Preview

To motivate the introduction of symplectic geometry in mechanics, we briefly consider Hamilton's equations. The starting point is Newton's second law, which states that a particle of mass $m > 0$ moving in a potential $V(q)$, $q = (q^1, q^2, q^3) \in \boldsymbol{R}^3$, moves along a curve $q(t)$ in $\boldsymbol{R}^3$ in such a way that $m\ddot{q} = -\,grad\,V(q)$. If we introduce the momentum $p_i = m\dot{q}^i$ and the energy $H(q,p) = (1/2m)\|p\|^2 + V(q)$, then Newton's law is equivalent to Hamilton's equations:

$$\begin{cases} \dot{q}^i = \partial H/\partial p_i \\ \dot{p}_i = -\partial H/\partial q^i, \qquad i = 1, 2, 3 \end{cases}$$

One proceeds to study this system of first-order equations for a general $H(q,p)$. To do this, we introduce the matrix $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, where I is the 3×3 identity, and note that the equations become $\dot{\xi} = J \cdot grad\,H(\xi)$, where $\xi = (q,p)$. (In complex notation, setting $z = q + ip$, they may be written as $\dot{z} = -2i\,\partial H/\partial \bar{z}$.)

Set $X_H = J \cdot grad\,H$. Then $\xi(t)$ satisfies Hamilton's equations iff $\xi(t)$ is an integral curve of X_H, that is, $\dot{\xi}(t) = X_H(\xi(t))$. The relationship between X_H and H can be rewritten as follows: introduce the skew-symmetric bilinear form ω

on $\boldsymbol{R}^3 \times \boldsymbol{R}^3$ defined by

$$\omega(v_1, v_2) = v_1 \cdot J \cdot v_2$$
$$v_1, v_2 \in \boldsymbol{R}^3 \times \boldsymbol{R}^3$$
$$v_1 = (x_1, y_1)$$
$$v_2 = (x_2, y_2)$$

[In complex notation on $C^3 \cong \boldsymbol{R}^3 \times \boldsymbol{R}^3$, $\omega(v_1, v_2) = -Im\langle v_1, v_2 \rangle$, where $v_1 = x_1 + iy_1$, $v_2 = x_2 + iy_2$, and $\langle \; , \; \rangle$ is the Hermitian inner product.]

Then we have, for all $\xi \in \boldsymbol{R}^3 \times \boldsymbol{R}^3$ and $v \in \boldsymbol{R}^3 \times \boldsymbol{R}^3$,

$$\omega(X_H(\xi), v) = dH(\xi) \cdot v$$

where $dH(q,p) = (\partial H/\partial q^i, \partial H/\partial p^i)$, a row vector in $\boldsymbol{R}^3 \times \boldsymbol{R}^3$, as is easily checked. One calls ω the *symplectic form* on $\boldsymbol{R}^3 \times \boldsymbol{R}^3$, and X_H the *Hamiltonian vector field* with energy H.

Suppose we make a change of coordinates $\eta = f(\xi)$, where $f: \boldsymbol{R}^3 \times \boldsymbol{R}^3 \to \boldsymbol{R}^3 \times \boldsymbol{R}^3$ is smooth. If $\xi(t)$ satisfies Hamilton's equations, the equations satisfied by $\eta(t) = f(\xi(t))$ are $\dot{\eta} = A\dot{\xi} = AJ\, grad_\xi H(\xi) = AJA^*\, grad_\eta H(\xi(\eta))$, where $(A)^{ij} = (\partial \eta^i / \partial \xi^j)$ is the Jacobian of f, and A^* is the transpose of A. The equations for η will be Hamiltonian with energy $K(\eta) = H(\xi(\eta))$ if and only if $AJA^* = J$. A transformation satisfying this condition is called *canonical* or *symplectic*, (or a symplectomorphism). In terms of the symplectic form ω, this condition, denoted $f^*\omega = \omega$, says the transformation f leaves ω unchanged.

The space $\boldsymbol{R}^3 \times \boldsymbol{R}^3$ of the ξ's is called the *phase space*. For a system of N particles we would use $\boldsymbol{R}^{3N} \times \boldsymbol{R}^{3N}$.

For many fundamental physical systems, the phase space is a manifold rather than Euclidean space. For instance, manifolds often arise when constraints are present. For example, the phase space for the motion of the rigid body is the tangent bundle of the group $SO(3)$ of 3×3 orthogonal matrices with determinant $+1$. (See Sect. 4.4 for details.) Not only are manifolds important in these examples, but their terminology and notation lead to a clearer understanding of the basic structure of mechanics.