Preface to the New Edition

The original preface which follows tells about the history of these notes and the missing chapters. This book is a slightly revised edition. Some footnotes and historical comments have been added in an attempt to compensate for the lack of references and attribution of credit in the original. There are two mathematical additions. One is a sketch of the analytic proof of the second inequality in Chapter VI. The other is several additional pages on Weil groups at the end of Chapter XV. They explain that what is there called a Weil group for a finite Galois extension K/F lacks an essential feature of a Weil group in Weil's sense, namely the homomorphism $W_{K,F} \to \mathrm{Gal}(K^{\mathrm{ab}}/F)$, but that we recover this once we construct a Weil group for $\bar{F}/F$ by passing to an inverse limit. There is also a sketch of an abstract version of Weil's proof of the existence and uniqueness of his $W_{K,F}$ for number fields.

I have not renumbered the chapters. After some preliminaries, the book still starts with Chapter V, but the mysterious references to the missing chapters have been eliminated. The book is now in TeX. The handwritten German letters are gone, and many typographical errors have been corrected. I thank Mike Rosen for his help with that effort. For the typos we've missed and other mistakes in the text, the AMS maintains a Web page with a list of errata at

`http://www.ams.org/bookpages/chel-366/`

I would like to thank the AMS for republishing this book, and especially Sergei Gelfand for his patience and help with the preparation of the manuscript.

For those unacquainted with the book, it is a quite complete account of the algebraic (as opposed to analytic) aspects of classical class field theory. The first four chapters, V–VIII, cover the basics of global class field theory, the cohomology of idèle classes, the reciprocity law and existence theorem, for both number fields and function fields. Chapters IX and X cover two more special topics, the structure and cohomology of the connected component of 1 in the idèle class group of a number field, and questions of local vs. global behavior surrounding the Grunwald–Wang theorem. Then there are two chapters on higher ramification theory, generalized local classfield theory, and explicit reciprocity laws. This material is beautifully covered also in [**21**]. For a recent report, see [**8**]. There is a nice generalization of our classical explicit formula in [**13**]. The last three chapters of the book cover abstract class field theory. The cohomological algebra behind the reciprocity law is common to both the local and global class field theory of number fields and function fields. Abstracting it led to the definition of a new algebraic structure, 'class formation', which embodies the common features of the four theories. The difference is in the proofs that the idèle classes globally, and the multiplicative groups locally, satisfy the axioms of a class formation. Chapter XIV concludes with a discussion of the reciprocity law and existence theorem for an abstract class formation. In the last chapter XV, Weil groups are defined for finite layers of an arbitrary class formation, and then, for topological class formations satisfying certain axioms which hold in the classical cases, a Weil group for the whole formation is constructed, by passage to an inverse limit, The class formation can be recovered from its Weil group, and the topological groups which occur as Weil groups are characterized by axioms.

The mathematics in this book is the result of a century of developement, roughly 1850–1950. Some history is discussed by Hasse in [**5**] and in several of the papers in [**18**]. The high point came in the 1920's with Takagi's proof that the finite abelian extensions of a number field are in natural one-to-one correspondence with the quotients of the generalized ideal class groups of that field, and Artin's proof several years later that an abelian Galois group and the corresponding ideal class group are canonically isomorphic, by an isomorphism which implied all known reciprocity laws. The flavor of this book is strongly influenced by the last steps in that history. Around 1950, the systematic use of the cohomology of groups by Hochschild, Nakayama and the authors shed new light. It enabled many theorems of the local class field theory of the 1930's to be transferred to the global theory, and led to the notion of class formation embodying the common features of both theories. At about the same time, Weil conceived the idea of Weil groups and proved their existence. With those two developments it is fair to say that the classical one-dimensional abelian class field theory had reached full maturity. There were still a few things to be worked out, such as the local and global duality theories, and the cohomology of algebraic tori, but it was time for new directions.

They soon came. For example:

- Higher dimensional class field theory;
- Non-abelian reciprocity laws and the Langlands program;
- Iwasawa theory;
- Leopold's conjecture;
- Abelian (and non-abelian) ℓ-adic representations;
- Lubin-Tate local theory, Hayes explicit theory for function fields, Drinfeld modules;
- Stark conjectures;
- Serre conjectures (now theorems).

Rather than say more or give references for these, I simply recommend what has become a universal reference, the internet. Searching any of the above topics is rewarding.

John Tate
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Preface

This is a chunk of the notes of the Artin–Tate seminar on class field theory given at Princeton University in 1951–1952, namely the part dealing with global class field theory (Chapters V through XII) and the part dealing with the abstract theory of class formations and Weil groups (Chapters XIII–XV). The first four chapters, which are not included, covered the cohomology theory of groups, the fundamentals of algebraic number theory, a preliminary discussion of class formations, and local class field theory. In view of these missing sections, the reader will encounter missing references and other minor flaws of an editorial nature, and also some unexplained notations. We have written a few pages below recalling some of these notations and outlining the local class field theory, in an attempt to reduce the "prerequisites" for reading these notes to a basic knowledge of the cohomology of groups and of algebraic theory, together with patience.

The reason for the long delay in publication was the ambition to publish a revised and improved version of the notes. This new version was to incorporate the advances in the cohomology theory of finite groups which grew out of the seminar and which led to the determination of the higher cohomology groups and to a complete picture of the cohomological aspects of the situation, as outlined in Tate's talk at the Amsterdam Congress in 1954. However this project was never completed and thus served only to prevent the publication of the most important part of the seminar, namely Chapters V through XII of these notes. That this material finally appears is due to the energies of Serge Lang, who took the original notes, continued to urge their publication, and has now made the arrangements for printing. It is a pleasure to express here our appreciation to him for these efforts.

Two excellent general treatments of class field theory, which complement these notes, have appeared during the past year, namely:

Cassels and Fröhlich, *Algebraic Number Theory*, Academic Press, London, 1967. (Distributed in the U.S. by the Thompson Publishing Company, Washington, D.C.).

Weil, *Basic Number Theory*, Springer-Verlag, Berlin/Heidelberg/New York, 1967.