Preface to the AMS Chelsea Edition

We are deeply grateful to the AMS for reissuing *Differential Topology* as part of its AMS Chelsea Book Series. Our elementary introduction to topology via transversality techniques has managed to stay in print for most of the thirty-six years since its original appearance, and we would like to thank Edward Dunne and his colleagues in Providence for ensuring its continuing availability (knock on wood) for the next thirty-six years. The techniques it highlights have, in some sense, a very 1970's flavor. The quixotic hopes of that decade, that singularity theory and catastrophe theory (of whose catastrophic demise the less said the better) would have a revolutionary impact on physics, chemistry, biology, economics, game theory, and investment strategies in the stock market, have proved largely unfounded. However, we have been pleased to find that our students today are, just as were the students of three decades ago, happy with the visceral, down-to-earth approach to topology espoused by books like ours and Milnor's wonderful *Topology from a Differential Viewpoint.* We hope (again knock on wood) that whatever the fashions in mathematics of the next thirty-six years, this will continue to be the case.

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Preface

The intent of this book is to provide an elementary and intuitive approach to differential topology. The topics covered are nowadays usually discussed in graduate algebraic topology courses as by-products of the big machinery, the homology and cohomology functors. For example, the Borsuk-Ulam theorem drops out of the multiplicative structure on the cohomology ring of projective space; the Lefschetz theorem comes from Poincaré duality and the Künneth theorem; the Jordan-Brouwer separation theorem follows from Alexander duality; and so on. We have two objections to this big-machinery approach: it obscures the elegant intuitive content of the subject matter, and it gives the student the impression that only big machines can do mathematics. We have attempted to make the results mentioned above and results like them (such as the Gauss-Bonnet theorem, the degree theorem, the Hopf theorem on vector fields) the main topic of our book rather than a mixed bag of interesting examples. In doing so we have abandoned algebraic topology altogether. Our point of view is that these theorems belong in a much more geometric realm of topology, namely intersection theory. Of course, intersection theory, properly done, requires its own apparatus: the transversality theorem. We must confess our sense of vulnerability to the charge that we have replaced one machine with another. Perhaps; *chacun á son goût*. The transversality arguments, it seems to us, can be visualized by the student, something which we feel cannot be honestly said of the singular homology functor.

This book is appropriate for a leisurely first year graduate course. We have also successfully taught a course based on the book to juniors and seniors. For undergraduates we suggest that certain topics be deleted—for example, the discussion of De Rham theory—and that the main emphasis be placed on mod 2 intersection theory, with some of the subjects from Chapter 3 presented mod 2 rather than with orientations. (The reader will notice, incidentally, that the section on De Rham theory is completely independent of the rest of the book. We have avoided using it even in our proof of the degree formula in Chapter 4, Section 8. We prove this formula using Stokes theorem rather than the theory of the top cohomology class, as in the usual treatment.)

The book is divided into four chapters. Chapter 1 contains the elementary theory of manifolds and smooth mappings. We define manifolds as subsets of Euclidean space. This has the advantage that manifolds appear as objects already familiar to the student who has studied calculus in $\mathbf{R}^2$ and $\mathbf{R}^3$; they are simply curves and sufaces generalized to higher dimensions. We also avoid confusing the student at the start with the abstract paraphernalia of charts and atlases. The most serious objection to working in Euclidean space is that it obscures the difference between properties intrinsic to the manifold and properties of its embedding. We have endeavored to make the student aware of this distinction, yet we have not scrupled to use the ambient space to make proofs more comprehensible. (See, for example, our use of the tubular neighborhood theorem in Chapter 2, Section 3). To provide cohesiveness to the elementary material, we have tried to emphasize the "stable" and "generic" quality of our definitions; whether this succeeds in making the basics more palatable, we leave to the reader's judgment.

The last two sections of Chapter 1 deal with Sard's theorem and some applications. Our most important use of Sard is in proving the transversality theorem in Chapter 2, but before doing so we use it to deduce the existence of Morse functions and to establish Whitney's embedding theorem. (Incidentally, it may seem pointless to prove the embedding theorem since our manifolds already sit in Euclidean space. We feel we have just placed the emphasis elsewhere: does there exist a k-dimensional mainfold so pathological that one cannot find a diffeomorphic copy of it inside Euclidean space of specified dimension N? Answer: no, provided $N \geq 2k + 1$.)

Chapter 2 begins by adding boundaries to manifolds. We classify one-manifolds and present Hirsch's proof of the Brouwer fixed-point theorem. Then the transversality theorem is derived, implying that transversal intersections are generic. Transversality permits us to define intersection numbers, and the one-manifold classification shows that they are homotopy invariants. At first we do intersection theorem mod 2, so that the student can become familiar with the topology without worrying about orientations. Moreover, mod 2 theory is the natural setting for the last two theorems of the chapter:

the Jordan-Brouwer separation theorem and the Borsuk-Ulam theorem. In each of the last three chapters we have included a section in which the student himself proves major theorems, with detailed guidance from the text. The Jordan-Brouwer separation theorem is the first of these. We found that our students received this enthusiastically, deriving real satisfaction from applying the techniques they had learned to significantly extend the theory.

In Chapter 3 we reconstruct intersection theory to include orientations. The Euler characteristic is defined as a self-intersection number and shown to vanish in odd dimensions. Next a primitive Lefschetz fixed-point theorem is proved and its use illustrated by an informal derivation of the Euler characteristics of compact surfaces. Translated into the vector field context, Lefschetz implies the Poincaré-Hopf index theorem. In an exercise section using the apparatus of the preceding discussions, the student proves the Hopf degree theorem and derives a converse to the index theorem. Finally, we relate the differential Euler characteristic to the combinational one.

Chapter 4 concerns forms and integration. The central result is Stokes theorem, which we do essentially as M. Spivak does in his *Calculus on Manifolds*. Stokes is used to prove an elementary but, we believe, largely underrated theorem: the degree formula relating integration to mappings. Finally, from this degree formula we derive the Gauss-Bonnet theorem for hypersurfaces in Euclidean space. Chapter 4 also includes an exercise section in which the student can construct De Rham cohomology and proves homotopy invariance. Although other problems relate cohomology to integration and intersection theory, the subject is treated essentially as an interesting aside to our primary discussion. (In particular, cohomology is not referred to anywhere else in the text.)

The original inspiration for this book was J. Milnor's lovely *Topology from the Differential Viewpoint*. Although our book in its present form involves a larger inventory of topics than Milnor's book, our debt to him remains clear.

We are indebted to Dan Quillen and John Mather for the elegant formulation of the transversality theorem given in Chapter 2, Section 3. We are also grateful to Jim King, Isadore Singer, Frank Warner, and Mike Cowan for valuable criticism, to Dennis Sullivan, Shlomo Sternberg, and Jim Munkres for many helpful conversations, and to Rena Themistocles and Phyllis Ruby for converting illegible scribble into typed manuscript. Most particularly, we are indebted to Barret O'Neill for an invaluable, detailed review of our first draft.

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