dimensions. Just as the study of regular pencils of matrices $A + \lambda B$ is based on Weierstrass' theory of elementary divisors, so the study of singular pencils is built upon Kronecker's theory of minimal indices, which is, as it were, a further development of Weierstrass's theory. By means of Kronecker's theory—the author believes that he has succeeded in simplifying the exposition of this theory—we establish in Chapter XII canonical forms of the pencil of matrices $A + \lambda B$ in the most general case. The results obtained there are applied to the study of systems of linear differential equations with constant coefficients.

In Chapter XIII we explain the remarkable spectral properties of matrices with non-negative elements and consider two important applications of matrices of this class: 1) homogeneous Markov chains in the theory of probability and 2) oscillatory properties of elastic vibrations in mechanics. The matrix method of studying homogeneous Markov chains was developed in the book [25] by V. I. Romanovskii and is based on the fact that the matrix of transition probabilities in a homogeneous Markov chain with a finite number of states is a matrix with non-negative elements of a special type (a 'stochastic' matrix).

The oscillatory properties of elastic vibrations are connected with another important class of non-negative matrices—the 'oscillation matrices.' These matrices and their applications were studied by M. G. Kreĭn jointly with the author of this book. In Chapter XIII, only certain basic results in this domain are presented. The reader can find a detailed account of the whole material in the monograph [7].

In Chapter XIV we compile the applications of the theory of matrices to systems of differential equations with variable coefficients. The central place (§§ 5-9) in this chapter belongs to the theory of the multiplicative integral (Produktintegral) and its connection with Volterra's infinitesimal calculus. These problems are almost entirely unknown in Soviet mathematical literature. In the first sections and in § 11, we study reducible systems (in the sense of Lyapunov) in connection with the problem of stability of motion; we also give certain results of N. P. Erugin. Sections 9-11 refer to the analytic theory of systems of differential equations. Here we clarify an inaccuracy in Birkhoff's fundamental theorem, which is usually applied to the investigation of the solution of a system of differential equations in the neighborhood of a singular point, and we establish a canonical form of the solution in the case of a regular singular point.

In § 12 of Chapter XIV we give a brief survey of some results of the fundamental investigations of I. A. Lappo-Danilevskii on analytic functions of several matrices and their applications to differential systems.

The last chapter, Chapter XV, deals with the applications of the theory of quadratic forms (in particular, of Hankel forms) to the Routh-Hurwitz problem of determining the number of roots of a polynomial in the right half-plane ($\operatorname{Re} z > 0$). The first sections of the chapter contain the classical treatment of the problem. In § 5 we give the theorem of A. M. Lyapunov in which a stability criterion is set up which is equivalent to the Routh-Hurwitz criterion. Together with the stability criterion of Routh-Hurwitz we give, in § 11 of this chapter, the comparatively little known criterion of Liénard and Chipart in which the number of determinant inequalities is only about half of that in the Routh-Hurwitz criterion.

At the end of Chapter XV we exhibit the close connection between stability problems and two remarkable theorems of A. A. Markov and P. L. Chebyshev, which were obtained by these celebrated authors on the basis of the expansion of certain continued fractions of special types in series of decreasing powers of the argument. Here we give a matrix proof of these theorems.

This, then, is a brief summary of the contents of this book.

F. R. Gantmacher

PUBLISHERS' PREFACE

THE PUBLISHERS WISH TO thank Professor Gantmacher for his kindness in communicating to the translator new versions of several paragraphs of the original Russian-language book.

The Publishers also take pleasure in thanking the VEB Deutscher Verlag der Wissenschaften, whose many published translations of Russian scientific books into the German language include a counterpart of the present work, for their kind spirit of cooperation in agreeing to the use of their formulas in the preparation of the present work.

No material changes have been made in the text in translating the present work from the Russian except for the replacement of several paragraphs by the new versions supplied by Professor Gantmacher. Some changes in the references and in the Bibliography have been made for the benefit of the English-language reader.