PREFACE.

THE important character of the extensive investigations into the theory of line-geometry renders it desirable that a treatise should exist for the purpose of presenting these investigations in a form easily accessible to the English student of mathematics. With this end in view, the present work on the Line Complex has been written.

The subject owes its origin to Plücker, who suggested the idea of taking the straight line as the element of space*. The straight line thus holds to the present subject the relationship in which the point and the plane stand to the older geometry. Types of coordinates of the line were introduced by Cayley and Grassmann; Plücker adopted a coordinate system which is a special form of them.

In his work the *Neue Geometrie des Raumes*, Plücker introduced the conception of a *complex of lines*, *i.e.* the ∞^3 lines which satisfy one given condition, so that one equation exists between the four coordinates of each line of a complex. He investigated in detail the linear and the quadratic complex; his work contains most of the chief properties of such complexes; in particular he shows that if any screw motion about a certain axis be given to the lines forming a linear complex, these lines still remain within the complex. He discovered the *polar properties*, viz. that the lines of a linear complex in any plane pass through a point, the *pole* of the plane; that the lines of the complex through any point lie in a plane, the *polar plane* of the point; and that if a point moves along any given line, its polar plane turns round another line called the *polar line* of the first line, the relationship between the two lines being reciprocal.

The greater part of the *Neue Geometrie* is concerned with the quadratic complex, of which it contains many of the leading properties; in particular, Plücker shows that while the lines of

* *System der Geometrie des Raumes*, Düsseldorf, (1846).

such a complex through any point form in general a quadric cone, there is a certain surface, the *Singular Surface* of the complex, for whose points these cones break up into two planes. Likewise the lines of the complex in any plane, which in general touch a conic, in the case of any tangent plane of the singular surface form two pencils. This surface is the one known as Kummer's surface; it is of the 4th degree and class and possesses 16 nodes and 16 singular tangent planes.

The next investigator in this field was Battaglini, who pursued still further the ideas of Plücker. He adopted as the general quadratic complex one which was afterwards shown to be a special case, viz. the complex formed by the lines for each of which the points of intersection with two given quadrics form a harmonic range; but many of his results apply also to the general complex.

The success of Plücker's researches was limited by the unsuitable (Cartesian) analysis he employed. The second important step in the development of the subject was due to Prof. Felix Klein, who, in his celebrated memoir in volume II. of the *Mathematische Annalen,* introduced the coordinate-system determined by six linear complexes in mutual involution*. By its adoption a simple and elegant analytical mode of treatment of line-geometry is rendered possible.

Klein further revealed the existence of a singly infinite series of quadratic complexes which have the same singular surface as any given quadratic complex. In his Dissertation (Bonn, 1868) he pointed to the method of Weierstrass for the canonization of two quadratic forms, as the appropriate instrument for classifying the quadratic complex; and this classification was carried out by Weiler. Another service rendered by Klein was his discovery of the analogue existing between the *lines* of three-dimensional space and the *points* of four-dimensional space, together with the equations embodying this relationship. His enunciation of the fact that line-geometry is point-geometry on a quadric contained

* On any line common to two linear complexes a (1, 1) correspondence of points is determined by the planes through the line, viz. by taking the poles of each plane for the two complexes. If a certain condition, connecting the constants of the equations of the two complexes, is satisfied, these pairs of points form an *involution.*

in point-space of five dimensions, offers a new point of view of the subject.

Other important contributions to the theory are introduced from time to time in the text: of these the most fundamental are contained in the investigations of Lie, in which he showed the connexion of line-geometry with sphere-geometry. He established a relationship between the lines and spheres of three-dimensional space of such a nature, that to two intersecting lines there correspond two spheres in contact; and he applied the ideas of both varieties of geometry to the investigation of various types of differential equations.

In the present work the analytical method of treatment with Klein coordinates has been generally adopted; but as it frequently happens that synthetic methods are appropriate, recourse to such has been occasionally made. Since the study of synthetic geometry has been less widely followed in this country than on the Continent, I have not thought it superfluous to insert, by way of Introduction, a short sketch of the simpler portions of that subject which have bearing on the context of the work.

The main object of investigation is, as has been stated, the properties of the line complex, and, in connexion with it, the characteristics of the system of ∞^2 lines common to any two complexes. To any set of ∞^2 lines the name *congruence* is attached; the study of such systems was extensively pursued at a period considerably before Plücker's discoveries took place. The chief property of a congruence is that each of its lines is bitangent to a surface, (including as a special case *two surfaces, a surface and a curve,* etc.). Through any point there pass a definite number m of the lines of a given congruence, and in any plane there lie a definite number n of its lines. *If the congruence is the complete intersection of two complexes,* $m = n$.

Though not necessarily included in the scope of this treatise, nevertheless, on account of its close connexion with the theory of the complex, a discussion has been given in Chapters XIV.—XVI. of the congruence (m, n), and in particular, of the congruences $(2, n)$, so elegantly treated by Kummer.

As regards the various authorities on this subject, the student is referred to the work of Prof. Gino Loria *Il passato ed il presente* *delle principali teorie geometriche*, which contains detailed references to the chief memoirs. A useful summary with references is given in Prôf. E. Pascal's *Repertorio di mathematiche superiori.* The comprehensive treatise of Prof. R. Sturm, *Die Gebilde ersten und zweiten Grades der Liniengeometrie*, is a storehouse of information; his method is, however, *exclusively* synthetic. An introduction to most of the leading ideas is given by Prof. G. Koenigs in his work *La géométrie réglée et ses applications.*

An interesting general account of Line Geometry given by Mr J. H. Grace in the Supplement to the *Encyclopaedia Britannica*, will be found very serviceable by the student of this subject.

I have thought it not desirable to include in this treatise a description of the important investigations of Prof. E. Study, on account of their distinctness in aim and method from those of the other writers who have built up this subject. I rather refer the reader to Prof. Study's work *Geometrie der Dynamen.*

It gives me much pleasure to express my gratitude to several friends for assistance generously given me; and especially to Mr J. H. Grace, M.A., Fellow of Peterhouse, Cambridge, who read the manuscript, and who, by his criticisms and suggestions, has greatly increased the value of the work. My colleague Mr G. W. Caunt, M.A., late Scholar of St Catharine's College, has read all the proofs; such accuracy as the book possesses is largely due to his carefulness. I am also under obligations to Mr P. W. Wood, B.A., Scholar of Emmanuel College, who has read the proofs and verified many of the examples.

Professor T. J. I'A. Bromwich, Fellow of St John's College, has kindly put at my disposal a collection of examples, most of which were made by him; they have been incorporated in the Miscellaneous Results and Exercises, and add greatly to the book's usefulness.

Finally, I feel it a pleasant duty to express my appreciation of the admirable manner in which the staff of the University Press have carried out the onerous task involved in the printing.

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