INTRODUCTION

What is an n-manifold? A popular answer is a recitation of the definition: an n-manifold is a metric space covered by open sets homeomorphic to Euclidean n-space E^n. From a foundational perspective, that answer merely suggests another question: what is the topological nature of E^n? Such questions form the central theme around which this book is organized. The real line is characterized by a short list of simple properties; the plane, by properties almost as simple. What about the other Euclidean spaces? The goal here is to explore the topological structure of E^n in case $n \geq 3$, almost invariably paying attention only to the cases $n \geq 5$.

The book is about decompositions, or partitions, of manifolds. The typical object of study will be the decomposition space, or quotient space, associated with a decomposition of some manifold. Technically speaking, the decompositions are all upper semicontinuous ones, meaning that the decomposition elements fit together in a fashion nice enough to ensure metrizability of the decomposition spaces. The latter can be fairly pathological; nevertheless, they are not totally removed from the more familiar world of Euclidean topology, for in this context they arrive on the scene equipped with an explicit connection, via the decomposition mapping, to manifolds. The decompositions themselves will be restricted somewhat, for it happens to be well known that every Peano continuum is the continuous image of a manifold, and the subject here will be more restricted than the study of Peano continua. Cell-like decompositions, in which the partition elements behave homotopically like points, will be the predominant topic. In n-manifolds such decompositions form the class for which it is reasonable to expect the product of E^1 with the decomposition space to be an $(n + 1)$-manifold.

There have been three distinct periods during which decomposition theory has flourished. The first of them, the early period, occurred in the first half of this century, in the 1920s and 1930s, led by R. L. Moore and G. T. Whyburn. Next, the classical period started in the early 1950s, headed by R. H. Bing; supported by S. Armentrout, it continued on through the 1960s and beyond. The current era, the modern period, began in 1977 with the work of R. D. Edwards, spurred by earlier results of J. W. Cannon.

The three periods are distinguished by their characteristic emphases on certain dimensions as the realm of study. During the first one the central results pertained to decompositions of the plane and of 2-manifolds; next, of 3-manifolds; and, last, of higher-dimensional manifolds. ("Higher-dimensional" usually means of dimension $n \geq 5$, because the $n = 4$ case so often demands its own, separate treatment.)

In addition to such chronological and numerical differences, the three are clearly distinguished by significant methodological differences. The explanation demands a bit of the standard notation: consider a nice decomposition (or partition) G of an n-manifold M and also the natural map $\pi: M \to M/G$ of M to the quotient space, called M/G. Usually one hopes to prove that, under sufficient conditions about the sets comprising G, M/G is topologically equivalent to M. The strategy of the early period was to use topological characterizations of the objects involved, the familiar plane, 2-sphere, or other 2-manifold, to deduce the desired equivalence. Customarily that plan did not work at all for 3-manifolds, due in part to the comparative difficulty of distinguishing one 3-manifold from another, but also due to the more complicated problem, which still remains unsatisfactorily resolved, of understanding when a space is a 3-manifold. The classical period got under way when Bing invented a new, workable strategy embodied in his shrinkability criterion, introduced here in Section 5. It posits the existence of a homeomorphism from M to itself sending the elements of G to sets of small size, the key feature, while simultaneously submitting to certain mild cover controls. When this shrinkability criterion holds, the quotient space M/G turns out to be homeomorphic to M, under a naturally arising function obtained as the limit of a sequence of such shrinking homeomorphisms. Accordingly, under Bing's strategy, one investigated the source manifold M to see whether the shrinkability criterion was valid. That strategy has proved effective for solving a multitude of decomposition problems, not just in 3-dimensional manifolds but also in higher-dimensional ones. The modern period began abruptly, not so much because of a successful attack on higher-dimensional manifolds, for powerful results about decompositions of higher-dimensional manifolds had already been discovered at the time, but because of a new strategy, a synthesis of its predecessors, developed by Edwards. He studied the decomposition map $\pi: M \to M/G$ with an eye toward approximating it by homeomorphisms. While the possibility of obtaining such approximations was a by-product of the shrinkability criterion, it was not a fundamental tenet of the prevailing philosophy. Operating with these new tactics, Edwards was able to approximate π via successive maps that were 1–1 over larger and larger subsets of M/G, culminating at the final stage in a limiting homeomorphism.

The essence of each period, for the most part, can be distilled into a single, typifying result. In the early period it was the famous theorem of R. L. Moore.

Theorem (Moore). *If G is an upper semicontinuous decomposition of the plane E^2 into continua, none of which separates E^2, then E^2/G is homeomorphic to E^2.*

From the classical period, the most difficult era to recapture in just one epitomizing theorem, a reasonable candidate is the following combination of the work of Bing [2] and Armentrout [6].

Theorem. *Let G be an upper semicontinuous decomposition of a 3-manifold M into cellular sets. Then M/G is homeomorphic to M if and only if G is shrinkable.*

Examples from Bing made it plain that not all nice decompositions of E^3 reproduce E^3, so several people, Bing among them, engaged in wholesale testing, particularly during the early portions of the period, testing of geometric conditions imposed on the decomposition elements to see which implied shrinkability and which did not. That effort generated a lot of empirical data, no one piece of which stands out as characteristic of the period, although the entire effort might. The theorem mentioned above reflects Bing's methodology characteristic of that era and also Armentrout's complementary contribution that, in order for M/G to be homeomorphic to M, G must be shrinkable. The modern period is set off by the following breakthrough result of Edwards, marking the change from the earlier era to the current one.

Theorem (cell-like approximation). *Let G denote an upper semicontinuous decomposition of an n-manifold M, $n \geq 5$, into cell-like sets. Then the decomposition map $\pi: M \to M/G$ can be approximated by homeomorphisms if and only if M/G is finite-dimensional and satisfies the following disjoint disks property: any two maps of the 2-cell B^2 to M/G can be approximated by maps having disjoint images.*

The point, of course, is that M/G is topologically equivalent to M when the latter two conditions hold; it is also the case that then G is shrinkable.

The paramount result treated in this book is Edwards's cell-like approximation theorem, established here as Theorem 24.3. Although Edwards's proof has been available in manuscript form and has been disseminated publicly in his lectures at the 1978 CBMS Regional Conference at Stillwater, Oklahoma, it has never been published. (A brief but splendid outline appears in Edwards [5].) A primary function of this book is to rectify that matter.

Structurally the book is divided into seven chapters. Chapter I, the preliminaries, introduces the basic terminology and studies some of the elementary consequences. Functioning throughout at a level of difficulty no higher than what is encountered in an elementary general topology text, it provides a hint of the methodology, though none of the major 2-dimensional results, prevalent during the early period of decomposition theory. Chapter II, which is more demanding, pushes ahead into the classical period. Delving into a wide variety of results and examples about decompositions of 3-manifolds, without attempting to be exhaustive, Chapter II presents a fairly large sample of Bing's 3-dimensional work. In particular, in Section 9 it sets forth several key examples of interesting, unusual, historically significant decompositions of 3-space, some of which yield 3-space again and others of which do not yield any manifold whatsoever; taken on the whole, these examples absolutely must be understood if one is to fully appreciate the pitfalls in this subject or to recognize potential circumvention techniques. Chapter II is not exclusively 3-dimensional in focus, however, for it also includes an elementary proof of the important result (also an immediate consequence of the cell-like approximation theorem studied later) that non-combinatorial triangulations of n-manifolds, $n \geq 5$, do exist. The unifying device is Bing's shrinkability criterion. At the beginning this part lays out several refined notions of shrinkability and at the end, based on the local contractibility of manifold homeomorphism groups, shows all of them to be equivalent in topological manifolds. Chapter III, somewhat more technical in nature, involves an investigation of properties preserved by the typical decompositions and sets the stage for the substantial effort in the sequel. Moving from the classical to the modern period, Chapter IV steadily builds up to its climax, the proof of the cell-like approximation theorem. A concluding, almost parenthetical note for this part demonstrates how the original aspects of the subject are reinvigorated by the most contemporary; it makes use of Edwards's methodology and the planar Schönflies theorem to derive the chief result of the early period, Moore's theorem. Chapter V deals with the consequences of Edwards's result, mainly for products involving such decomposition spaces, either with a line or with another such space. Positive in tone, it stresses the conditions under which a decomposition under consideration is shrinkable. By contrast, Chapter VI is more negative, setting forth techniques for constructing pathological decompositions of high-dimensional manifolds. Finally, Chapter VII treats the far-reaching, grander applications of decomposition theory to the rest of geometric topology. Not at all self-contained, nowhere close to it, Chapter VII displays the power, centrality, and diversity potentially available in what has gone on before. It does so by suddenly bringing into play several big results from other branches of geometric topology, something done as infrequently as possible in the earlier parts of the book.

This brings us to the twin issues of textual scope and exegetical style. Two limitations confine the material to manageable size. The first, of course, pertains to the subject that titles the book; the emphasis involves the part of geometric topology concerning decompositions. As stated in the Preface, this book strives to organize linearly decomposition theory, not all of geometric topology. Accordingly, in the interest of efficiency, occasionally one will encounter, with little explanation or justification, invocation of some profound result from one of these collateral topics (e.g., the Kirby–Edwards local contractibility theorem in Section 13, the Bing–Kister isotopy theorem in Section 21, and the Lickorish–Siebenmann PL regular neighborhood classification theorem in Section 38). While invocation of a *deus ex machina* can detract from the stark beauty of a rigorous, orderly mathematical development, the demands to maintain finiteness and to make progress seem to allow no other course. Construed positively, this practice can highlight for the reader the significance of certain major theorems while identifying subjects for future study.

The second limitation pertains to the residence of the decompositions to be considered. Almost invariably these decompositions live in finite-dimensional manifolds, largely in those of dimension $n \geq 5$. The classical $n = 2$ case, which presents little difficulty, receives scant attention. The $n = 3$ case, which carries a great deal of significance and which provides both satisfaction and motivation to the visual imagination, receives much more. We study the salient examples in painstaking detail. Full exposition of this case, however, entails a variety of powerful but uniquely 3-dimensional techniques, which by choice we avoid, preferring instead to move forward into the less well-charted world of higher-dimensional manifolds. That is why we give no proof, for instance, of Armentrout's homeomorphic approximation theorem, alluded to earlier, or of the Denman–Starbird theorem that upper semicontinuous decompositions of E^3 into points and countably many starlike-equivalent compacta are shrinkable. We pay even scarcer attention to the $n = 4$ case, encountering the 4-dimensional world mostly through results that hold in other dimensions as well.* At another end of the spectrum is the $n = \infty$ case, which also is totally ignored here. While some methods for studying decompositions of infinite-dimensional manifolds are similar to those developed for the $n \geq 5$ case, many others are intrinsic to that subject.

* For more information about this highly intriguing situation, probably the toughest finite-dimensional case of them all, keep an eye out for a forthcoming book by M. H. Freedman and F. Quinn.

T. A. Chapman's book [2] and H. Toruńczyk's characterizations [1, 2] provide a good introduction.

What background is needed to be able to read this material? At the outset, nothing beyond an introduction to point-set topology. As the text progresses, increasingly more collateral material is brought into play, primarily taken from classical dimension theory and the beginnings of algebraic topology and of PL topology. It probably is imperative that the reader know about the PL regular neighborhood theorem and something about general position, or transversality, techniques. Perhaps the single best reference is T. B. Rushing's *Topological Embeddings*—indeed, we conceived this text as a kind of companion volume to Rushing's. The PL preliminaries laid out there should suffice for this as well, and the engulfing methodology presented there in elaborate and careful detail is extremely valuable for efforts here involving higher-dimensional manifolds.

In light of the above, the ideal personal reference library would contain:

Dimension Theory, by W. Hurewicz and H. Wallman
Introduction to Piecewise-Linear Topology, by C. P. Rourke and B. J. Sanderson
Topological Embeddings, by T. B. Rushing
Algebraic Topology, by E. H. Spanier

For later use, upon completion of this text, it might also include:

Lectures on Hilbert Cube Manifolds, by T. A. Chapman

Designed as a text, this book includes many problems, ranging from simple to fairly difficult. Some preview future subjects, others require filling in steps omitted from a proof, and still others call for reapplication of techniques developed in the body of the text. None of them, except by accident, should be impossibly hard; those at the level of recent thesis problems are suggested only after substantial groundwork has been laid. The reader is urged to do the problems, as the optimal method to begin reworking this particular canvas to reflect one's own insight and vision.

Two closing remarks about typographical shortcuts. First, within any one section specification of a result from another section is given, for example, as "Lemma 30.4", referring to Lemma 4 of Section 30. Specification of any result from the same section is made without using the section number. Second, bibliographic references are given by author's name plus the number of the item in the list of References here, as in, say, "(Bing [5])". When it is totally obvious that it is Bing's work being discussed, the reference may be given omitting "Bing" and appearing solely as "[5]".