PREFACE TO THE SECOND EDITION

This is a reprinting with no essential changes of the First Edition of this book. I have taken the opportunity to correct some minor errors and to bring the list of suggestions for further reading (in Appendix III) up to date.

I would make the following more definite suggestions than I did in the Preface of the First Edition, as to how this book could be used in a variety of courses. First of all, with a one-semester course specifically set aside for the foundations of the number systems, I would omit Section 5.2 on polynomials in several variables, the subsection on Sturm's theorem for real polynomials on pages 278-285, Section 8.3 on the explicit representation of roots of complex polynomials, all of Chapter 9 on algebraic fields and field extensions, and both Appendices I and II; moreover, I would recommend assigning Chapter 1 as reading for the students, with a quick review by the instructor. In a one-quarter course I would omit further Section 5.1 on polynomial functions and polynomial forms, giving the latter a brief informal explanation instead, Section 6.3 on solutions of algebraic equations in fields, (possibly) Section 6.4 on polynomials over a field, and Section 7.5 on algebraic and transcendental numbers. If the book is used as part of a two-quarter course on modern algebra or analysis at the intermediate level, the topics to be omitted or meshed with the principal material for the course would depend on the other text to be used. Whatever the syllabus planned, I would tell students that this is a book which calls for careful reading, with active attention to details of definition and proof.

The present book basically takes for granted the non-constructive set-theoretical foundation of mathematics, which is tacitly if not explicitly accepted by most working mathematicians but which I have since come to reject. Still, whatever one's foundational views, students must be trained in this approach in order to understand modern mathematics. Moreover, most of the material of the present book can be modified so as to be acceptable under alternative constructive and semi-constructive viewpoints, as has been demonstrated in more advanced texts and research articles. Unfortunately, there is to my knowledge currently no text at the level of intermediate undergraduate study which serves to explain the rationale for such alternative approaches and to show in detail how the successive number systems are treated in them. In lieu of such, even a brief explanation by the instructor to the students of how and why constructive foundations for mathematics differ from set-theoretical foundations could be useful. The constructive viewpoints I have in mind are described in the book *Constructivism in Mathematics: An Introduction,* Volume 1, by A. S. Troelstra and D. van Dalen, North-Holland, 1988. The semi-constructive viewpoint is that of *predicativity,* which was first developed by Hermann Weyl in his 1918 monograph *Das Kontinuum* (reprinted by Chelsea Publishing Company in *Das Kontinuum und andere Monographien*), of which there is now a translation under the title *The Continuum,* by S. Pollard and T. Bole, Thomas Jefferson University Press/ University Press of America, 1987. Weyl's work has only been brought up to date in research articles, the most recent of which is one I have written entitled "Weyl vindicated: *Das Kontinuum* 70 years later," which appears in the volume *Temi e prospettive della logica e della filosofia della scienza contemporanee,* Vol. 1, edited by C. Cellucci and G. Sambin, Clueb, Bologna, 1989, pages 55-93.

I wish to thank Mr. Aaron Galuten of the Chelsea Publishing Company for his invitation to publish a second edition of this book, as I believe it continues to serve well many worthwhile purposes.

SOLOMON FEFERMAN

Stanford, California
February 1989

PREFACE TO THE FIRST EDITION

The subject matter of this book is the successive construction and development of the basic number systems of mathematics, namely the positive integers, integers, rational numbers, real numbers, and complex numbers. It is a subject that many mathematicians feel should be learned by every serious student in this field. Preferably, he should do this as soon as possible after his first course in mathematical analysis (calculus)—either before or during his introduction to more rigorous treatments of analysis and algebra.

Despite the significance of this subject in a mathematical education, there does not seem to be any special provision for its study in most American universities. Sometimes a hasty review of the material is given in intermediate courses on algebra or analysis. Another approach often taken in these courses is to begin with the real number system as axiomatically given, rather than to develop its properties from more basic notions and results.

We believe this situation has come about for several reasons. First of all, the (now) classical presentations of this material have a curious isolation from the rest of mathematics. The ideas and methods employed seem to have a "once only" character and lack the sense of interrelatedness of most other important mathematical concepts. Second, the rate at which knowledge is increasing makes it imperative that the student of mathematics hasten his mastery of the main parts of his field. Finally, and in tune with the "abstractness" of modern mathematics, there is a growing tendency to present all its parts axiomatically.

As a result of these circumstances, there is often a gap in the student's education between his "concrete" computational work in the calculus and his more advanced work. It is true that modern abstract analysis and algebra have developed as the proper means to encompass, and then to advance beyond, the particular notions and results concerning the classical number systems uncovered before this century. However, a firm grasp of the significant particular cases provides the best basis for an appreciation of the newer developments.

It thus seems to us that the subject of this book provides the most appropriate material for this transition period in the student's mathematical education. We have tried to give here a presentation which is on the one hand up to date, complete, and rigorous, and on the other hand constantly motivated with reference to both the student's background and the needs of modern mathematics.

We believe the approach taken here makes the text adaptable to a variety of teaching situations. It can be used for a one-quarter or a one-semester course specifically set aside for this material and demanding no prerequisites at this level. It can be the text for the first part of a conventional intermediate course in algebra or analysis, with certain sections omitted or merely sketched, according to the specifications of the course and the taste of the instructor. It might also be used as a basic reference work for such a course or as the text for a reading course, which the student would master by independent study. It is with this last possibility especially in mind that we have chosen to make this book self-contained and to pursue clarity and completeness, rather than conciseness.

S. F.

Stanford, California
October 1963