Preface

In this book we study complete Riemannian manifolds by developing techniques for comparing the geometry of a general manifold M with that of a simply connected model space of constant curvature M_H. A typical conclusion is that M retains particular geometrical properties of the model space under the assumption that its sectional curvature K_M, is bounded between suitable constants. Once this has been established, it is usually possible to conclude that M retains topological properties of M_H as well.

The distinction between strict and weak bounds on K_M is important, since this may reflect the difference between the geometry of say the sphere and that of Euclidean space. However, it is often the case that a conclusion which becomes false when one relaxes the condition of strict inequality to weak inequality can be shown to fail only under very special circumstances. Results of this nature, which are known as rigidity theorems, generally require a delicate global argument. Here are some examples which will be treated in more detail in Chapter 8.

Topological Theorem. If M is a complete manifold such that $K_M \geq \delta > 0$, then M has finite fundamental group.

Geometrical Antecedent. If M is a complete manifold such that $K_M \geq \delta > 0$, then the diameter of its universal covering space $\widetilde{M}$ is $\leq \Pi/\sqrt{\delta}$. In particular, $\widetilde{M}$ is compact.

Even if we assume M to be compact, the preceding statements are false if only $K_M \geq 0$. However, we can show the following.

Rigid Topological Theorem. Let M be a compact manifold such that $K_M \geq 0$. Then there is an exact sequence

$$0 \rightarrow \Phi \rightarrow \Pi_1(M) \rightarrow B \rightarrow 0$$

where is a finite group and B is a crystallographic group on $\mathbb{R}_k$ for some $k \leq \dim M$, and therefore satisfies an exact sequence

$$0 \rightarrow \mathbb{Z}_k \rightarrow B \rightarrow \Psi \rightarrow 0$$

where Ψ is a finite group.

Rigid Geometrical Antecedent. Let M be a compact manifold such that $K_M \geq 0$. Then $\widetilde{M}$ splits isometrically as $\overline{M} \times \mathbb{R}_k$ (same k as above), where $\overline{M}$ is compact and $\mathbb{R}_k$ has its standard flat metric. Thus, if $K_M \geq 0$, $\widetilde{M}$ may not be compact, but it is at worst the isometric product of a compact manifold and Euclidean space. The infinite part of $\Pi_1(M)$ comes precisely from the Euclidean factor.

The reader of this book should have a basic knowledge of differential geometry and algebraic topology, at least the equivalent of a one term course is each. Our purpose is to provide him with a fairly direct route to some interesting geometrical theorems, without his becoming bogged down in a detailed study of connections and tensors. In keeping with this approach, we have limited ourselves primarily to those techniques which arise as outgrowths of the second variation formula and to some extent of Morse theory.

In Chapter 1 we have included a rapid treatment of the more elementary material on which the later chapters are based. Of course, we do not recommend that the less knowledgeable reader regard this as a comprehensive introduction to Riemannian geometry. Likewise, Chapters 3 and 4 are provided in part for the convenience of the reader. In Chapter 3 which deals with homogeneous spaces, we also summarize without proof the relevant material on Lie groups. In Chapter 4 the main theorems of Morse theory are stated, again mostly without proofs. An exception, however, is Lemma 4.11, which is perhaps less standard than the other material. For the unproven results in both chapters, excellent references are readily available. Our main geometrical tools, the Rauch Comparison Theorems and the more global Toponogov Theorem, are discussed in Chapters 1 and 2 respectively. Chapter 5 deals with closed geodesics and the injectivity radius of the exponential map. Chapters 6-9 form the core of our study. Chapter 6 contains the Sphere Theorem – M simply connected and $1 \geq K_M > 1/4$ implies M homeomorphic to a sphere – as well as Berger's rigidity theorem which covers the case $1 \geq K_M \geq 1/4$. The last three chapters deal with material of recent origin. Chapter 7 is primarily concerned with the differentiable version of the Sphere Theorem. Chapter 8 takes up the structure theory of complete noncompact manifolds of nonnegative curvature, while Chapter 9 gives some results on the fundamental group of compact manifolds of nonpositive curvature.

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