CHAPTER ONE

Valuations of a Field

A *valuation* of a field k is a real-valued function $|x|$, defined for all $x \in k$, satisfying the following requirements:

(1) $|x| \geqslant 0$; $|x| = 0$ if and only if $x = 0$,

(2) $|xy| = |x|\,|y|$,

(3) If $|x| \leqslant 1$, then $|1 + x| \leqslant c$, where c is a constant; $c \geqslant 1$.

(1) and (2) together imply that a valuation is a homomorphism of the multiplicative group k^* of non-zero elements of k into the positive real numbers.

If this homomorphism is trivial, i.e. if $|x| = 1$ for all $x \in k^*$, the valuation is also called *trivial.*

1. Equivalent Valuations

Let $|\ |_1$ and $|\ |_2$ be two functions satisfying conditions (1) and (2) above; suppose that $|\ |_1$ is non-trivial. These functions are said to be *equivalent* if $|a|_1 < 1$ implies $|a|_2 < 1$. Obviously for such functions $|a|_1 > 1$ implies $|a|_2 > 1$; but we can prove more.

Theorem 1: Let $|\ |_1$ and $|\ |_2$ be equivalent functions, and suppose $|\ |_1$ is non-trivial. Then $|a|_1 = 1$ implies $|a|_2 = 1$.

Proof: Let $b \neq 0$ be such that $|b|_1 < 1$. Then $|a^n b|_1 < 1$; whence $|a^n b|_2 < 1$, and so $|a|_2 < |b|_2^{-1/n}$. Letting $n \to \infty$, we have $|a|_2 \leqslant 1$. Similarly, replacing a in this argument by $1/a$, we have $|a|_2 \geqslant 1$, which proves the theorem.

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Corollary: For non-trivial functions of this type, the relation of equivalence is reflexive, symmetric and transitive.

There is a simple relation between equivalent functions, given by

Theorem 2: If $|\ |_1$ and $|\ |_2$ are equivalent functions, and $|\ |_1$ is non-trivial, then $|a|_2 = |a|_1^{\alpha}$ for all $a \in k$, where α is a fixed positive real number.

Proof: Since $|\ |_1$ is non-trivial, we can select an element $c \in k^*$ such that $|c|_1 > 1$; then $|c|_2 > 1$ also.

Set $|a|_1 = |c|_1^{\gamma}$, where γ is a non-negative real number. If $m/n > \gamma$, then $|a|_1 < |c|_1^{m/n}$, whence $|a^n/c^m|_1 < 1$. Then $|a^n/c^m|_2 < 1$, from which we deduce that $|a|_2 < |c|_2^{m/n}$. Similarly, if $m/n < \gamma$, then $|a|_2 > |c|^{m/n}$. It follows that $|a|_2 = |c|_2^{\gamma}$. Now, clearly,

$$\gamma = \frac{\log |a|_1}{\log |c|_1} = \frac{\log |a|_2}{\log |c|_2}.$$

This proves the theorem, with

$$\alpha = \frac{\log |c|_2}{\log |c|_1}.$$

In view of this result, let us agree that the equivalence class defined by the trivial function shall consist of this function alone.

Our third condition for valuations has replaced the classical "Triangular Inequality" condition, viz., $|a + b| \leqslant |a| + |b|$. The connection between this condition and ours is given by

Theorem 3: Every valuation is equivalent to a valuation for which the triangular inequality holds.

Proof. (1) When the constant $c = 2$, we shall show that the triangular inequality holds for the valuation itself.

Let $|a| \leqslant |b|$.

Then

$$\left|\frac{a}{b}\right| \leqslant 1 \Rightarrow \left|1 + \frac{a}{b}\right| \leqslant 2 \Rightarrow |a + b| \leqslant 2|b| = 2\max(|a|, |b|).$$

Similarly

$$|a_1 + a_2 + a_3 + a_4| \leqslant 4\max(|a_1|, \cdots, |a_4|)$$

and

$$| a_1 + \cdots + a_{2^r} | \leqslant 2^r \max (| a_1 |, \cdots, | a_{2^r} |) .$$

Now given $a_1, a_2, \cdots, a_n$, we can find an integer r such that $n \leqslant 2^r < 2n$. Hence

$$\begin{aligned} | a_1 + \cdots + a_n | &= | a_1 + \cdots + a_n + 0 + \cdots + 0 | \\ &\leqslant 2^r \max_\nu | a_\nu | < 2n \max_\nu | a_\nu | . \end{aligned}$$

In particular, if we set all the $a_\nu = 1$, we have $| n | \leqslant 2n$. We may also weaken the above inequality, and write

$$| a_1 + \cdots + a_n | \leqslant 2n(| a_1 | + \cdots + | a_n |) .$$

Consider

$$\begin{aligned} | a + b |^n &= \left| a^n + \binom{n}{1} a^{n-1}b + \cdots + b^n \right| \\ &\leqslant 2(n+1) \left\{ | 1 | \, | a |^n + \left| \binom{n}{1} \right| \, | a |^{n-1} | b | + \cdots + | b |^n \right\} \\ &\leqslant 4(n+1) \left\{ | a |^n + \binom{n}{1} | a |^{n-1} | b | + \cdots + | b |^n \right\} \\ &= 4(n+1) \{ | a | + | b | \}^n . \end{aligned}$$

Hence

$$| a + b | \leqslant \sqrt[n]{4(n+1)} \, (| a | + | b |) .$$

Letting $n \to \infty$ we obtain the desired result.

We may note that, conversely, the triangular inequality implies that our third requirement is satisfied, and that we may choose $c = 2$.

(2) When $c > 2$, we may write $c = 2^\alpha$. Then it is easily verified that $| \ |^{1/\alpha}$ is an equivalent valuation for which the triangular inequality is satisfied.

2. The Topology Induced by a Valuation

Let $| \ |$ be a function satisfying the axioms (1) and (2) for valuations. In terms of this function we may define a topology in k by prescribing the fundamental system of neighborhoods of each element $x_0 \in k$ to be the sets of elements x such that $| x - x_0 | < \epsilon$. It is clear that equivalent functions induce the same topology in k, and that the trivial function induces the discrete topology.

There is an intimate connection between our third axiom for valuations and the topology induced in k.

Theorem 4: The topology induced by $| \ |$ is Hausdorff if and only if axiom (3) is satisfied.

Proof: (1) If the topology is Hausdorff, there exist neighborhoods separating 0 and -1. Thus we can find real numbers a and b such that if $| x | \leqslant a$, then $| 1 + x | \geqslant b$.

Now let x be any element with $| x | \leqslant 1$; then either $| 1 + x | \leqslant 1/a$ or $| 1 + x | > 1/a$. In the latter case, set $y = - x/(1 + x)$; then

$$| y | = \frac{| x |}{| 1 + x |} \leqslant \frac{1}{a^{-1}} = a ;$$

hence

$$| 1 + y | = \left| \frac{1}{1 + x} \right| \geqslant b ,$$

i.e. $| 1 + x | \leqslant 1/b$. We conclude, therefore, that if $| x | \leqslant 1$, then $| 1 + x | \leqslant \max (1/a, 1/b)$, which is axiom (3).

(2) The converse is obvious if we replace $| \ |$ by the equivalent function for which the triangular inequality holds.

It should be remarked that the field operations are continuous in the topology induced on k by a valuation.

3. Classification of Valuations

If the constant c of axiom (3) can be chosen to be 1, i.e. if $| x | \leqslant 1$ implies $| 1 + x | \leqslant 1$, then the valuation is said to be *non-archimedean*. Otherwise the valuation is called *archimedean*. Obviously the valuations of an equivalence class are either all archimedean or all non-archimedean. For nonarchimedean valuations we obtain a sharpening of the triangular inequality:

Theorem 5: For non-archimedean valuations,

$$| a + b | \leqslant \max (| a | , | b |) .$$

Proof. Let $| a | \leqslant | b |$; then $| a/b | \leqslant 1$. It follows that $| 1 + a/b | \leqslant 1$, whence

$$| a + b | \leqslant | b | = \max (| a | , | b |) .$$

Corollary 5.1: If $| a | < | b |$, then $| a + b | = | b |$.

Proof:

$$| b | = | - a + (a - b) | \leqslant \max (| a | , | a + b |) ,$$

by the Theorem. By hypothesis, $| b |$ is not $\leqslant | a |$, so that we have $| b | \leqslant | a + b |$. But using the theorem again we have

$$| a + b | \leqslant \max (| a | , | b |) = | b | .$$

Thus if $| a | < | b |$, then $| a + b | = | b |$.

We notice that this equality does not necessarily hold when $| a | = | b |$; for example, if $b = - a$, we have $| a + b | = 0 < | a |$. In general we have

$$| a_1 + \cdots + a_n | \leqslant | a_1 | ,$$

where $| a_1 | = \max_\nu | a_\nu |$; and

$$| a_1 + \cdots + a_n | = | a_1 |$$

if, for every $\nu > 1$, $| a_\nu | < | a_1 |$. This last result is frequently used in the following form:

Corollary 5.2: Suppose it is known that $| a_\nu | \leqslant | a_1 |$ for all ν, and that $| a_1 + \cdots + a_n | < | a_1 |$. Then for some $\nu > 1$, $| a_\nu | = | a_1 |$.

We now give a necessary and sufficient condition for a field to be non-archimedean:

Theorem 6: A valuation is non-archimedean if and only if the values of the rational integers are bounded.

Proof: (1) The necessity of the condition is obvious, for if the valuation is non-archimedean, then

$$| m | = | 1 + 1 + \cdots + 1 | \leqslant | 1 | .$$

(2) To prove the sufficiency of the condition we consider the equivalent valuation for which the triangular inequality is satisfied. Obviously the values of the integers are bounded in this valuation also; say $| m | \leqslant D$. Consider

$$\begin{aligned} | a + b |^n = | (a + b)^n | &= \left| a^n + \binom{n}{1} a^{n-1}b + \cdots + b^n \right| \\ &\leqslant | 1 | | a^n | + \left| \binom{n}{1} \right| | a |^{n-1} | b | + \cdots + | 1 | | b |^n \\ &\leqslant D\{| a |^n + | a |^{n-1} | b | + \cdots + | b |^n\} \\ &\leqslant D(n + 1) \{\max (| a | , | b |)\}^n . \end{aligned}$$

Hence

$$| a + b | \leqslant \sqrt[n]{D(n + 1)} \max (| a | , | b |) .$$

Letting $n \to \infty$, we have the desired result.

Corollary: A valuation of a field of characteristic $p > 0$ is non-archimedean.

We may remark that if k_0 is a subfield of k, then a valuation of k is (non-)archimedean on k_0 is (non-)archimedean on the whole of k. In particular, if the valuation is trivial on k_0, it is non-archimedean on k.

4. The Approximation Theorem

Let $\{a_n\}$ be a sequence of elements of k; we say that a is the limit of this sequence with respect to the valuation if

$$\lim_{n \to \infty} | a - a_n | = 0 .$$

The following examples will be immediately useful:

(a) If $| a | < 1$, then

$$\lim_{n \to \infty} a^n = 0 .$$

For $| a^n - 0 | = | a |^n \to 0$ as $n \to \infty$.

(b) If $| a | < 1$, then

$$\lim_{n \to \infty} \frac{a^n}{1 + a^n} = 0.$$

(c) If $| a | > 1$, then

$$\lim_{n \to \infty} \frac{a^n}{1 + a^n} = 1.$$

For

$$\left| \frac{a^n}{1 + a^n} - 1 \right| = \left| \frac{1}{1 + a^n} \right| = \left| \frac{\left(\frac{1}{a}\right)^n}{1 + \left(\frac{1}{a}\right)^n} \right| \to 0 \quad \text{as} \quad n \to \infty .$$

We now examine the possibility of finding a relation between non-equivalent valuations; we shall show that if the number of valuations considered is finite, no relation of a certain simple type is possible.

Theorem 7: Let $| \ |_1, \cdots, | \ |_n$ be a finite number of inequivalent non-trivial valuations of k. Then there is an element $a \in k$ such that $| a |_1 > 1$, and $| a |_\nu < 1$ $(\nu = 2, \cdots, n)$.

Proof. First let $n = 2$. Then since $| \ |_1$ and $| \ |_2$ are nonequivalent, there certainly exist elements b, $c \in k$ such that $| b |_1 < 1$ and $| b |_2 \geqslant 1$, while $| c |_1 \geqslant 1$ and $| c |_2 < 1$. Then $a = c/b$ has the required properties.

The proof now proceeds by induction. Suppose the theorem is true for $n - 1$ valuations; then there is an element $b \in k$ such that $| b |_1 > 1$, and $| b |_\nu < 1$ $(\nu = 2, \cdots, n - 1)$. Let c be an element such that $| c |_1 > 1$ and $| c |_n < 1$. We have two cases to consider:

Case 1: $| b |_n \leqslant 1$. Consider the sequence $a_r = cb^r$. Then $| a_r |_1 = | c |_1 | b |_1^r > 1$, while $| a_r |_n = | c |_n | b |_n^r < 1$; for sufficiently large r, $| a_r |_\nu = | c |_\nu | b |_\nu^r < 1$ $(\nu = 2, \cdots, n - 1)$. Thus a_r is a suitable element, and the theorem is proved in this case.

Case 2: $| b |_n > 1$. Here we consider the sequence

$$a_r = \frac{cb^r}{1 + b^r} .$$

This sequence converges to the limit c in the topologies induced by $| \ |_1$ and $| \ |_n$. Thus $a_r = c + \eta_r$ where $| \eta_r |_1$ and $| \eta_r |_n \to 0$ as $r \to \infty$. Hence for r large enough, $| a_r |_1 > 1$ and $| a_r |_n < 1$.

For $\nu = 2, \cdots, n-1$, the sequence a_r converges to the limit 0 in the topology induced by $| \ |_\nu$. Hence for large enough values of r, $| a_r |_\nu < 1$ ($\nu = 2, \cdots, n-1$). Thus a_r is a suitable element, for r large enough, and the theorem is proved in this case also.

Corollary: With the conditions of the theorem, there exists an element a which is close to 1 in $| \ |_1$ and close to 0 in $| \ |_\nu$ ($\nu = 2, \cdots, n-1$).

Proof. If b is an element such that $| b |_1 > 1$ and $| b |_\nu < 1$ ($\nu = 2, \cdots, n-1$), then $a_r = b^r/(1 + b^r)$ satisfies our requirements for large enough values of r.

Theorem 8: (*The Approximation Theorem*): Let $| \ |_1, \cdots, | \ |_n$ be a finite number of non-trivial inequivalent valuations. Given any $\epsilon > 0$, and any elements a_ν ($\nu = 1, \cdots, n$), there exists an element a such that $| a - a_\nu |_\nu < \epsilon$.

Proof. We can find elements b_i ($i = 1, \cdots, n$) close to 1 in $| \ |_i$ and close to zero in $| \ |_\nu$ ($\nu \neq i$).

Then $a = a_1 b_1 + \cdots + a_n b_n$ is the required element.

Let us denote by $(k)_i$ the field k with the topology of $| \ |_i$ imposed upon it. Consider the Cartesian product $(k)_1 \times (k)_2 \times \cdots \times (k)_n$. The elements $(a, a, \cdots, a)$ of the diagonal form a field k_D isomorphic to k. The Approximation Theorem states that k_D is everywhere dense in the product space. The theorem shows clearly the impossibility of finding a non-trivial relation of the type

$$\prod_{\nu=1}^{n} | x |_\nu^{c_\nu} = 1 ,$$

with real constants c_ν.

5. Examples

Let k be the quotient field of an integral domain $\mathfrak{o}$; then it is easily verified that a valuation $| \ |$ of k induces a function $\mathfrak{o}$ (which we still denote by $| \ |$), satisfying the conditions

(1) $|a| \geqslant 0$; $|a| = 0$ if and only if $a = 0$,

(2) $|ab| = |a|\,|b|$,

(3) $|a + b| \leqslant 0 \max(|a|, |b|)$.

Suppose, conversely, that we are given such a function on $\mathfrak{o}$. Then if $x = a/b$ ($a, b \in \mathfrak{o}$, $x \in k$), we may define $|x| = |a|/|b|$; $|x|$ is well-defined on k, and obviously satisfies our axioms (1) and (2) for valuations. To show that axiom (3) is also satisfied, let $|x| \leqslant 1$, i.e. $|a| \leqslant |b|$. Then

$$|1 + x| = \frac{|a + b|}{|b|} \leqslant \frac{e \max(|a|, |b|)}{|b|} = e\,.$$

Hence if k is the quotient field of an integral domain $\mathfrak{o}$, the valuations of k are sufficiently described by their actions on $\mathfrak{o}$.

First Example: Let $k = R$, the field of rational numbers; k is then the quotient field of the ring of integers $\mathfrak{o}$.

Let m, n be integers > 1, and write m in the n-adic scale:

$$m = a_0 + a_1 n + \cdots + a_r n^r$$

$$\left(0 \leqslant a_i < n;\ n^r \leqslant m, \quad \text{i.e.} \quad r \leqslant \frac{\log m}{\log n}\right).$$

Let $|\ |$ be a valuation of R; suppose $|\ |$ replaced, if necessary, by the equivalent valuation for which the triangular inequality holds. Then $|a_\nu| < n$, and we have

$$|m| \leqslant \left(\frac{\log m}{\log n} + 1\right) n \cdot \{\max(1, |n|)\}^{\log m/\log n}.$$

Using this estimate for $|m|^s$, extracting the sth root, and letting $s \to \infty$, we have

$$|m| \leqslant \{\max(1, |n|)\}^{\log m/\log n}.$$

There are now two cases to consider.

Case 1: $|n| > 1$ for all $n > 1$. Then

$$|m| \leqslant |n|^{\log m/\log n} : |m|^{1/\log m} \leqslant |n|^{1/\log n}.$$

Since $|m| > 1$ also, we may interchange the roles of m and n, obtaining the reversed inequality. Hence

$$|m|^{1/\log m} = |n|^{1/\log n} = e^\alpha,$$

where α is a positive real number. It follows that

$$| n | = e^{\alpha \log n} = n^{\alpha},$$

so that $| \ |$ is in this case equivalent to the ordinary "absolute value", $| x | = \max (x, - x)$.

Case 2: There exists an integer $n > 1$ such that $| n | \leqslant 1$. Then $| m | \leqslant 1$ for all $m \in \mathfrak{o}$. If we exclude the trivial valuation, we must have $| n | < 1$ for some $n \in \mathfrak{o}$; clearly the set of all such integers n forms an ideal (p) of $\mathfrak{o}$. The generator of this ideal is a prime number; for if $p = p_1 p_2$, we have $| p | = | p_1 | \, | p_2 | < 1$, and hence (say) $| p_1 | < 1$. This $p_1 \in (p)$, i.e. p divides p_1; but p_1 divides p; hence p is a prime. If $| p | = c$, and $n = p^{\nu} b$, $(p, b) = 1$, then $| n | = c^{\nu}$. Every non-archimedean valuation is therefore defined by a prime number p.

Conversely, let p be a prime number in $\mathfrak{o}$, c a constant, $0 < c < 1$. Let $n = p^{\nu} b$, $(p, b) = 1$, and define the function $| \ |$ by setting $| n | = c^{\nu}$. It is easily seen that this function satisfies the three conditions for such functions on $\mathfrak{o}$, and hence leads to a valuation on R. This valuation can be described as follows: let x be a non-zero rational number, and write it as $x = p^{\nu} y$, where the numerator and denominator are prime to p. Then $| x | = c^{\nu}$.

Second Example. Let k be the field of rational functions over a field F: $k = F(x)$. Then k is the quotient field of the ring of polynomials $\mathfrak{o} = F[x]$. Let $| \ |$ be a valuation of k which is trivial on F; $| \ |$ will thus be non-archimedean. We have again two cases to consider.

Case 1: $| x | > 1$. Then if

$$f(x) = c_0 + c_1 x + \cdots + c_n x^n ,$$

$c_n \neq 0$, we have

$$| f(x) | = | x |^n = | x |^{\deg f(x)} .$$

Conversely, if we select a number $c > 1$ and set

$$| f(x) | = c^{\deg f(x)} ,$$

our conditions for functions on $\mathfrak{o}$ are easily verified. Hence this function yields a valuation of k described as follows: let $a = f(x)/g(x)$, and define

$$\deg a = \deg f(x) - \deg g(x)\,.$$

Then $|a| = c^{\deg a}$. Obviously the different choices of c lead only to equivalent valuations.

Case 2: $|x| < 1$. Then for any $f(x) \in \mathfrak{o}$, $|f(x)| \leqslant 1$. If we exclude the trivial valuation, we must have $|f(x)| < 1$ for some $f(x) \in \mathfrak{o}$. As in the first example, the set of all such polynomials is an ideal, generated by an irreducible polynomial $p(x)$. If $|p(x)| = c$, and $f(x) = (p(x))^\nu g(x)$, $(p(x), g(x)) = 1$, then $|f(x)| = c^\nu$.

Conversely, if $p(x)$ is an irreducible polynomial, it defines a valuation of this type. This is shown in exactly the same way as in the first example.

In both cases, the field $k = F(x)$ and the field $k = R$ of the rational numbers, we have found equivalence classes of valuations, one to each prime p (in the case of $F(x)$, one to each irreducible polynomial) with one exception, an equivalence class which does not come from a prime. To remove this exception we introduce in both cases a "symbolic" prime, the so-called *infinite prime*, p_∞ which we associate with the exceptional equivalence class. So $|a|_{p_\infty}$ stands for the ordinary absolute value in the case $k = R$, and for $c^{\deg a}$ in the case $k = F(x)$. We shall now make a definite choice of the constant c entering in the definition of the valuation associated with a prime p.

(I) $k = R$. (a) $p \neq p_\infty$. We choose $c = 1/p$. If, therefore, $a \neq 0$, and $a = p^\nu b$, where the numerator and denominator of b are prime to p, then we write $|a|_p = (1/p)^\nu$. The exponent ν is called the ordinal of a at p and is denoted by $\nu = \text{ord}_p\, a$.

(b) $p = p_\infty$. Then let $|a|_{p\infty}$ denote the ordinary absolute value.

(II) $k = F(x)$. Select a fixed number d, $0 < d < 1$.

(a) $p \neq p_\infty$, so that p is an irreducible polynomial; write $c = d^{\deg p}$. If $a \neq 0$ we write as in case I(a), $a = p^\nu b$, $\nu = \text{ord}_\nu\, a$, and so we define $|a|_p = c^\nu = d^{\deg p \cdot \text{ord}_p a}$.

(b) $p = p_\infty$, so that $|a| = c^{\deg a}$. where $c > 1$. We choose $c = 1/d$, and so define $|a|_{p_\infty} = d^{-\deg a}$.

In all cases we have made a definite choice of $|a|_p$ in the equivalence class corresponding to p; we call this $|a|_p$ the *normal valuation* at p.

The case where $k = F(x)$, where F is the field of all complex numbers, can be generalized as follows. Let D be a domain on the Gauss sphere and k the field of all functions meromorphic in D. If $x_0 \in D$, $x_0 \neq \infty$, and $f(x) \in k$, we may write

$$f(x) = (x - x_0)^{\operatorname{ord}_{x_0} f(x)}\, g(x),$$

where $g(x) \in k$ ($g(x_0) \neq 0$ or ∞), and define a valuation by

$$|f(x)|_{x_0} = c^{\operatorname{ord}_{x_0} f(x)}.$$

If $x_0 = \infty$, we write

$$f(x) = \left(\frac{1}{x}\right)^{\operatorname{ord}_\infty f(x)} g(x)\,,$$

where $g(x) \in k$ ($g(\infty) \neq 0$ or ∞), and define

$$|f(x)|_\infty = c^{\operatorname{ord}_\infty f(x)}\,.$$

This gives for each $x_0 \in D$ a valuation of k — axioms (1) and (2) are obviously satisfied, while axiom (3) follows from

$$\begin{aligned} |f|_{x_0} \leqslant 1 &\Rightarrow f \text{ is regular at } x_0 \\ &\Rightarrow 1 + f \text{ is regular at } x_0 \\ &\Rightarrow |1 + f|_{x_0} \leqslant 1. \end{aligned}$$

The valuation $|\ |_{x_0}$ obviously describes the behavior of $f(x)$ at the point x_0:

If $|f|_{x_0} < 1$, or $\operatorname{ord}_{x_0}(f(x)) = n > 0$, then $f(x)$ has a zero of order n at x_0.

If $|f|_{x_0} > 1$, or $\operatorname{ord}_{x_0} f(x) = n < 0$, then $f(x)$ has a pole or order $-n$ at x_0.

If $|f|_{x_0} = 1$, then $f(x)$ is regular and non-zero at x_0.

Should D be the whole Gauss sphere, we have $k = F(x)$; in this case the irreducible polynomials are linear of type $(x - x_0)$. The valuation $|f(x)|_{x-x_0}$ as defined previously is now the valuation denoted by $|f(x)|_{x_0}$; the valuation given by $|f(x)|_{p_\infty}$ is now denoted by $|f(x)|_\infty$. We see that to each point of the Gauss sphere there corresponds one of our valuations.

It was in analogy to this situation, that we introduced in the case $k = R$, the field of rational numbers, the "infinite prime" and associated it with the ordinary absolute value.

We now prove a theorem which establishes a relation between the normal valuations at all primes p:

Theorem 9: In both cases, $k = R$ and $k = F(x)$, the product $\Pi_p \mid a \mid_p = 1$.

We have already remarked that a relation of this form cannot be obtained for any finite number of valuations.

Proof: Only a finite number of primes (irreducible polynomials) divide a given rational number (rational function). Hence $|a|_p = 1$ for almost all (i.e. all but a finite number of) primes p, and so the product $\Pi_p \mid a \mid_p$ is well-defined.

If we write $\phi(a) = \Pi_p \mid a \mid_p$ we see that $\phi(ab) = \phi(a)\,\phi(b)$; thus it suffices to prove the result for $a = q$, where q is a prime (irreducible polynomial).

For $q \in R$,

$$\phi(q) = |q|_q \, |q|_\infty = \left(\frac{1}{q}\right)^{\mathrm{ord}_q q} \cdot q = \frac{1}{q} \cdot q = 1.$$

For $q \in k(x)$,

$$\phi(q) = d^{\deg q \cdot \mathrm{ord}_q q} \cdot d^{-\deg q} = d^{\deg q} \cdot d^{-\deg q} = 1.$$

This completes the proof.

We notice that this is essentially the only relation of the form $\Pi \mid a \mid_p^{e_p} = 1$. For if $\psi(a) = \Pi \mid a \mid_p^{e_p} = 1$, we have for each prime q

$$\psi(q) = |q|_q^{e_q} \, |q|_\infty^{e_\infty}.$$

But $|q|_q \, |q|_\infty = 1$ by the theorem; hence

$$\psi(q) = |q|_\infty^{e_\infty - e_q} = 1.$$

Thus $e_q = e_\infty$, and our relation is simply a power of the one established before.

The product formula has a simple interpretation in the classical case of the field of rational functions with complex coefficients. In this case

$$\phi(a) = d \text{ number of zeros-number of poles} = 1;$$

so a rational function has as many zeros as poles.

Now that the valuations of the field of rational numbers have been determined, we can find the best constant c for our axiom (3).

Theorem 10: For any valuation, we may take

$$c = \max(|1|, |2|).$$

Proof. (1) When the valuation is non-archimedean,

$$c = 1 = |1| \geqslant |2|.$$

(2) When the valuation is archimedean, k must have characteristic zero; hence k contains R, the field of rational numbers. The valuation is archimedean on R, and hence is equivalent to the ordinary absolute value; suppose that for the rational integers n we have $|n| = n^\beta$. Write $c = 2^\alpha$; then

$$|a + b| \leqslant 2^\alpha \max(|a|, |b|).$$

By the method of Theorem 3, we can deduce

$$|a_1 + \cdots + a_m| \leqslant (2m)^\alpha \max |a_\nu|.$$

As a special case of this we have

$$|a + b|^m \leqslant (2(m+1))^\alpha \max_\nu \left(\left| \binom{m}{\nu} \right| |a|^\nu |b|^{m-\nu} \right).$$

Now

$$\left| \binom{m}{\nu} \right| \leqslant \binom{m}{\nu}^\beta \leqslant 2^{m\beta},$$

since

$$\sum \binom{m}{\nu} = (1+1)^m = 2^m.$$

Hence we have

$$| a + b |^m \leqslant (2(m + 1))^\alpha 2^{m\beta}(\max (| a | , | b |))^m .$$

Taking the m-th root, and letting $m \to \infty$, we obtain

$$| a + b | \leqslant 2^\beta \max (| a | , | b |) .$$

Since

$$2^\beta = | 2 | = \max (| 1 | , | 2 |) ,$$

our theorem is proved in this case also. Since the constant c for an extension field is the same as for the prime field contained in it, it follows that if the valuation satisfies the triangular inequality on the prime field, then it does so also on the extension field.

6. Completion of a Field

Let $| \ |$ be a valuation of a field k; replace $| \ |$, if necessary, by an equivalent valuation for which the triangular inequality holds. A sequence of elements $\{a_\nu\}$ is said to form a *Cauchy sequence* with respect to $| \ |$ if, corresponding to every $\epsilon > 0$ there exists an integer N such that for $\mu, \nu \geqslant N$, $| a_\mu - a_\nu | < \epsilon$.

A sequence $\{a_\nu\}$ is said to form a *null-sequence* with respect to $| \ |$ if, corresponding to every $\epsilon > 0$, there exists an integer N such that for $\nu \geqslant N$, $| a_\nu | < \epsilon$.

k is said to be *complete* with respect to $| \ |$ if every Cauchy sequence with respect to $| \ |$ converges to a limit in k. We shall now sketch the process of forming the completion of a field k.

The Cauchy sequences form a ring P under termwise addition and multiplication:

$$\{a_\nu\} + \{b_\nu\} = \{a_\nu + b_\nu\} , \qquad \{a_\nu\} \{b_\nu\} = \{a_\nu b_\nu\} .$$

It is easily shown that the null-sequences form a maximal ideal N in P; hence the residue class ring P/N is a field $\tilde{k}$.

The valuation $| \ |$ of k naturally induces a valuation on $\tilde{k}$; we still denote this valuation by $| \ |$. For if a $\in \tilde{k}$ is defined by the residue class of P/N containing the sequence $\{a_\nu\}$, we define $| \alpha |$ to be $\lim_{\nu \to \infty} | a_\nu |$. To justify this definition we must prove

(a) that if $\{a_\nu\}$ is a Cauchy sequence, then so is $\{| a_\nu |\}$,

(b) that if $\{a_\nu\} \equiv \{b_\nu\} \bmod N$, then $\lim | a_\nu | = \lim | b_\nu |$,

(c) that the valuation axioms are satisfied.

The proofs of these statements are left to the reader.

If $a \in k$, let a' denote the equivalence class of Cauchy sequences containing $(a, a, a, \cdots)$; $a' \in \tilde{k}$. If $a' = b'$, then the sequence $((a - b), (a - b), \cdots) \in N$, so that $a = b$. Hence the mapping ϕ of k into $\tilde{k}$ defined by $\phi(a) = a'$ is (1, 1); it is easily seen to be an isomorphism under which valuations are preserved: $| a' | = | a |$. Let $k' = \phi(k)$; we shall now show that k' is everywhere dense in $\tilde{k}$. To this end, let α be an element of $\tilde{k}$ defined by the sequence $\{a_\nu\}$. We shall show that for large enough values of ν, $| \alpha - a_\nu' |$ is as small as we please. The elemnt $\alpha - a_\nu'$ is defined by the Cauchy sequence $\{(a_1 - a_\nu), (a_2 - a_\nu), \cdots\}$, and

$$| \alpha - a_\nu' | = \lim_{\mu \to \infty} | a_\mu - a_\nu |;$$

but since $\{a_\nu\}$ is a Cauchy sequence, this limit may be made as small as we please by taking ν large enough.

Finally we prove that $\tilde{k}$ is complete. Let $\{\alpha_\nu\}$ be a Cauchy sequence in $\tilde{k}$. Since k' is everywhere dense in $\tilde{k}$, we can find a sequence $\{a_\nu'\}$ in k' such that $| a_\nu' - \alpha_\nu | < 1/\nu$. This means that $\{(a_\nu' - \alpha_\nu)\}$ is a null-sequence in $\tilde{k}$; hence $\{a_\nu'\}$ is a Cauchy sequence in $\tilde{k}$. Since absolute values are preserved under the mapping ϕ, $\{a_\nu\}$ is a Cauchy sequence in k. This defines an element $\beta \in \tilde{k}$ such that $\lim_{\nu\to\infty} | a_\nu' - \beta | = 0$. Hence $\lim_{\nu\to\infty} | \alpha_\nu - \beta | = 0$, i.e. $\beta = \lim \alpha_\nu$, and so $\tilde{k}$ is complete.

We now agree to identify the elements of k' with the corresponding elements of k; then $\tilde{k}$ may be regarded as an extension of k. When k is the field of rational numbers, the completion under the ordinary absolute value ("the completion at the infinite prime") is the real number field; the completion under the valuation corresponding to a finite prime p ("the completion at p") is called the *field of p-adic numbers.*