Preface to the Second Edition

When the publishers of this book asked me to revise and update my problem book for a second edition, I had to decide how much to change, taking into consideration the fast development of the field (but also that the first edition was out of print). Combinatorics has grown a lot in the last decade, especially in those fields interacting with other branches of mathematics, like polyhedral combinatorics, algebraic combinatorics, combinatorial geometry, random structures and, most significantly, algorithmic combinatorics and complexity theory. (The theory of computing has so many applications in combinatorics, and vice versa, that sometimes it is difficult to draw the border between them.) But combinatorics is a discipline on its own right, and this makes this collection of exercises (subject to some updating) still valid.

I decided not to change the structure and main topics of the book. Any conceptual change (like introducing algorithmic issues consistently, together with an analysis of the algorithms and the complexity classification of the algorithmic problems) would have meant writing a new book. I could not resist, however, to working out a series of exercises on random walks on graphs, and their relations to eigenvalues, expansion properties, and electrical resistance (this area has classical roots but has grown explosively in the last few years). So Chapter 11 became substantially longer.

In some other chapters I also found lines of thought that have been extended in a natural and significant way in the last years. Altogether, I have added about 60 new exercises (more if you count subproblems), simplified several solutions, and corrected those errors that I became aware of.

In the preface of the first edition, I said that I plan a second volume on important topics left out, like matroids, polyhedral combinatorics, lattice geometry, block designs, etc. These topics have grown enormously since then, and to cover all of them would certainly need more than a single volume. I still love the procedure of selecting key results in various fields and analyzing them so that their proofs can be broken down to steps adding one idea at a time, thus creating a series of exercises leading up to a main result. (This love was revigorated while working on this new edition.) But writing a new volume is at the moment beyond my time and energy constraints.

In the meanwhile, many monographs were published on these topics, and several of these (in the first line, A. Recski's book *Matroid Theory and its Applications in Electric Network Theory and Statics*, Akadémiai Kiadó–Springer Verlag, 1989) contain extensive and very carefully compiled lists of problems and exercises.

Acknowledgements. I have received many remarks, corrections, and suggestions for improvements from my collegues; many of these were based on experience while teaching a course based on this book. Needless to say how pleased I felt by their interest in my work, and how grateful I am to these collegues for taking the trouble of formulating these remarks and sending them to me. Virtually all of them were right, and I have implemented almost all of these comments while revising the text (a few concerned research results and further topics related to the material, and were beyond the scope of the book). I am particularly grateful to J. Burghduff, A. Frank, F. Galvin, D. E. Knuth, and I. Tomescu for their extensive and thoughtful comments. My special thanks are due to D. E. Knuth and D. Aldous for reflecting on the new series of exercises, and in fact so fast that I could implement their remarks in the final version of the revised manuscript.

I also feel indebted to Ms. K. Fried for the very careful and expert typing into TeX (and for discovering many errors in the first edition while doing so), and to G. Bacsó and T. Csizmazia for their help in the proofreading and their thoughtful observations made during this work.

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Preface

Having vegetated on the fringes of mathematical science for centuries, combinatorics has now burgeoned into one of the fastest growing branches of mathematics — undoubtedly so if we consider the number of publications in this field, its applications in other branches of mathematics and in other sciences, and also, the interest of scientists, economists and engineers in combinatorial structures. The mathematical world had been attracted by the success of algebra and analysis and only in recent years has it become clear, due largely to problems arising from economics, statistics, electrical engineering and other applied sciences, that combinatorics, the study of finite sets and finite structures, has its own problems and principles. These are independent of those in algebra and analysis but match them in difficulty, practical and theoretical interest and beauty.

Yet the opinion of many first-class mathematicians about combinatorics is still in the pejorative. While accepting its interest and difficulty, they deny its depth. It is often forcefully stated that combinatorics is a collection of problems, which may be interesting in themselves but are not linked and do not constitute a theory. It is easy to obtain new results in combinatorics or graph theory because there are few techniques to learn, and this results in a fast-growing number of publications.

The above accusations are clearly characteristic of any field of science at an early stage of its development — at the stage of collecting data. As long as the main questions have not been formulated and the abstractions to a general level have not been carried through, there is no way to distinguish between interesting and less interesting results — except on an aesthetic basis, which is, of course, too subjective. Those techniques whose absence has been disapproved of above await their discoverers. So underdevelopment is not a case against, but rather for, directing young scientists toward a given field.

In my opinion, combinatorics is now growing out of this early stage. There *are* techniques to learn: enumeration techniques, matroid theory, the probabilistic method, linear programming, block design constructions, etc. There *are* branches which consist of theorems forming a hierarchy and which contain central structure theorems forming the backbone of study: connectivity of graphs (network flows) or factors of graphs, just to pick two examples from graph theory. There *are* notions abstracted from many non-trivial results, which unify large parts of the theory, such as matroids or the concept of good characterization (see below). My feeling that it is no longer possible to obtain significant results without the knowledge of these facts, concepts and techniques. (Of course, exceptions may occur, since the field is destined to cover such a large part of the world of mathematics that entirely new problems may still arise.)

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The reader will forgive I hope the insertion of some general ideas which tend to play the role of systematizing and unifying concepts. The first of these is the notion of the class NP.† A property T of graphs is in NP if we are able to efficiently prove (exhibit) T if it holds. (Technically, "efficiently" means that the length of the proof is bounded by a polynomial in the size of the graph.) For example, if a graph G is Hamiltonian, we can exhibit this by specifying a Hamilton cyclein G. This notion leads us to the notion of a *good characterization*, or — in the language of computational complexity theory — of the class NP∩co-NP, formulated by J. Edmonds. A property T of graphs is in NP∩co-NP if (making use of the different formulations of the definition and the equivalent condition) we are able to efficiently prove T if it holds and to efficiently disprove if it does not. For example, Kuratowski's classical characterization of planar graphs provides a good characterization of planarity: if a graph is planar, we easily establish this by drawing it in the plane; if it is non-planar, we can show this by exhibiting one of Kuratowski's graphs in it (see problem 5.37). A good characterization reflects a deep underlying logical duality of the property and, as the reader may convince himself by comparing good and "non-good" characterizations occurring throughout this book, often amounts to "the" solution of the problem. Of course, this does not mean that "non-good" characterizations may not be deep and useful theorems.

The existence of good characterizations tends to go hand in hand with the existence of good decision algorithms (by a "good" or "efficient" algorithm we mean one whose running time in the worst case is only a polynomial in the input data; this again does not directly affect its practical value). Several combinatorial properties are known for which there exists a polynomial-time algorithm to decide whether a given structure has the property, but its existence is by no means obvious (e.g. having a 1-factor). An interesting theoretical result, due to S. A. Cook, R. M. Karp and L. A. Levin is the following. Several properties of graphs (for example, the existence of Hamiltonian circuits, independence number, chromatic number, the existence of a kernel etc.) are equivalent in the sense that if any of them could be solved by a polynomial-time algorithm, then all of them could, and in this case also the "good" algorithmic solvability of very many other

† For a detailed description see e.g. A. V. Aho, J. E. Hopcroft, J. D. Ullman, *The Design and Analysis of Computer Algorithms,* Addison–Wesley, 1974, Chap. 10, or M.R. Garey and D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness,* Freeman, San Francisco, 1979.

problems in different fields of mathematics would follow (just to mention a very distant one: testing an n-digit number for primarility). These "most difficult" problems in NP are called NP-*complete*. It is unlikely that all of these could be solved efficiently, but there is as yet no proof of this. This is the famous P$\neq$NP problem of computer science.

Another idea which has proved very fruitful is that combinatorial optimization problems can generally be formulated as linear programming problems with integrality constraints. If one could disregard the constraints, the Duality Theorem of linear programming would provide the solution. So the solution of these problems is connected to investigating the effect of the integrality constraints on the behavior of optima. For example, if we can show that they do not change the optimum, we obtain a *minimax theorem*. Several instances of this idea can be found in § 13 (Hypergraphs).

Last but not least we mention the use of *linear algebra*. This ranges from applications of matrix calculus to the introduction of homology and cohomology groups. A common background of many applications of linear algebra is *Matroid Theory*, which is now a flourishing branch of combinatorics itself.

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The main purpose of this book is to provide help in learning existing techniques in combinatorics. The most effective (but admittedly very time-consuming) way of learning such techniques is to solve (appropriately chosen) exercises and problems.

This book presents all the material in the form of problems and series of problems (apart from some general comments at the beginning of each chapter). We hope that it will be useful to those students who intend to start research in graph theory, combinatorics or their applications, and for those who feel that combinatorial techniques might help them with their work in other branches of mathematics, management science, electrical engineering and so on. For background, only the elements of linear algebra, group theory, probability and calculus are needed.

When selecting the material I have had to restrict the topics covered. I feel that a more detailed analysis of a few basic notions is more useful than touching of all possible fields of research. So in this volume only enumeration problems, graphs and set-systems are discussed. Some fields have had to be completely omitted: random structures (here the reader is advised to read the book *Probabilistic Methods in Combinatorics* by P. Erdős and J. Spencer, Akadémiai Kiadó, Budapest and Academic Press, New York–London, 1974), integer programming, matroids (combinatorial geometries), finite geometries, block designs, lattice geometry, etc. I hope eventually to write a sequel to this volume covering some of these latter topics.

The book consists of three major parts: Problems, Hints and Solutions. A reader with less experience may read the hint given to a problem before trying to solve it; those problems with one or two asterisks are thought to be difficult and the reader may read the hint given right away unless he is prepared to sacrifice several days to the problem (some of them are worth it, I venture to say). Even having solved a problem the reader is advised to compare his solution with the one given here: it may be that the idea occurring in our solution will be basic in a later series of problems. Here it should be pointed out that problems come in series and previous problems often serve as steps to the last, deepest result in the series. Also note that the solution often uses notation or properties introduced in the hint.

As references I have preferred to give those where further development of the subject can best be seen. Thus, for those results reproduced in textbooks or monographs, usually the latter are given as reference. No reference means either that the assertion of the exercise is so well known that it would have been impossible or superfluous to trace it back or that the problem is believed to be new. A list of those textbooks and monographs most often cited is given at the end of the volume. A dictionary containing the definitions of combinatorial concepts used in the book and a list of symbols as well as an Author Index and a Subject Index are included.

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