PREFACE TO THE SECOND EDITION

THE second edition of this book consists largely of a reproduction of the first edition, with additional theorems and examples. The arrangement of the first seven chapters, as well as of Chapter IX., has undergone very little alteration: to the eighth chapter a discussion of the solution of linear differential equations of the second order has been added. Chapter X. of the first edition ("Complex Series and Products") has been broken up into two chapters, X. and XI., the first of these containing the general theory of complex series and products, and the second dealing with special series and functions. The principal new feature of the latter chapter is a discussion of elliptic function formulae.

Chapter XI. of the first edition ("Non-Convergent and Asymptotic Series") now becomes Chapter XII. Here the entire discussion of the theory of summable series, apart from the historical introduction, has been omitted, as Dr. Bromwich felt that an adequate account of the subject with its later developments would require more space than could be given to it in the present volume. The part of the chapter devoted to asymptotic series has been enlarged, and contains, among other new matter, an exposition of the asymptotic expansions of the Bessel functions. Room has also been found for a discussion of trigonometrical series, including Stokes's transformation and Gibbs's phenomenon.

The alterations in Appendix I. are slight, but Appendix II. has been expanded to make room for an account of Napier's invention of logarithms. Appendix III. ("Infinite Integrals and Gamma Functions") was originally written in connection with the discussion of summable series, and might therefore have been omitted. As, however, this Appendix contains much material not otherwise accessible in English text-books, it has been decided to include a verbatim reprint of it in this edition. The set of "Harder Miscellaneous Examples" has also been omitted, but some of these examples will be found in the collections of examples throughout the book.

Dr. Bromwich, unfortunately, has not been able to supervise the passing of the final proofs of the new edition through the press. When these came into my hands the entire book, except Appendix III., was already in type, and my share of the work has been confined to matters of detail. Many errors have been eliminated, and it is hoped that the work has not suffered seriously from the absence, at the final stages, of the guiding hand of the author.

THOMAS M. MACROBERT.

GLASGOW, *October*, 1925.

Note.—The numbering of some of the articles referred to in the Preface to the First Edition has been altered in the Second Edition: Arts. 19, 20, 23, 149, 150, 151, 163 become Arts. 21, 22, 20, 143, 144, 145, 161 and Art. 83 is now replaced by Arts. 86 and 87.

PREFACE TO THE FIRST EDITION

This book is based on courses of lectures on Elementary Analysis given at Queen's College, Galway, during each of the sessions 1902-1907. But additions have naturally been made in preparing the manuscript for press: in particular the whole of Chapter XI. and the greater part of the Appendices have been added. In selecting the subject-matter, I have attempted to include proofs of all theorems stated in Pringsheim's article, *Irrationalzahlen und Konvergenz unendlicher Prozesse,** with the exception of theorems relating to continued fractions.

In Chapter I. a preliminary account is given of the notions of a limit and of convergence. I have not in this chapter attempted to supply arithmetic proofs of the fundamental theorems concerning the existence of limits, but have allowed their truth to rest on an appeal to the reader's intuition, in the hope that the discussion may thus be made more attractive to beginners. An arithmetic treatment will be found in Appendix I., where Dedekind's definition of irrational numbers is adopted as fundamental; this method leads at once to the monotonic principle of convergence (Art. 149), from which the existence of extreme limits † is deduced (Arts. 5, 150); it is then easy to establish the general principle of convergence (Art. 151).

In the remainder of the book free use is made of the notation and principles of the Differential and Integral Calculus; I have for some time been convinced that beginners should not attempt to study Infinite Series in any detail until after they have mastered the

* *Encyklopädie der Mathematischen Wissenschaften*, Bd. I., A, 3 and G, 3 (pp. 47 and 1121).

† Not only here, but in many other places, the proofs and theorems have been made more concise by a systematic use of these maximum and minimum limits.

differentiation and integration of the simpler functions, and the geometrical meaning of these operations.

The use of the Calculus has enabled me to shorten and simplify the discussion of various theorems (for instance, Arts. 11, 61, 62), and to include other theorems which must have been omitted otherwise (for instance, Arts. 45, 46, and the latter part of 83).

It will be noticed that from Art. 11 onwards, free use is made of the equation

$$\frac{d}{dx}(\log x)=\frac{1}{x},$$

although the limit of $(1+1/\nu)^\nu$ (from which this equation is commonly deduced) is not obtained until Art. 57. To avoid the appearance of reasoning in a circle, I have given in Appendix II. a treatment of the theory of the logarithm of a real number, starting from the equation

$$\log a=\int_1^a \frac{dx}{x}.$$

The use of this definition of a logarithm goes back to Napier, but in modern teaching its advantages have been overlooked until comparatively recently. An arithmetic proof that the integral represents a definite number will be found in Art. 163, although this fact would naturally be treated as axiomatic when the subject is approached for the first time.

In Chapter V. will be found an account of Pringsheim's theory of double series, which has not been easily accessible to English readers hitherto.

The notion of uniform convergence usually presents difficulties to beginners; for this reason it has been explained at some length, and the definition has been illustrated by Osgood's graphical method. The use of Abel's and Dirichlet's names for the tests given in Art. 44 is not strictly historical, but is intended to emphasise the similarity between the tests for uniform convergence and for simple convergence (Arts. 19, 20).

In obtaining the fundamental power-series and products constant reference is made to the principle of uniform convergence, and particularly to Tannery's theorems (Art. 49); the proofs are thus simplified and made more uniform than is otherwise possible. Considerable use is also made of Abel's theorem (Arts. 50, 51, 83) on the continuity of power-series, a theorem which, in spite of its importance, has usually not been adequately discussed in text-books.

Chapter XI. contains a tolerably complete account of the recently developed theories of non-convergent and asymptotic series; the treatment has been confined to the arithmetic side, the applications to function-theory being outside the scope of the book. As might be expected, a systematic examination of the known results has led to some extensions of the theory (see, for instance, Arts. 118-121, 123, and parts of 133).

The investigations of Chapter XI. imply an acquaintance with the convergence of infinite integrals, but when the manuscript was being prepared for printing no English book was available from which the necessary theorems could be quoted.* I was therefore led to write out Appendix III., giving an introduction to the theory of integrals; here special attention is directed to the points of similarity and of difference between this theory and that of series. To emphasise the similarity, the tests of convergence and of uniform convergence (Arts. 169, 171, 172) are called by the same names as in the case of series; and the traditional form of the Second Theorem of Mean Value is replaced by inequalities (Art. 166) which are more obviously connected with Abel's Lemma (Arts. 23, 80). To illustrate the general theory, a short discussion of Dirichlet's integrals and of the Gamma integrals is given; it is hoped that these proofs will be found both simple and rigorous.

The examples (of which there are over 600) include a number of theorems which could not be inserted in the text, and in such cases references are given to sources of further information.

Throughout the book I have made it my aim to keep in view the practical applications of the theorems to every-day work in analysis. I hope that most double-limit problems, which present themselves *naturally*, in connexion with integration of series, differentiation of integrals, and so forth, can be settled without difficulty by using the results given here.

Mr. G. H. Hardy, M.A., Fellow and Lecturer of Trinity College, has given me great help during the preparation of the book; he has

* While my book has been in the press, three books have appeared, each of which contains some account of this theory: Gibson's *Calculus* (ch. xxi., 2nd ed.), Carslaw's *Fourier Series and Integrals* (ch. iv.), and Pierpont's *Theory of Functions of a Real Variable* (chs. xiv., xv.).

read all the proofs, and also the manuscript of Chapter XI. and the Appendices. I am deeply conscious that the value of the book has been much increased by Mr. Hardy's valuable suggestions and by his assistance in the selection and manufacture of examples.

The proofs have also been read by Mr. J. E. Bowen, B.A., Senior Scholar of Queen's College, Galway, 1906-1907; and in part by Mr. J. E. Wright, M.A., Fellow of Trinity College, and Professor at Bryn Mawr College, Pennsylvania. The examples have been verified by Mr. G. N. Watson, B.A., Scholar of Trinity College, who also read the proofs of Chapter XI. and Appendix III. To these three gentlemen my best thanks are due for their careful work.

T. J. I'A. BROMWICH.

CAMBRIDGE, *December*, 1907.