

# INDEX OF SPECIAL INTEGRALS, PRODUCTS, AND SERIES.

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\* Commonly attributed to Cauchy; it occurs in Maclaurin's *Fluxions*, 1742, Art. 350.

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