IV

THE FROBENIUS ALGEBRA

4.1. Groups and algebras

A SIMILARITY between the properties of groups and algebras, especially the property of non-commutative multiplication, suggests a connexion between the two theories. In fact, if we take an abstract group, and further define operations of addition and scalar multiplication, we obtain a special type of algebra which is called after Frobenius, the founder of the theory, a *Frobenius algebra* (**1**).†

Let $S_1, S_2,..., S_h$ be the operations of a group H. Then the algebra with basis $e_1, e_2,..., e_h$ and multiplication table

$$e_i e_j = e_k \quad \text{whenever } S_i S_j = S_k,$$

so that

$$\gamma_{ijk} = 1 \quad \text{if } S_i S_j = S_k,$$
$$\gamma_{ijk} = 0 \quad \text{if } S_i S_j \neq S_k,$$

is called the *group algebra*, or the *Frobenius algebra* of the group.

There is a logical distinction between the elements of the group and the basal elements of the Frobenius algebra. Henceforward, however, we shall use the same symbol S_i for both purposes. No logical confusion arises, as the basal elements do in any case form a representation of the group, and the exposition is thus made simpler.

It is necessary to distinguish between an *element of the algebra*, which is any element of the form $\sum \xi_i S_i$, and a *group element*, which must be one of the basal elements of the algebra S_i.

We replace S_h by S_0, and use this symbol to denote the identical element of the group. From the multiplication table we obtain

$$\gamma_{j0j} = 1, \qquad \gamma_{jij} = 0 \quad (i \neq 0).$$

We thus obtain the following result.

The trace of S_0 is h, and the trace of every other group element is zero.

Thus the trace of $X = \sum \xi_i S_i$ is $h\xi_0$.

THEOREM. *The Frobenius algebra of a group is expressible as a direct sum of simple matrix algebras.*

This is the important theorem of this section. The proof follows immediately from the following lemma.

LEMMA. *The Frobenius algebra of a group contains no properly nilpotent element.*

† Bold-face numbers refer to notes at end of book.

Let $X = \sum \xi_i S_i$ be any element of the algebra, and let

$$\bar{X} = \sum \bar{\xi}_i S_i^{-1},$$

where $\bar{\xi}_i$ is the conjugate complex number of ξ_i. Then the trace of $X\bar{X}$, namely $h \sum \xi_i \bar{\xi}_i$, is clearly a positive number. It follows that $X\bar{X}$ cannot be nilpotent, and X cannot be properly nilpotent. This proves the lemma, and, consequently, the theorem.

Elements which commute with every element of the algebra

Let the group H contain p classes, $C_0, C_1, \ldots, C_{p-1}$, of which C_0 is the class containing the identity. We shall use the word *class* and the symbol C_ρ also to denote the sum of the elements of the class.

Let S_i be any group element of the class C_ρ of order h_ρ. Then, from § 3.6, if S_j runs through the h elements of H, the h elements $S_j^{-1}S_iS_j$ consist of the class C_ρ repeated h/h_ρ times. Hence

$$\sum_j S_j^{-1}S_iS_j = hC_\rho/h_\rho.$$

Thus, if X is any element $\sum \xi_i S_i$ of the algebra, $\sum_j S_j^{-1}XS_j$ is a linear function of the classes.

If an element X of the algebra commutes with every element of the algebra, then

$$S_j^{-1}XS_j = X$$

and

$$\frac{1}{h}\sum_j S_j^{-1}XS_j = X.$$

It follows that X is a linear function of the classes. Conversely, a linear function of the classes commutes with every element of the algebra.

An element of a Frobenius algebra which commutes with every other element of the algebra is a linear function of the classes, and, conversely, a linear function of the classes commutes with every element of the algebra.

Such elements form a linear set of order p.

Let the Frobenius algebra be equivalent to the direct sum of q simple matrix sub-algebras $\Gamma_1, \Gamma_2, \ldots, \Gamma_q$, and let the modulus of the sub-algebra Γ_i be ϵ_i.

The only matrices which commute with every matrix of order n^2 are the scalar multiples of the unit matrix. Thus, if an element commutes with every element of the Frobenius algebra, its representation in each sub-algebra is a scalar multiple of the modulus, and the element may be expressed in the form $\sum \lambda_i \epsilon_i$, λ_i being scalar. The order of the linear set of such elements is q, and it follows that $p = q$.

The number of simple matrix sub-algebras in the Frobenius algebra of a group is equal to the number of classes.

The classes can clearly be expressed linearly in terms of the moduli of the sub-algebras, and conversely.

$$C_\rho = \sum \psi_\rho^{(i)} \epsilon_i, \tag{4.1; 1}$$

$$\epsilon_i = \sum \phi_\rho^{(i)} C_\rho. \tag{4.1; 2}$$

Denote the class containing the inverses of the elements in the class C_ρ by $C_{\rho'}$. By substituting from (4.1; 1) and (4.1; 2) respectively we have

$$\begin{aligned} \epsilon_i C_\rho &= \sum \psi_\rho^{(j)} \epsilon_i \epsilon_j = \psi_\rho^{(i)} \epsilon_i \\ &= \sum \phi_\sigma^{(i)} C_\sigma C_\rho. \end{aligned} \tag{4.1; 3}$$

Now the trace of $C_\sigma C_\rho$ is hh_ρ if $\sigma = \rho'$, but zero otherwise. Also the trace of ϵ_i is $f^{(i)2}$, where $f^{(i)}$ is the degree of the matrix sub-algebra, i.e. the number of rows or columns.

Taking the trace of (4.1; 3), we have

$$\psi_\rho^{(i)} f^{(i)2} = hh_\rho \phi_{\rho'}^{(i)}. \tag{4.1; 4}$$

Again, taking the trace of (4.1; 2) and remembering that the trace of C_0 is h, but the trace of every other class is zero, we have

$$f^{(i)2} = h\phi_0^{(i)}. \tag{4.1; 5}$$

Substituting in (4.1; 4), we obtain

$$\psi_\rho^{(i)} = h_\rho \phi_{\rho'}^{(i)} / \phi_0^{(i)},$$

whence (4.1; 1) may be rewritten

$$C_\rho = \sum h_\rho \phi_{\rho'}^{(i)} \epsilon_i / \phi_0^{(i)}. \tag{4.1; 6}$$

4.2. The group characters

The Frobenius algebra is equivalent to the direct sum of p simple matrix sub-algebras. Any group element S_i may thus be expressed as a sum of elements in each of the sub-algebras. These are called the *representations* of S_i in the various sub-algebras. There are an infinity of ways of expressing each sub-algebra as a simple matrix algebra, but the spur of the matrix is independent of the mode of representation (§ 2.9, Theorem V).

DEFINITION. *The spur of the matrix representation of S_i in the sub-algebra Γ_j is called the characteristic of S_i, and is written $\chi^{(j)}(S_i)$. The set of characteristics of the h group elements corresponding to the sub-algebra Γ_j is called a group character, and is written $\chi^{(j)}$.*

Since the h_ρ elements of a class C_ρ are transforms of one another, the matrix representations in a given sub-algebra have the same spur. Hence the characteristics of the h_ρ elements are equal. This value will be referred to as the *characteristic of the class* C_ρ, and may be written $\chi_\rho^{(i)}$. $\chi_0^{(i)} = f^{(i)}$ is called the *degree* of the character.

There are thus p^2 distinct numbers which are the characteristics of the p classes corresponding to the p sub-algebras. These numbers may be arranged in a square table which we call the *table of characters*. It will be seen from Chapter IX that almost all the properties of a group may be deduced from this table of characters.

The group characters satisfy very important orthogonal relations, which we proceed to obtain.

The spur of the representation in Γ_i of the class C_ρ is evidently $h_\rho \chi_\rho^{(i)}$. Hence the trace of this element, which is the trace of $\epsilon_i C_\rho$, is

$$h_\rho f^{(i)} \chi_\rho^{(i)}.$$

From (4.1; 4) we obtain

$$h_\rho f^{(i)} \chi_\rho^{(i)} = h h_\rho \phi_{\rho'}^{(i)},$$

so that

$$\phi_\rho^{(i)} = f^{(i)} \chi_{\rho'}^{(i)}/h. \tag{4.2; 1}$$

Substituting in equations (4.1; 2), (4.1; 6), we obtain

$$\epsilon_i = \sum f^{(i)} \chi_{\rho'}^{(i)} C_\rho/h, \tag{4.2; 2}$$

$$C_\rho = \sum h_\rho \chi_\rho^{(i)} \epsilon_i/f^{(i)}. \tag{4.2; 3}$$

In the left-hand side of the equations

$$\epsilon_i^2 = \epsilon_i, \qquad \epsilon_i \epsilon_j = 0$$

substitute from (4.2; 2) and take the trace. Remembering that the trace of $C_\rho C_\sigma$ is hh_ρ if $\sigma = \rho'$, and zero otherwise, we have

$$\sum f^{(i)2} \chi_\rho^{(i)} \chi_{\rho'}^{(i)} h h_\rho/h^2 = f^{(i)2},$$

and

$$\sum f^{(i)} f^{(j)} \chi_\rho^{(i)} \chi_{\rho'}^{(j)} h h_\rho/h^2 = 0.$$

Thus

$$\left.\begin{aligned} \sum_\rho h_\rho \chi_\rho^{(i)} \chi_{\rho'}^{(i)} &= h, \\ \sum_\rho h_\rho \chi_\rho^{(i)} \chi_{\rho'}^{(j)} &= 0 \quad (i \neq j). \end{aligned}\right\} \tag{4.2; 4}$$

Again, substituting from (4.2; 3) in $C_\rho C_\sigma$ and taking the trace, we have

$$\begin{aligned} \sum h_\rho h_\sigma \chi_\rho^{(i)} \chi_\sigma^{(i)} f^{(i)2}/f^{(i)2} &= hh_\rho \quad \text{if } \sigma = \rho', \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Hence

$$\left.\begin{aligned} h_\rho \sum_i \chi_\rho^{(i)} \chi_{\rho'}^{(i)} &= h, \\ \sum \chi_\rho^{(i)} \chi_\sigma^{(i)} &= 0 \quad (\sigma \neq \rho'). \end{aligned}\right\} \tag{4.2; 5}$$

Equations (4.2; 4) and (4.2; 5) constitute the orthogonal relations referred to above.

EXAMPLE. Consider the symmetric group of permutations on the three symbols α, β, γ. The six elements of the group include the identity I, the three elements

$$(\alpha\beta), \qquad (\beta\gamma), \qquad (\alpha\gamma),$$

which form the class (2 1), which we will denote by C_1; and the two elements

$$(\alpha\beta\gamma), \qquad (\alpha\gamma\beta)$$

which form the class (3), which we shall denote by C_2.

The multiplication table of the classes, which are, of course, commutative, is as follows:

$$C_0^2 = C_0, \qquad C_0 C_1 = C_1, \qquad C_0 C_2 = C_2,$$
$$C_1^2 = 3C_0+3C_2, \qquad C_1 C_2 = 2C_1, \qquad C_2^2 = 2C_0+C_2.$$

Hence, clearly, the moduli of the three simple matrix algebras of the Frobenius algebra are respectively

$$\tfrac{1}{6}(C_0+C_1+C_2),$$
$$\tfrac{1}{6}(C_0-C_1+C_2),$$
$$\tfrac{1}{3}(2C_0-C_2);$$

these being the irreducible idempotents of the algebra of the classes.

The six elements

$$x = \tfrac{1}{6}[I+(\alpha\beta)+(\beta\gamma)+(\gamma\alpha)+(\alpha\beta\gamma)+(\alpha\gamma\beta)],$$
$$y = \tfrac{1}{6}[I-(\alpha\beta)-(\beta\gamma)-(\gamma\alpha)+(\alpha\beta\gamma)+(\alpha\gamma\beta)],$$
$$z_{11} = \tfrac{1}{3}[I+(\alpha\beta)-(\alpha\gamma)-(\alpha\beta\gamma)],$$
$$z_{22} = \tfrac{1}{3}[I+(\alpha\gamma)-(\alpha\beta)-(\alpha\gamma\beta)],$$
$$z_{12} = \tfrac{1}{3}[(\beta\gamma)-(\alpha\gamma)+(\alpha\gamma\beta)-(\alpha\beta\gamma)],$$
$$z_{21} = \tfrac{1}{3}[(\beta\gamma)-(\alpha\beta)+(\alpha\beta\gamma)-(\alpha\gamma\beta)]$$

form a basis for the Frobenius algebra, and satisfy

$$x^2 = x, \qquad y^2 = y, \qquad xy = yx = 0,$$
$$xz_{ij} = z_{ij}x = yz_{ij} = z_{ij}y = 0,$$
$$z_{ij}z_{jk} = z_{ik}, \qquad z_{ij}z_{kp} = 0 \quad (j \neq k;\, i, j, k, p = 1, 2).$$

The Frobenius algebra is thus exhibited as the direct sum of three sub-algebras, two being simply isomorphic with the complex numbers, and the third being a simple matrix algebra of order 4.

Now putting $z = z_{11}+z_{22}$, the modulus of the third sub-algebra, denote $\lambda_{11}z_{11}+\lambda_{12}z_{12}+\lambda_{21}z_{21}+\lambda_{22}z_{22}$ by $\begin{bmatrix}\lambda_{11}, & \lambda_{12}\\ \lambda_{21}, & \lambda_{22}\end{bmatrix}z$. Then we can solve the above equations for the group elements in terms of x, y, and z.

$$I = x+y+\begin{bmatrix}1 & \\ & 1\end{bmatrix}z,$$

$$(\alpha\beta) = x-y+\begin{bmatrix}1, & -1\\ & -1\end{bmatrix}z,$$

$$(\beta\gamma) = x-y+\begin{bmatrix} & 1\\ 1 & \end{bmatrix}z,$$

$$(\gamma\alpha) = x-y+\begin{bmatrix}-1 & \\ -1, & 1\end{bmatrix}z,$$

$$(\alpha\beta\gamma) = x+y+\begin{bmatrix} & -1\\ 1, & -1\end{bmatrix}z,$$

$$(\alpha\gamma\beta) = x+y+\begin{bmatrix}-1, & 1\\ -1 & \end{bmatrix}z.$$

The characters are the spurs of the matrix coefficients of x, y, and z, and may be expressed in table form:

Class . .	(1^3)	(2 1)	(3)
Order . .	1	3	2
	1	1	1
	1	−1	1
	2	0	−1

It is easily verified that these characters satisfy the orthogonal relations (4.2; 4) and (4.2; 5).

4.3. Matrix representations and group matrices

If to each element S_i of a group there corresponds a matrix M_i such that $M_i M_j = M_k$ whenever $S_i S_j = S_k$, the matrices M_i are said to form a *matrix representation* of the group.

The matrices M_i need not all be distinct. To several elements of the group may correspond identical matrices, so that the representation is not simply, but multiply isomorphic with the group. For example, M_i may be the one-rowed unit matrix for every element of the group. But to each group element S_i there must correspond a unique matrix M_i.

The matrix $\sum \xi_i M_i$ which corresponds to the general element $\sum \xi_i S_i$ of the Frobenius algebra is called a *group matrix*. It exhibits in the form of one matrix the complete matrix representation of the group. Each element of the matrix is a linear function of the ξ_i's.

Clearly the sets of linear substitutions to which the matrices M_i correspond form a representation of the abstract group as a group of linear substitutions. The theories of these two types of representation are equivalent.

Since the Frobenius algebra is equivalent to the direct sum of p simple matrix algebras, to each group element there corresponds a matrix in each sub-algebra. The p sub-algebras thus give p matrix representations of the group. We shall show that every *irreducible* matrix representation of the group is *equivalent* to one of these p representations, and every representation is equivalent to a *direct sum* of these representations, each being repeated any number of times, or omitted.

If X is a group matrix of order n^2, and T is any fixed matrix of order n^2, then $T^{-1}XT$ is also a group matrix, for

$$T^{-1}M_i T . T^{-1}M_j T = T^{-1}M_i M_j T,$$

and the matrices $T^{-1}M_i T$ are simply isomorphic with the matrices M_i.

The group matrices X and $T^{-1}XT$ are said to be *equivalent*, and the corresponding matrix representations are said to be *equivalent*.

If a group matrix X is equivalent to a matrix of the form

$$\left[\begin{array}{c|c} X_1 & Y \\ \hline 0 & X_2 \end{array}\right], \tag{4.3; 1}$$

where X_1 and X_2 are square matrices, X is said to be *reducible*.

If, also, it can be arranged so that the rectangular matrix Y in the top right-hand corner is zero as well as the rectangular matrix in the bottom left-hand corner, then X is said to be *completely reducible*.

In this case X_1 and X_2 are group matrices, and X is said to be equivalent to the *direct sum* of the group matrices X_1 and X_2.

The group matrices corresponding to the representations obtained in the Frobenius algebra are obviously *irreducible*, for as the corresponding sub-algebra is a *simple* matrix algebra, the f^2 elements of the group matrix will be linearly independent. As we shall show that the general group matrix is equivalent to a direct sum of irreducible group matrices corresponding to representations obtained in the Frobenius algebra, it will follow that for finite groups *reducibility implies complete reducibility*.

We assume therefore that in (4.3; 1), $Y = 0$, and by *reducible* we shall mean *completely reducible*.

The matrix representation corresponding to X is also said to be *reducible*, and equivalent to the *direct sum* of the representations corresponding to X_1 and X_2.

Now let $X = \sum \xi_i M_i$ be any group matrix.

If the determinant $|X|$ is identically zero, let M_0 be the matrix corresponding to the identity. Since

$$M_0^2 = M_0,$$

M_0 may be transformed into diagonal form. After transformation let

$$M_0 = \text{diag}(1^r, 0^{n-r}).$$

Since $X = M_0 X M_0$, X has now a square matrix of order r^2 in the top left-hand corner bordered by $n-r$ rows and columns of zeros. Clearly we may ignore the zeros and consider only the square matrix of order r^2. The determinant of this matrix is not identically zero, for the coefficient of ξ_0^r is unity. We assume henceforward that X is a square matrix of non-vanishing determinant, in which the coefficient of ξ_0 is the unit matrix.

Let $\epsilon_1, \epsilon_2,..., \epsilon_p$ be the moduli of the simple matrix sub-algebras of the Frobenius algebra. Let $m_1, m_2,..., m_p$ be the corresponding matrices in the representation, some of which may be identically zero.

Since $\epsilon_i^2 = \epsilon_i$ and $\epsilon_i \epsilon_j = 0$ $(i \neq j)$, we have also

$$m_i^2 = m_i, \qquad m_i m_j = 0 \quad (i \neq j).$$

It follows from § 1.8, Theorem VI, that the matrices $m_1, m_2,..., m_p$ may be transformed simultaneously into diagonal form. Let us assume them to be so transformed, and that

$$m_1 = \text{diag}(1^{a_1}, 0^{r-a_1}),$$
$$m_2 = \text{diag}(0^{a_1}, 1^{a_2}, 0^{r-a_1-a_2}),$$
$$\cdots\cdots\cdots\cdots$$

Now if x is any number of the Frobenius algebra,

$$x = \sum \epsilon_i x \epsilon_i.$$

Hence $$X = \sum m_i X m_i.$$
It follows that X is of the form

$$X = \begin{bmatrix} X_1 & & & 0 \\ & X_2 & & \\ & & \ddots & \\ 0 & & & X_p \end{bmatrix},$$

and X is equivalent to the direct sum of the group matrices X_1, $X_2,\ldots, X_p$.

We consider now one of these group matrices X_i. In this representation the matrix corresponding to ϵ_i is the unit matrix, and the matrix corresponding to ϵ_j $(j \neq i)$ is identically zero.

Let the corresponding sub-algebra of the Frobenius algebra be a simple matrix algebra of order f^2 with basal units $e_{\alpha\beta}$ $(1 \leqslant \alpha, \beta \leqslant f)$ such that
$$e_{\alpha\beta}\, e_{\beta\gamma} = e_{\alpha\gamma},$$
$$e_{\alpha\beta}\, e_{\gamma\delta} = 0 \quad (\beta \neq \gamma).$$
Let the corresponding matrices in the representation corresponding to the group matrix X_i be $\psi_{\alpha\beta}$.

Now $$\epsilon_i = e_{11}+e_{22}+\ldots+e_{ff}$$
and consequently $$I = \psi_{11}+\psi_{22}+\ldots+\psi_{ff}.$$
Also $$\psi_{jj}^2 = \psi_{jj}, \qquad \psi_{jj}\psi_{kk} = 0 \quad (j \neq k).$$
Hence the matrices $\psi_{11}, \psi_{22},\ldots, \psi_{ff}$ may be transformed simultaneously into diagonal form, and since they may be transformed into one another, they are of equal rank ν such that
$$f\nu = r.$$
We will assume that after such a transformation
$$\psi_{11} = \mathrm{diag}(1^\nu, 0^{r-\nu}),$$
$$\psi_{22} = \mathrm{diag}(0^\nu, 1^\nu, 0^{r-2\nu}),$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Since $\psi_{\alpha\beta} = \psi_{\alpha\alpha}\psi_{\alpha\beta}\psi_{\beta\beta}$, it is clear that the matrix $\psi_{\alpha\beta}$ has non-zero terms only in the rows for which $\psi_{\alpha\alpha}$ has unity in the leading diagonal, and in the columns for which $\psi_{\beta\beta}$ has unity in the leading diagonal.

The element $\sum \xi_j S_j$ of the Frobenius algebra may be expressed in the form

$$\sum \xi_j S_j = \sum \zeta_{\alpha\beta} e_{\alpha\beta} + \eta,$$

where η belongs to sub-algebras other than Γ_i. Hence

$$X_i = \sum \zeta_{ij} \psi_{ij},$$

and this may be expressed as a matrix with matrix elements

$$X_i = [\zeta_{st} A_{st}];$$

A_{st} being the ν-rowed square matrix picked out from ψ_{st}, from the rows for which ψ_{ss} has unity in the leading diagonal, and the columns for which ψ_{tt} has unity in the leading diagonal.

The matrices A_{st} satisfy

$$A_{\alpha\beta} A_{\beta\gamma} = A_{\alpha\gamma},$$
$$A_{\alpha\alpha} = I_\nu.$$

Now let $$Y = \operatorname{diag}(A_{11}, A_{21},..., A_{f1}),$$

so that $$Y^{-1} = \operatorname{diag}(A_{11}, A_{12},..., A_{1f}).$$

Then $$Y^{-1}X_i Y = [\zeta_{st} A_{1s} A_{st} A_{t1}]$$
$$= [\zeta_{st} I_\nu],$$

which by a rearrangement of rows and columns becomes

$$\begin{bmatrix} [\zeta_{st}] & & 0 & \\ & [\zeta_{st}] & & \\ & & [\zeta_{st}] & \\ & 0 & & \ddots \end{bmatrix},$$

and X_i is equivalent to the representation of the simple matrix algebra Γ_i repeated ν times.

Any group matrix X is equivalent to a direct sum of the irreducible group matrices which correspond to the simple matrix sub-algebras of the Frobenius algebra, each of these group matrices being repeated any number of times, or omitted.

The spur of X is a linear function of the ξ's, say $\sum \xi_j \phi(S_j)$. The spur of the irreducible group-matrix corresponding to Γ_i is $\sum \xi_j \chi^{(i)}(S_j)$.

Hence $$\phi(S_j) = \sum \nu_i \chi^{(i)}(S_j).$$

The coefficient ν_i, being the number of times the representation is repeated, is a positive integer.

Any linear function of the characters with positive integral coefficients is called a *compound character.*

The spurs of the matrices in any matrix representation of the group is a compound character of the group, the coefficient of any simple character being the number of times the corresponding representation is repeated in the equivalent direct sum of irreducible representations.

Given the compound character $\phi(S_j)$, the coefficients of the simple characters may be found as follows.

Let
$$\phi(S_j) = \sum \nu_i \chi^{(i)}(S_j).$$
Then
$$\sum \chi^{(i)}(S_j^{-1})\phi(S_j) = \sum \nu_k \chi^{(i)}(S_j^{-1})\chi^{(k)}(S_j) = h\nu_i,$$
so that
$$\nu_i = \frac{1}{h}\sum \chi^{(i)}(S_j^{-1})\phi(S_j) = \frac{1}{h}\sum h_\rho \chi_{\rho'}^{(i)} \phi_\rho. \qquad (4.3; 2)$$

Given any group H of order h, there is a regular permutation representation of degree h (§ 3.8) which may be regarded as a group of linear substitutions, and a corresponding representation by permutation matrices. The corresponding group matrix is called the *regular group matrix*. The spur of the regular group matrix is clearly $h\xi_0$. For the corresponding compound character ϕ we have
$$\phi(S_0) = h,$$
$$\phi(S_i) = 0 \quad (i \neq 0).$$
If $\phi = \sum \nu_i \chi^{(i)}$, then
$$h\nu_i = \sum h_\rho \chi_{\rho'}^{(i)} \phi_\rho = h\chi_0^{(i)} = hf^{(i)}.$$

The regular group matrix is equivalent to the direct sum of all the irreducible group matrices, each being repeated as many times as the degree of the character.

Schur (**2**) has developed the theory of group matrices from a standpoint independent of the Frobenius algebra. He defines a group character as the set of spurs of the matrices in a matrix representation. He proves the orthogonal properties directly, and obtains the complete set of independent irreducible representations by the reduction of the regular group matrix, proving the completeness of the set so obtained by means of the orthogonal relations.

It is necessary in this development to prove specifically that reducibility implies complete reducibility, a theorem which is effectively equivalent to the theorem that a Frobenius algebra possesses no properly nilpotent element.

His method is capable of extension to continuous matrix groups, to which the method of the Frobenius algebra is not. An account is given in Chapter XI. The proof given there that reducibility implies complete reducibility is applicable to finite groups. We give now Schur's proof of the orthogonal properties for finite groups.

Let $X = \sum \xi_i M_i = [\sum \xi_i a^i_{st}]$ be an irreducible group matrix of order f^2. Let $X' = \sum \xi_i M'_i = [\sum \xi_i a^{i'}_{st}]$ be any independent irreducible group matrix of order $\leqslant f^2$. We suppose it to be bordered by rows and columns of zeros if the order is $< f^2$.

Firstly, if P is a constant matrix such that $PX = XP$, then P is a scalar multiple of the unit matrix. If P is not of this form, then, transforming the matrices so that P is in canonical form, it is immediately apparent that X is simultaneously reduced, contrary to hypothesis.

Secondly, if P is a constant matrix such that $PX = X'P$, then P is non-singular and X and X' are equivalent.

If P is non-singular, then X and X' are equivalent, since

$$X = P^{-1}X'P.$$

If P is singular, let its rank be r. Then there is a matrix Q of rank $f-r$ such that

$$PQ = 0.$$

Thus

$$PXQ = X'PQ = 0.$$

Now if P is a singular matrix, we can find a non-singular matrix A such that $P_1 = AP$ is idempotent. Transform P first into canonical form. For each submatrix in the canonical form corresponding to a non-zero characteristic root there is a reciprocal. For a zero characteristic root we may multiply the submatrix by a permutation matrix which will reduce it to the form $\text{diag}(0, 1^{\nu-1})$. The result follows readily. Similarly, we may find a non-singular matrix B such that $Q_1 = QB$ is idempotent.

Then

$$P_1 X Q_1 = 0,$$

and transforming P_1 and Q_1 simultaneously into diagonal form, X is thereby reduced, contrary to hypothesis.

Let $U = [u_{st}]$ be a fixed matrix of order f^2, and let

$$V = \sum M_i^{-1} U M_i, \qquad (4.3;3)$$

$$V_1 = \sum M_i'^{-1} U M_i, \qquad (4.3;4)$$

each summed for all operations of the group.

Now
$$M_j^{-1} V M_j = \sum M_j^{-1} M_i^{-1} U M_i M_j = V,$$

since, as M_i runs through all the operations of the group, so does $M_i M_j$. Thus V commutes with each of the matrices M_i, and hence with the group matrix X. Since X is irreducible, V is a scalar multiple of the modulus.

In a similar manner

$$M_j'^{-1} V_1 M_j = \sum M_j'^{-1} M_i'^{-1} U M_i M_j = V_1,$$

$$V_1 M_j = M_j' V_1,$$

and hence
$$V_1 = 0,$$

so that
$$\sum M_i'^{-1} U M_i = 0. \qquad (4.3;5)$$

Let $M_i^{-1} = [a_{st}^i]^{-1} = [\alpha_{st}^i]$, and let $\delta_{ii} = 1$, $\delta_{ij} = 0$, $i \neq j$. Then equation (4.3; 3) may be written

$$\sum [\alpha_{st}^i][u_{st}][a_{st}^i] = k[\delta_{st}],$$

so that
$$\sum_{iqr} \alpha_{pq}^{(i)} u_{qr} a_{rv}^{(i)} = k\delta_{pv},$$

where k is a scalar.

Put $p = v$, and sum for all p. Since

$$\sum a_{rp}^{(i)} \alpha_{pq}^{(i)} = \delta_{rq},$$

we have
$$h \sum u_{qq} = kf,$$

so that
$$k = \frac{h}{f} \sum u_{qq}.$$

Hence
$$\sum_{iqr} \alpha_{pq}^{(i)} u_{qr} a_{rv}^{(i)} = \frac{h}{f} \sum u_{qq} \delta_{pv}.$$

Equating coefficients of u_{qr} we have

$$\sum_i \alpha_{pq}^{(i)} a_{rv}^{(i)} = \frac{h}{f} \delta_{pv} \delta_{qr}. \qquad (4.3;6)$$

Hence if χ is the character of the group corresponding to the representation, so that
$$\chi(S_i) = \sum_q a_{qq}^{(i)},$$

then
$$\sum_i \chi(S_i)\chi(S_i^{-1}) = \sum_{iqr} \alpha_{qq}^{(i)} a_{rr}^{(i)} = \sum \frac{h}{f} \delta_{qr} \delta_{qr} = h. \qquad (4.3;7)$$

Similarly, from the equation

$$\sum M_i'^{-1} U M_i = 0$$

we obtain

$$\sum a_{pq}^{(i)} \alpha_{rv}^{(i)'} = 0 \tag{4.3; 8}$$

and

$$\sum_i \chi(S_i)\chi'(S_i^{-1}) = 0. \tag{4.3; 9}$$

A generalization of the orthogonal relations is easily obtained by this method. We have

$$\sum \chi(S_i)\chi(S_i^{-1}S_j) = \sum a_{pp}^{(i)} \alpha_{qr}^{(i)} a_{rq}^{(j)} = \sum \frac{h}{f} a_{pp}^{(j)} = \frac{h}{f}\chi(S_j). \tag{4.3; 10}$$

Similarly,

$$\sum \chi(S_i)\chi'(S_i^{-1}S_j) = 0. \tag{4.3; 11}$$

4.4. Characteristic units

An idempotent element A of the Frobenius algebra of a group is called a *characteristic unit* (**3**).

Let $A = \sum A_i$, where A_i is an element of the simple matrix algebra Γ_i. If the rank of A_i is r_i, it can be transformed into the form

$$A_i = \text{diag}(1^{r_i}, 0^{f-r_i}).$$

The characteristic unit A is associated with the compound character

$$\phi = \sum r_i \chi^{(i)}.$$

A characteristic unit associated with a simple character $\chi^{(i)}$ is called a *primitive characteristic unit.* It is a primitive idempotent of the sub-algebra Γ_i.

I. *A primitive characteristic unit corresponding to the character* $\chi^{(i)}$ *has an aggregate of* $h_\rho \chi_\rho^{(i)}/h$ *elements from the class* C_ρ.

Since all primitive idempotents of the same simple matrix algebra have the same canonical form, they are transforms of one another, and have the same aggregate of elements from each class. But the modulus of the sub-algebra, ϵ_i, may be expressed as the sum of $f^{(i)}$ primitive idempotents, and from the equation

$$h\epsilon_i = \sum f^{(i)} \chi_\rho^{(i)} C_\rho$$

the theorem follows.

Clearly, from the representation in diagonal form, any characteristic unit may be expressed as a sum of primitive characteristic units. The corresponding compound character will be the sum of the corresponding simple characters.

II. *A characteristic unit corresponding to any character* ϕ, *simple or compound, has an aggregate of* $h_\rho \phi_\rho/h$ *elements from the class* C_ρ.

III. *Two characteristic units corresponding to the same character ϕ are transforms of one another.*

They have the same canonical form in each sub-algebra.

IV. *The product of two primitive characteristic units is either zero, nilpotent, or a multiple of a primitive characteristic unit.*

If the characteristic units belong to different sub-algebras, then the product is zero. If they belong to the same sub-algebra, the rank of each of the two corresponding matrices is unity. If the product is not zero, it must be of rank unity also. Its reduced characteristic equation must be a quadratic with one zero root, i.e.

$$x^2-\lambda x = 0.$$

If $\lambda = 0$, it is nilpotent. If $\lambda \neq 0$, it is a multiple of an idempotent.

4.5. The relations between the characters of a group and those of a subgroup

Let the group H of order h have a subgroup G of order g. Any matrix representation of the group H is clearly also a matrix representation of the group G, for the matrices corresponding to the subgroup G are a subset of the matrices corresponding to the group H.

By taking the spurs of the matrices of a simple matrix representation of H, it follows that every simple character of H is a character, simple or compound, of G.

Let $\chi^{(i)}$ represent any simple character of H, and $\phi^{(j)}$ any simple character of G. Then

$$\chi^{(i)} = \sum g_{ij}\,\phi^{(j)}. \tag{4.5; 1}$$

Now let C_ρ be any class of H, of order h_ρ, and let g_ρ of the elements of this class belong to G, forming the classes C_{ρ_1}, C_{ρ_2},... of G, of orders g_{ρ_1}, g_{ρ_2},... respectively.

Equation (4.5; 1) holds for each of these classes, so that

$$\chi^{(i)}_\rho = \sum g_{ij}\,\phi^{(j)}_{\rho_1} = \sum g_{ij}\,\phi^{(j)}_{\rho_2} = \dots .$$

We thus have

$$\sum g_{ij}\,g_{\rho_1}\phi^{(j)}_{\rho_1}\,\phi^{(k)}_{\rho_1'} = g_{\rho_1}\chi^{(i)}_\rho\,\phi^{(k)}_{\rho_1'},$$

$C_{\rho_1'}$ denoting the class of the inverses of the elements of the class C_{ρ_1}.

Summing with respect to all the classes of G, we obtain

$$gg_{ik} = \sum g_{\rho_1}\chi^{(i)}_\rho\phi^{(k)}_{\rho_1'}; \tag{4.5; 2}$$

the right-hand side being summed for all the classes of G, C_ρ being the class of H which contains the elements of the class C_{ρ_1}.

Hence
$$\sum_i g_{ik} g h_\sigma \chi_{\sigma'}^{(i)} = \sum g_{\rho_1} \phi_{\rho_1'}^{(k)} h_\sigma \chi_\rho^{(i)} \chi_{\sigma'}^{(i)}$$
$$= h \sum g_{\sigma_1} \phi_{\sigma_1'}^{(k)},$$
the last summation being confined to the classes C_{σ_1} of G which correspond to the given class C_σ of H. Thus
$$\sum_i g_{ik} \chi_{\rho'}^{(i)} = \sum \frac{h g_{\rho_1}}{g h_\rho} \phi_{\rho_1'}^{(k)}.$$

Taking the inverse classes, and combining the two equations, we obtain Frobenius's formulae (**4**) expressing the relations between the characters of a group and those of a subgroup.
$$\left.\begin{aligned} \chi_\rho^{(i)} &= \sum_j g_{ij} \phi_{\rho_1}^{(j)}, \\ \sum_i g_{ij} \chi_\rho^{(i)} &= \sum \frac{h g_{\rho_1}}{g h_\rho} \phi_{\rho_1}^{(j)}, \end{aligned}\right\} \qquad (4.5;3)$$
the last summation being with respect to those classes C_{ρ_1}, $C_{\rho_2}, \ldots$ of G into which the class C_ρ of H separates. The coefficients g_{ij} are positive integers and are the same for both equations.

The fact that $\sum h g_{\rho_1} \phi_{\rho_1}^{(j)} / g h_\rho$ is a compound character of H, i.e. a linear function of the characters of H with *positive integral* coefficients, may be deduced otherwise from the concept of characteristic units.

A characteristic unit of a subgroup G is necessarily a characteristic unit of H, as it is idempotent, and the group algebra of G is a subalgebra of the group algebra of H.

But a characteristic unit of G corresponding to the character $\phi^{(j)}$ has an aggregate of $g_{\rho_1} \phi_{\rho_1}^{(j)} / g$ elements from the class C_{ρ_1} of G, and a characteristic unit of H corresponding to the character $\chi^{(i)}$ has an aggregate of $h_\rho \chi_\rho^{(i)} / h$ elements from the class C_ρ.

Hence
$$\sum g_{\rho_1} \phi_{\rho_1}^{(j)} / g = \sum h_\rho \chi_\rho^{(i)} / h,$$
the summation on the left being with respect to the classes C_{ρ_1} of G which correspond to the same class C_ρ of H, and the summation on the right being with respect to the simple characteristic units of H of which the given characteristic unit of G is the sum. The result follows.