

# Table of Contents

<b>Preface</b>	vii
<b>Basic Terminology</b>	xxix
<b>1 Matchings in bipartite graphs</b>	
1.0. Introduction	1
1.1. The Theorems of König, P. Hall and Frobenius	4
Box 1A. NP-properties, good characterizations and minimax theorems	8
1.2. A bipartite matching algorithm: the Hungarian Method	12
Box 1B. On algorithms	16
1.3. Deficiency, surplus and a glimpse of matroid theory	17
Box 1C. Matroids	27
1.4. Some consequences of bipartite matching theorems	29
<b>2 Flow theory</b>	
2.0. Introduction	41
2.1. The Max-Flow Min-Cut Theorem	42
2.2. Flow algorithms	46
Box 2A. Searching a graph	53
Box 2B. Numbers in algorithms	59
2.3. Flow-equivalent trees	61
2.4. Applications of flow theory to matching theory	68
2.5. Matchings, flows and measures	74
<b>3 Size and structure of maximum matchings</b>	
3.0. Introduction	83
3.1. Tutte's theorem, Gallai's lemma and Berge's formula	84
Box 3A. Matching matroids and matroid duality	92

3.2.	The Gallai-Edmonds Structure Theorem . . . . .	93
3.3.	Toward a calculus of barriers . . . . .	102
3.4.	Sufficient conditions for matchings of a given size . . . . .	110
<b>4</b>	<b>Bipartite graphs with perfect matchings</b>	
4.0.	Introduction . . . . .	121
4.1.	Elementary graphs and their ear structure . . . . .	122
4.2.	Minimal elementary bipartite graphs . . . . .	127
4.3.	Decomposition into elementary bipartite graphs . . . . .	137
<b>5</b>	<b>General graphs with perfect matchings</b>	
5.0.	Introduction . . . . .	143
5.1.	Elementary graphs: elementary properties . . . . .	145
5.2.	The canonical partition $\mathcal{P}(G)$ . . . . .	150
5.3.	Saturated graphs and cathedrals . . . . .	159
5.4.	Ear structure of 1-extendable graphs . . . . .	174
5.5.	More about factor-critical and bicritical graphs	195
<b>6</b>	<b>Some graph-theoretical problems related to matchings</b>	
6.0.	Introduction . . . . .	213
6.1.	2-matchings and 2-covers . . . . .	213
6.2.	2-bicritical and regularizable graphs . . . . .	217
6.3.	Matchings, 2-matchings and the König Property . . . . .	220
	Box 6A. Reducibility problems and NP-completeness . . . . .	226
6.4.	Hamilton cycles and 2-matchings . . . . .	228
6.5.	The Chinese Postman Problem . . . . .	231
6.6.	Optimum paths, cycles, joins and cuts . . . . .	243
	Box 6B. Packing paths, cycles, joins and cuts . . . . .	253

<b>7</b>	<b>Matching and linear programming</b>	
7.0.	Introduction . . . . .	255
	Box 7A. Cones, polytopes and polyhedra, and other preliminaries from linear programming . . . . .	256
	Box 7B. Linear programming algorithms . . . . .	261
7.1.	Linear programming and matching in bigraphs . . . . .	266
	Box 7C. The Hoffman-Kruskal Theorem and other conditions of integrality . . . . .	272
7.2.	Matchings and fractional matchings . . . . .	273
7.3.	The matching polytope . . . . .	274
	Box 7D. Cutting planes . . . . .	283
7.4.	Chromatic index . . . . .	285
	Box 7E. Good characterizations other than the Farkas Lemma . . . . .	290
7.5.	Fractional matching polytopes and cover polyhedra . . . . .	291
7.6.	The dimension of the perfect matching polytope . . . . .	292
	Box 7F. The dimension of a “good” polytope . . . . .	303
<b>8</b>	<b>Determinants and matchings</b>	
8.0.	Introduction . . . . .	307
8.1.	Permanents . . . . .	309
8.2.	The method of variables . . . . .	315
8.3.	The Pfaffian and the number of perfect matchings . . . . .	318
8.4.	Probabilistic enumeration of perfect matchings . . . . .	330
	Box 8A. Probabilistic methods in graph theory . . . . .	331
8.5.	Matching polynomials . . . . .	333
8.6.	More on the number of perfect matchings . . . . .	345
8.7.	Two applications to physical science . . . . .	349

<b>9</b>	<b>Matching algorithms</b>	
9.0.	Introduction . . . . .	357
9.1.	The Edmonds Matching Algorithm . . . . .	358
9.2.	Weighted matching . . . . .	369
9.3.	An algorithm based upon the Gallai-Edmonds Theorem . . . . .	376
9.4.	A linear programming algorithm for matching . . . . .	379
<b>10</b>	<b>The <math>f</math>-factor problem</b>	
10.0.	Introduction . . . . .	383
10.1.	Reduction principles . . . . .	385
10.2.	A structure theory for $f$ -factors . . . . .	388
10.3.	Realization of degree sequences . . . . .	404
<b>11</b>	<b>Matroid matching</b>	
11.0.	Introduction . . . . .	409
11.1.	Formulations of the Matroid Matching Problem . . . . .	409
	Box 11A. Oracles . . . . .	413
	Box 11B. Minimizing submodular set functions . . . . .	417
11.2.	The main theorem of polymatroid matching . . . . .	426
11.3.	Matching in special polymatroids . . . . .	433
<b>12</b>	<b>Vertex packing and covering</b>	
12.0.	Introduction . . . . .	443
12.1.	Critical graphs . . . . .	445
12.2.	Vertex packing polytopes . . . . .	456
12.3.	Hypergraph matching . . . . .	466
12.4.	Vertex packing in claw-free graphs . . . . .	471
	Box 12A. Bounds on the independence number, or: can anything be done with NP-complete problems? . . . . .	480

<b>Appendix: Developments in matching theory since this book was first published</b>	
A1	The matching structure . . . . . A1
A1.1	A calculus of barriers . . . . . A1
A1.2	Brick and brace decomposition . . . . . A1
A1.3	Removable lines . . . . . A2
A1.4	Ear-decompositions . . . . . A2
A1.5	The matching lattice . . . . . A3
A1.6	Equimatchable graphs . . . . . A4
A1.7	Extendability . . . . . A4
A2	Sampling and counting matchings . . . . . A6
A2.1	The number of matchings in regular bipartite graphs . . . . . A6
A2.2	Monte-Carlo sampling and approximate counting . . . . . A6
A2.3	Properties of random matchings . . . . . A8
A3	Improvements on the determinant method . . . . . A9
A3.1	Explicit substitution on the Tutte matrix . . . . . A9
A3.2	Substituting random exponentials . . . . . A9
A3.3	Random substitutions and the number of perfect matchings . . . . . A10
A3.4	Pfaffian graphs . . . . . A10
A4	Matching in parallel . . . . . A11
A5	Facets of the stable set polytope and $\alpha$ -critical graphs . . . . . A12
A6	Factorization . . . . . A13
A7	Structures generalizing matchings and matroids . . . . . A13
A7.1	Path matching . . . . . A13
A7.2	Jump systems . . . . . A14
	References . . . . . A15
	<b>References . . . . . 483</b>
	<b>Index of Terms . . . . . 527</b>
	<b>Index of Symbols . . . . . 539</b>
	<b>Errata in the book . . . . . 545</b>