Preface

Representation theory is the study of concrete realizations of the axiomatic systems of abstract algebra. It originated in the study of permutation groups, and algebras of matrices. The theory of group representations was developed in an astonishingly complete and useful form by Frobenius in the last two decades of the nineteenth century. Both Frobenius and Burnside realized that group representations were sure to play an important part in the theory of abstract finite groups. The first book to give a systematic account of representation theory appeared in 1911 (Burnside [4]) and contained many results on abstract groups which were proved using group characters. Perhaps the most famous of these is Burnside's theorem that a finite group whose order has at most two distinct prime divisors must be solvable. Recently, a purely group-theoretic proof of Burnside's theorem has been obtained by Thompson. The new proof is of course important for the structure theory of groups, but it is at least as complicated as the original proof by group characters.

The second stage in the development of representation theory, initiated by E. Noether [1] in 1929, resulted in the absorption of the theory into the study of modules over rings and algebras. The representation theory of rings and algebras has led to new insights in the classical theory of semi-simple rings and to new investigations of rings with minimum condition centering around Nakayama's theory of Frobenius algebras and quasi-Frobenius rings.

Another major development in representation theory is R. Brauer's work on modular representations of finite groups. Like the original work of Frobenius, Brauer's theory has many significant applications to the theory of finite groups. At the same time it draws on the representation theory of algebras and suggests new problems on modules and rings with minimum condition. It also emphasizes the fundamental importance of number-theoretical questions in group theory and representation theory.

During the past decade there has been increased emphasis on integral representations of groups and rings, motivated to some extent by questions arising from homological algebra. This theory of integral representations has been a fruitful source of problems and conjectures both in homological algebra and in the arithmetic of non-commutative rings.

The purpose of this book is to give, in as self-contained a manner as possible, an up-to-date account of the representation theory of finite groups and associative rings and algebras. This book is not intended to be encyclopedic in nature, nor is it a historical listing of the entire theory. We have instead concentrated on what seem to us to be the most important and fruitful results and have included as much preliminary material as necessary for their proofs.

In addition to the classical work given in Burnside's book [4], we have paid particular attention to the theory of induced characters and induced representations, quasi-Frobenius rings and Frobenius algebras, integral representations, and the theory of modular representations. Much of this material has heretofore been available only in research articles. We have concentrated here on general methods and have built the theory solidly on the study of modules over rings with minimum condition. Enough examples and problems have been included, however, to help the research worker who needs to compute explicit representations for particular groups. We have included some applications of group representations to the structure theory of finite groups, but a definitive account of these applications lies outside the scope of this book. In Section 92 we have given a survey of the present literature dealing with these applications and have included in this book all the representation-theoretic prerequisites needed for reading this literature, though not all the purely group-theoretic background which might be necessary.

No attempt has been made to orient the reader toward physical applications. For these we may refer the reader to recent books and articles dealing with that part of group theory relevant to physics, and in particular to Wigner [1], Gelfand-Sapiro [1], Lomont [1], and Boerner [1].

It has also been necessary to omit the vast literature on representations of the symmetric group. Fortunately the reader is now able to consult the excellent book on this topic by Robinson [1].

Many of the results on group representations have been generalized to infinite groups and also to infinite-dimensional representations of topological groups. We have felt that these generalizations do not properly fall within the scope of this book and, in fact, would require a lengthy separate presentation.

The book has been written in the form of a textbook; a preliminary version has been used in several courses. We have assumed that the reader is familiar with the following topics, which are usually treated in a "standard" first-year graduate course in algebra: elementary group theory, commutative rings, elementary number theory, rudiments of Galois theory, vector spaces, and linear transformations.

We are confident that the expert as well as the student will find something of interest in this book. We offer no apology, however, for writing to be understood by a reader unfamiliar with the subject. In keeping with this objective, we have not always presented results in their greatest generality, and we have included details which will sometimes seem tedious to the experienced reader. After serious deliberation, we decided not to introduce the full machinery of homological algebra. Although it would have simplified several sections of the book, we felt that many readers were not likely to be well-grounded in homological algebra, and this book was not intended to be a first course in the subject.

The first three chapters are written at the level of a first-year graduate course and include introductory material as well as background for later chapters. Much of this material may be skimmed rapidly or omitted entirely at a first reading, though Sections 9–13 should be read with care.

Chapters IV–VII form a unit containing the structure theory of semi-simple rings with minimum condition, and the applications of this theory to group representations and characters.

Chapters IV, VIII, IX, and X form a unit on rings with minimum condition and finite-dimensional algebras. Chapter IV develops the theory of the radical and semi-simplicity by the perhaps old-fashioned method of calculations with idempotents, because idempotents furnish the main tool in the study of non-semi-simple rings and algebras in Chapters VIII and IX.

Chapters III and XI form a more or less self-contained account of algebraic number theory and integral representations of groups. Some knowledge of earlier chapters is needed, especially in Sections 77–78.

Chapter XII is devoted to the theory of modular representations and requires aknowledge of parts of all the preceding chapters. The exact prerequisites for reading Chapter XII are given at the beginning of the chapter.

For the reader whose main interest is in representations of finite groups, we may suggest the following sections for a first brief reading: 9–13, 23–27, 30–34, 38–40, 43–46, 49–50, 54–55, 61, 82–92.

These sections are to some extent self-contained, provided that the reader is willing to postpone to the second reading the proofs of some of the results needed from other sections.

Exercises are included at the end of almost every section. Some provide easy checks on the reader's comprehension of the text; others are intended to challenge his abilities. Many are important results in their own right and may occasionally be referred to when needed in later sections.

Sections are numbered consecutively throughout the book. A cross reference to (a.b) refers to Section a and to the bth numbered item in that section.

There is a fairly large bibliography of works which are either directly relevant to the text or offer supplementary material of interest. An attempt has been made to give credit for some of the major methods and theorems, but we have stopped far short of trying to trace each theorem to its source.

We are indebted to many persons and organizations for assisting us with this work. Our students, friends, colleagues, and families have listened to us lecture on these subjects, read portions of the manuscript and proof sheets, made suggestions and corrections, and given us encouragement. We are deeply appreciative of their kind help. Our interest in this subject was stimulated by a seminar conducted at the Institute for Advanced Study in 1954–1955. We are indebted to the participants in that seminar for their help and to the Institute for making possible the preparation of mimeographed seminar notes. It is a pleasure to acknowledge the generous support we have received for the work on this book from the Office of Naval Research. Finally we are grateful to Interscience Publishers for publishing it and giving us their patient and friendly cooperation.

Charles W. Curtis
Irving Reiner

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