Foreword

Lars Ahlfors often spoke of his excitement as a young student listening to Rolf Nevanlinna's lectures on the new theory of meromorphic functions. It was, as he writes in his collected papers, his "first exposure to live mathematics." In his enormously influential research papers and in his equally influential books, Ahlfors shared with the reader, both professional and student, that excitement.

The present volume derives from lectures given at Harvard over many years, and the topics would now be considered quite classical. At the time the book was published, in 1973, most of the results were already decades old. Nevertheless, the mathematics feels very much alive and still exciting, for one hears clearly the voice of a master speaking with deep understanding of the importance of the ideas that make up the course.

Moreover, several of those ideas originated with or were cultivated by the author. The opening chapter on Schwarz's lemma contains Ahlfors' celebrated discovery, from 1938, of the connection between that very classical result and conformal metrics of negative curvature. The theme of using conformal metrics in connection with conformal mapping is elucidated in the longest chapter of the book, on extremal length. It would be hard to overstate the impact of that method, but until the book's publication there were very few places to find a coherent exposition of the main ideas and applications. Ahlfors credited Arne Beurling as the principal originator, and with the publication of Beurling's collected papers [2] one now has access to some of his own reflections.

Extremal problems are a recurring theme, and this strongly influences the choices Ahlfors makes throughout the book. Capacity is often discussed in relation to small point sets in function theory, with implications for existence theorems, but in that chapter Ahlfors has a different goal, aiming instead for the solution of a geometric extremal problem on closed subsets of the unit circle. The method of harmonic measure appeals to the Euclidean geometry of a domain and parts of its boundary to systematize the use of the maximum principle. Here Ahlfors concentrates on two problems, Milloux's problem, as treated in Beurling's landmark thesis, and a precise version of Hadamard's three circles theorem in a form given by Teichmüller. Nowhere else is there an accessible version of Teichmüller's solution. The chapter on harmonic measure provides only a small sample of a large circle of ideas, developed more systematically in the recent book [7].

Ahlfors devotes four short chapters to discussions of extremal problems for univalent functions, with focus on Loewner's parametric method and Schiffer's variational method. The material on coefficient estimates is now quite dated, following the proof of the Bieberbach conjecture by Louis de Branges [3] and its subsequent adaptation [6] appealing to the classical form of Loewner's differential equation. However, the methods of Loewner and Schiffer have broad applications in geometric function theory and their relevance is undiminished. More detailed treatments have since appeared [8,4], but Ahlfors' overview still brings these ideas to life. In recent years, Loewner's method has stepped into the limelight again with Oded Schramm's discovery of the stochastic Loewner equation and its connections with mathematical physics.

The final two chapters give an introduction to Riemann surfaces, with topological and analytical background supplied to support a proof of the uniformization theorem. In the author's treatment, as in all treatments, the main difficulty is in the parabolic case. Overall, the reader is encouraged to consult other sources for more details, for example [5].

We close with Ahlfors' own words from an address in 1953 at a conference celebrating the centennial of Riemann's dissertation [1]:

> Geometric function theory of one variable is already a highly developed branch of mathematics, and it is not one in which an easily formulated classical problem awaits its solution. On the contrary it is a field in which the formulation of essential problems is almost as important as their solution; it is a subject in which methods and principles are all-important, while an isolated result, however pretty and however difficult to prove, carries little weight.

The reader can learn much of this from the present volume. Furthermore, Ahlfors' remarks came around the time that quasiconformal mappings and, later, Kleinian groups began to flower, fields in which he was the leader. What a second volume those topics would have made!

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1

APPLICATIONS OF SCHWARZ'S LEMMA

1-1 THE NONEUCLIDEAN METRIC

The fractional linear transformation

$$S(z) = \frac{az + b}{\bar{b}z + \bar{a}} \tag{1-1}$$

with $|a|^2 - |b|^2 = 1$ maps the unit disk $\Delta = \{z; |z| < 1\}$ conformally onto itself. It is also customary to write (1-1) in the form

$$S(z) = e^{i\alpha} \frac{z - z_0}{1 - \bar{z}_0 z} \tag{1-2}$$

which has the advantage of exhibiting $z_0 = S^{-1}(0)$ and $\alpha = \arg S'(0)$.

Consider $z_1, z_2 \in \Delta$ and set $w_1 = S(z_1)$, $w_2 = S(z_2)$. From (1-1) we obtain

$$w_1 - w_2 = \frac{z_1 - z_2}{(\bar{b}z_1 + \bar{a})(\bar{b}z_2 + \bar{a})}$$

$$1 - \bar{w}_1 w_2 = \frac{1 - \bar{z}_1 z_2}{(b\bar{z}_1 + a)(\bar{b}z_2 + \bar{a})},$$

and hence
$$\left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| = \left|\frac{w_1 - w_2}{1 - \bar{w}_1 w_2}\right|. \tag{1-3}$$

We say that
$$\delta(z_1,z_2) = \left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| \tag{1-4}$$

is a *conformal invariant*. Comparison of (1-2) and (1-4) shows that $\delta(z_1,z_2) < 1$, a fact that can also be read off from the useful identity
$$1 - \delta(z_1,z_2)^2 = \frac{(1 - |z_1|^2)(1 - |z_2|^2)}{|1 - \bar{z}_1 z_2|^2}.$$

If z_1 approaches z_2, (1-3) becomes
$$\frac{|dz|}{1 - |z|^2} = \frac{|dw|}{1 - |w|^2}.$$

This shows that the Riemannian metric whose element of length is
$$ds = \frac{2|dz|}{1 - |z|^2} \tag{1-5}$$

is invariant under conformal self-mappings of the disk (the reason for the factor 2 will become apparent later). In this metric every rectifiable arc γ has an invariant length
$$\int_\gamma \frac{2|dz|}{1 - |z|^2},$$

and every measurable set E has an invariant area
$$\iint_E \frac{4dx\,dy}{(1 - |z|^2)^2}.$$

The shortest arc from 0 to any other point is along a radius. Hence the geodesics are circles orthogonal to $|z| = 1$. They can be considered straight lines in a geometry, the *hyperbolic* or *noneuclidean* geometry of the disk.

The noneuclidean distance from 0 to $r > 0$ is
$$\int_0^r \frac{2dr}{1 - r^2} = \log \frac{1 + r}{1 - r}.$$

Since $\delta(0,r) = r$, it follows that the noneuclidean distance $d(z_1,z_2)$ is connected with $\delta(z_1,z_2)$ through $\delta = \tanh\,(d/2)$.

The noneuclidean geometry can also be carried over to the half plane

$H = \{z = x + iy; y > 0\}$. The element of length that corresponds to the choice (1-5) is

$$ds = \frac{|dz|}{y}, \tag{1-6}$$

and the straight lines are circles and lines orthogonal to the real axis.

1-2 THE SCHWARZ–PICK THEOREM

The classic Schwarz lemma asserts the following: If f is analytic and $|f(z)| < 1$ for $|z| < 1$, and if $f(0) = 0$, then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Equality $|f(z)| = |z|$ with $z \neq 0$ or $|f'(0)| = 1$ can occur only for $f(z) = e^{i\alpha}z$, α a real constant.

There is no need to reproduce the well-known proof. It was noted by Pick that the result can be expressed in invariant form.

Theorem 1-1 An analytic mapping of the unit disk into itself decreases the noneuclidean distance between two points, the noneuclidean length of an arc, and the noneuclidean area of a set.

The explicit inequalities are

$$\frac{|f(z_1) - f(z_2)|}{|1 - \overline{f(z_1)}f(z_2)|} \leq \frac{|z_1 - z_2|}{|1 - \bar{z}_1 z_2|}$$
$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

Nontrivial equality holds only when f is a fractional linear transformation of the form (1-1).

Pick does not stop with this observation. He also proves the following more general version which deserves to be better known.

Theorem 1-2 Let $f\colon \Delta \to \Delta$ be analytic and set $w_k = f(z_k)$, $k = 1, \ldots, n$. Then the Hermitian form

$$Q_n(t) = \sum_{h,k=1}^{n} \frac{1 - w_h\bar{w}_k}{1 - z_h\bar{z}_k} t_h\bar{t}_k$$

is positive definite (or semidefinite).

PROOF We assume first that f is analytic on the closed disk. The function $F = (1 + f)/(1 - f)$ has a positive real part, and if $F = U + iV$

we have the representation

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} U(e^{i\theta})\, d\theta + iV(0).$$

This gives

$$F(z_h) + \overline{F(z_k)} = \frac{1}{\pi} \int_0^{2\pi} \frac{1 - z_h \bar{z}_k}{(e^{i\theta} - z_h)(e^{-i\theta} - \bar{z}_k)} U\, d\theta,$$

and hence

$$\sum_{h,k=1}^{n} \frac{F_h + \bar{F}_k}{1 - z_h \bar{z}_k} t_h \bar{t}_k = \frac{1}{\pi} \int_0^{2\pi} \left| \sum_1^n \frac{t_k}{e^{i\theta} - z_k} \right|^2 U\, d\theta \geq 0.$$

Here $F_h + \bar{F}_k = 2(1 - w_h \bar{w}_k)/(1 - w_h)(1 - \bar{w}_k)$. The factors in the denominator can be incorporated in t_h, $\bar{t}_k$, and we conclude that $Q_n(t) \geq 0$. For arbitrary f we apply the theorem to $f(rz)$, $0 < r < 1$, and pass to the limit.

Explicitly, the condition means that all the determinants

$$D_k = \begin{vmatrix} \dfrac{1 - |w_1|^2}{1 - |z_1|^2} & \cdots & \dfrac{1 - w_1 \bar{w}_k}{1 - z_1 \bar{z}_k} \\ \cdots & \cdots & \cdots \\ \dfrac{1 - w_k \bar{w}_1}{1 - z_k \bar{z}_1} & \cdots & \dfrac{1 - |w_k|^2}{1 - |z_k|^2} \end{vmatrix}$$

are ≥ 0. It can be shown that these conditions are also sufficient for the interpolation problem to have a solution. If $w_1, \ldots, w_{n-1}$ are given and $D_1, \ldots, D_{n-1} \geq 0$, the condition on w_n will be of the form $|w_n|^2 + 2 \operatorname{Re}(a w_n) + b \leq 0$. This means that w_n is restricted to a certain closed disk. It turns out that the disk reduces to a point if and only if $D_{n-1} = 0$.

The proof of the sufficiency is somewhat complicated and would lead too far from our central theme. We shall be content to show, by a method due to R. Nevanlinna, that the possible values of w_n fill a closed disk. We do not prove that this disk is determined by $D_n \geq 0$.

Nevanlinna's reasoning is recursive. For $n = 1$ there is very little to prove. Indeed, there is no solution if $|w_1| > 1$. If $|w_1| = 1$ there is a unique solution, namely, the constant w_1. If $|w_1| < 1$ and f_1 is a solution, then

$$f_2(z) = \frac{f_1(z) - w_1}{1 - \bar{w}_1 f_1(z)} : \frac{z - z_1}{1 - \bar{z}_1 z} \tag{1-7}$$

is regular in Δ, and we have proved that $|f_2(z)| \leq 1$. Conversely, for any such function f_2 formula (1-7) yields a solution f_1.

For $n = 2$ the solutions, if any, are among the functions f_1 already determined, and $f_2(z_2)$ must be equal to a prescribed value $w_2^{(2)}$. There are the same alternatives as before, and it is clear how the process continues. We are trying to construct a sequence of functions f_k of modulus ≤ 1 with certain prescribed values $f_k(z_k) = w_k^{(k)}$ which can be calculated from $w_1, \ldots, w_k$. If $|w_k^{(k)}| > 1$ for some k, the process comes to a halt and there is no solution. If $|w_k^{(k)}| = 1$, there is a unique f_k, and hence a unique solution of the interpolation problem restricted to $z_1, \ldots, z_k$. In case all $|w_k^{(k)}| < 1$, the recursive relations

$$f_{k+1}(z) = \frac{f_k(z) - w_k^{(k)}}{1 - \bar{w}_k^{(k)} f_k(z)} : \frac{z - z_k}{1 - \bar{z}_k z} \qquad k = 1, \ldots, n$$

lead to all solutions f_1 of the original problem when f_{n+1} ranges over all analytic functions with $|f_{n+1}(z)| \leq 1$ in Δ.

Because the connection between f_k and f_{k+1} is given as a fractional linear transformation, the general solution is of the form

$$f_1(z) = \frac{A_n(z) f_{n+1}(z) + B_n(z)}{C_n(z) f_{n+1}(z) + D_n(z)},$$

where A_n, B_n, C_n, D_n are polynomials of degree n determined by the data of the problem. We recognize now that the possible values of $f(z)$ at a fixed point do indeed range over a closed disk.

This solution was given in R. Nevanlinna [42]. The corresponding problem for infinitely many z_k, w_k was studied by Denjoy [17], R. Nevanlinna [43], and more recently Carleson [13].

1-3 CONVEX REGIONS

A set is convex if it contains the line segment between any two of its points. We wish to characterize the analytic functions f that define a one-to-one conformal map of the unit disk on a convex region. For simplicity such functions will be called convex univalent (Hayman [27]).

Theorem 1-3 An analytic function f in Δ is convex univalent if and only if

$$\operatorname{Re} \frac{z f''(z)}{f'(z)} \geq -1 \tag{1-8}$$

for all $z \in \Delta$. When this is true the stronger inequality

$$\left| \frac{z f''(z)}{f'(z)} - \frac{2|z|^2}{1 - |z|^2} \right| \leq \frac{2|z|}{1 - |z|^2} \tag{1-9}$$

is also in force.

Suppose for a moment that f is not only convex univalent but also analytic on the closed disk. It is intuitively clear that the image of the unit circle has a tangent which turns in the positive direction when $\theta = \arg z$ increases. This condition is expressed through $\partial/\partial\theta \arg df \geq 0$. But $\arg df = \arg f' + \arg dz = \arg f' + \theta + \pi/2$, and the condition becomes $\partial/\partial\theta (\arg f' + \theta) = \operatorname{Re} (zf''/f' + 1) \geq 0$ for $|z| = 1$. By the maximum principle the same holds for $|z| < 1$.

Although this could be made into a rigorous proof, we much prefer an idea due to Hayman. We may assume that $f(0) = 0$. If f is convex univalent, the function

$$g(z) = f^{-1}\left[\frac{f(\sqrt{z}) + f(-\sqrt{z})}{2}\right]$$

is well defined, analytic, and of absolute value <1 in Δ. Hence $|g'(0)| \leq 1$. But if $f(z) = a_1 z + a_2 z^2 + \cdots$, then $g(z) = (a_2/a_1)z + \cdots$, and we obtain $|a_2/a_1| \leq 1$, $|f''(0)/f'(0)| \leq 2$. This is (1-9) for $z = 0$.

We apply this result to $F(z) = f[(z + c)/(1 + \bar{c}z)]$, $|c| < 1$, which maps Δ on the same region. Simple calculations give

$$\frac{F''(0)}{F'(0)} = \frac{f''(c)}{f'(c)}(1 - |c|^2) - 2\bar{c},$$

and we obtain (1-9) and its consequence (1-8).

The proof of the converse is less elegant. It is evidently sufficient to prove that the image of $\Delta_r = \{z; |z| < r\}$ is convex for every $r < 1$. The assumption (1-8) implies that $\arg df$ increases with θ on $|z| = r$. Since f' is never zero, the change of $\arg df$ is 2π. Therefore, we can find θ_1 and θ_2 such that $\arg df$ increases from 0 to π on $[\theta_1, \theta_2]$ and from π to 2π on $[\theta_2, \theta_1 + 2\pi]$. If $f(re^{i\theta}) = u(\theta) + iv(\theta)$, it follows that v increases on the first interval and decreases on the second. Let v_0 be a real number between the minimum $v(\theta_1)$ and the maximum $v(\theta_2)$. Then $v(\theta)$ passes through v_0 exactly once on each of the intervals, and routine use of winding numbers shows that the image of Δ_r intersects the line $v = v_0$ along a single segment. The same reasoning applies to parallels in any direction, and we conclude that the image is convex.

The condition $|f''(0)/f'(0)| \leq 2$ has an interesting geometric interpretation. Consider an arc γ in Δ that passes through the origin and whose image is a straight line. The curvature of γ is measured by $d(\arg dz)/|dz|$. By assumption $d(\arg df) = 0$ along γ so that $d(\arg dz) = -d \arg f'$. The curvature is thus a directional derivative of $\arg f'$, and as such it is at most $|f''/f'|$ in absolute value. We conclude that the curvature at the origin is at most 2.

This result has an invariant formulation. If the curvature at the origin is ≤ 2, the circle of curvature intersects $|z| = 1$. But the circle of curvature is the circle of highest contact. A conformal self-mapping preserves circles and preserves order of contact. Circles of curvature are mapped on circles of curvature, and our result holds not only at the origin, but at any point.

Theorem 1-4 Let γ be a curve in Δ whose image under a conformal mapping on a convex region is a straight line. Then the circles of curvature of γ meet $|z| = 1$.

This beautiful result is due to Carathéodory.

1-4 ANGULAR DERIVATIVES

For $|a| < 1$ and $R < 1$ let $K(a,R)$ be the set of all z such that

$$\left| \frac{z - a}{1 - \bar{a}z} \right| < R.$$

Clearly, $K(a,R)$ is an open noneuclidean disk with center a and radius d such that $R = \tanh (d/2)$.

Let $K_n = K(z_n,R_n)$ be a sequence of disks such that $z_n \to 1$ and

$$\frac{1 - |z_n|}{1 - R_n} \to k \neq 0, \infty. \tag{1-10}$$

We claim that the K_n tend to the *horocycle* K_∞ defined by

$$\frac{|1 - z|^2}{1 - |z|^2} < k. \tag{1-11}$$

The horocycle is a disk tangent to the unit circle at $z = 1$.

The statement $K_n \to K_\infty$ is to be understood in the following sense: (1) If $z \in K_n$ for infinitely many n, then $z \in \bar{K}_\infty$, the closure of K_∞; (2) if $z \in K_\infty$, then $z \in K_n$ for all sufficiently large n. For the proof we observe that $z \in K_n$ is equivalent to

$$\frac{|1 - \bar{z}_n z|^2}{1 - |z|^2} < \frac{1 - |z_n|^2}{1 - R_n{}^2}. \tag{1-12}$$

If this is true for infinitely many n, we can go to the limit and obtain (1-11) by virtue of (1-10), except that equality may hold. Conversely, if

(1-11) holds, then

$$\lim_{n\to\infty} \frac{|1 - \bar{z}_n z|^2}{1 - |z|^2} < k$$

while

$$\lim_{n\to\infty} \frac{1 - |z_n|^2}{1 - R_n{}^2} = k,$$

so that (1-12) must hold for all sufficiently large n.

After these preliminaries, let f be analytic and $|f(z)| < 1$ in Δ. Suppose that $z_n \to 1$, $f(z_n) \to 1$, and

$$\frac{1 - |f(z_n)|}{1 - |z_n|} \to \alpha \neq \infty. \tag{1-13}$$

Given $k > 0$ we choose R_n so that $(1 - |z_n|)/(1 - R_n) = k$; this makes $0 < R_n < 1$ provided $1 - |z_n| < k$. With the same notation

$$K_n = K(z_n, R_n)$$

as above, we know by Schwarz's lemma that $f(K_n) \subset K'_n = K(w_n, R_n)$ where $w_n = f(z_n)$. The K_n converge to the horocycle K_∞ with parameter k as in (1-11), and because $(1 - |w_n|)/(1 - R_n) \to \alpha k$, the K'_n converge to K'_∞ with parameter αk. If $z \in K_\infty$, it belongs to infinitely many K_n. Hence $f(z)$ belongs to infinitely many K'_n and consequently to $\bar{K}'_\infty$. In view of the continuity it follows that

$$\frac{|1 - z|^2}{1 - |z|^2} \leq k \qquad \text{implies} \qquad \frac{|1 - f(z)|^2}{1 - |f(z)|^2} \leq \alpha k.$$

This is known as *Julia's lemma*.

Since k is arbitrary, the same result may be expressed by

$$\frac{|1 - f(z)|^2}{1 - |f(z)|^2} \leq \alpha \frac{|1 - z|^2}{1 - |z|^2},$$

or by

$$\beta = \sup \left[\frac{|1 - f(z)|^2}{1 - |f(z)|^2} : \frac{|1 - z|^2}{1 - |z|^2} \right] \leq \alpha.$$

In particular, α is never 0, and if $\beta = \infty$, there is no finite α.

Let us now assume $\beta < \infty$ and take $z_n = x_n$ to be real. Then

$$|1 - w_n|^2 < \beta \frac{1 - x_n}{1 + x_n},$$

and the condition $w_n \to 1$ is automatically fulfilled. Furthermore,

$$\beta \geq \frac{|1 - w_n|^2}{1 - |w_n|^2} \frac{1 + x_n}{1 - x_n} \geq \frac{1 + x_n}{1 + |w_n|} \frac{|1 - w_n|}{1 - x_n} \geq \frac{1 + x_n}{1 + |w_n|} \frac{1 - |w_n|}{1 - x_n}$$

so that (1-13) implies $\alpha \leq \beta$. Hence $\alpha = \beta$ for arbitrary approach along the real axis, and we conclude that

$$\lim_{x \to 1} \frac{1 - |f(x)|}{1 - x} = \lim_{x \to 1} \frac{|1 - f(x)|}{1 - x} = \beta. \tag{1-14}$$

Since $\beta \neq 0, \infty$, the equality of these limits easily implies $\arg [1 - f(x)] \to 0$, and with this information (1-14) can be improved to

$$\lim_{x \to 1} \frac{1 - f(x)}{1 - x} = \beta. \tag{1-15}$$

We have proved (1-14) and (1-15) only if $\beta \neq \infty$. However, if $\beta = \infty$, we know that (1-13) can never hold with a finite α. Hence (1-14) is still true, and for $\beta = \infty$ (1-14) implies (1-15).

So far we have shown that the quotient $[1 - f(z)]/(1 - z)$ always has a radial limit. We shall complete this result by showing that the quotient tends to the same limit when $z \to 1$ subject to a condition $|1 - z| \leq M(1 - |z|)$. The condition means that z stays within an angle less than π, and the limit is referred to as an angular limit.

Theorem 1-5 Suppose that f is analytic and $|f(z)| < 1$ in Δ. Then the quotient

$$\frac{1 - f(z)}{1 - z}$$

always has an angular limit for $z \to 1$. This limit is equal to the least upper bound of

$$\frac{|1 - f(z)|^2}{1 - |f(z)|^2} : \frac{|1 - z|^2}{1 - |z|^2},$$

and hence either $+\infty$ or a positive real number. If it is finite, $f'(z)$ has the same angular limit.

PROOF We have to show that β is an angular limit. If $\beta = \infty$, no new reasoning is needed, for we conclude as before that

$$\lim_{z \to 1} \frac{1 - |f(z)|}{1 - |z|} = \infty,$$

and when $|1 - z| \leq M(1 - |z|)$ this implies

$$\lim_{z \to 1} \frac{1 - f(z)}{1 - z} = \infty.$$

The case of a finite β can be reduced to the case $\beta = \infty$. The definition of β as a least upper bound implies

$$\mathrm{Re}\,\frac{1+z}{1-z} \le \beta\,\mathrm{Re}\,\frac{1+f(z)}{1-f(z)}.$$

Therefore we can write

$$\beta\,\frac{1+f}{1-f} - \frac{1+z}{1-z} = \frac{1+F}{1-F} \tag{1-16}$$

with $|F| < 1$. Because β cannot be replaced by a smaller number, it is clear that the function F must fall under the case $\beta = \infty$ so that $(1-z)/(1-F) \to 0$ in every angle. It then follows from (1-16) that $(1-f)/(1-z)$ has the angular limit β.

From (1-16) we have further

$$\beta f'(1-f)^{-2} - (1-z)^{-2} = F'(1-F)^{-2}.$$

We know by Schwarz's lemma that $|F'|/(1-|F|^2) \le 1/(1-|z|^2)$. With this estimate, together with $|1-z| \le M(1-|z|)$, we obtain

$$\left| \beta f'(z)\left[\frac{1-z}{1-f(z)}\right]^2 - 1 \right| \le 2M^2\,\frac{1-|z|}{1-|F|} \to 0,$$

and from this we conclude that $f'(z) \to \beta$.

When $\beta \ne \infty$, it is called the angular derivative at 1. In this case the limit $f(1) = 1$ exists as an angular limit, and β is the angular limit of the difference quotient $[f(z) - f(1)]/(z-1)$ as well as of $f'(z)$. The mapping by f is conformal at $z = 1$ provided we stay within an angle.

The theorem may be applied to $f_1(z) = e^{-i\delta}f(e^{-i\gamma}z)$ with any real γ and δ, but it is of no interest unless $f(z) \to e^{i\delta}$ as $z \to e^{i\gamma}$ along a radius. In that case the difference quotient $[f(z) - e^{i\delta}]/(z - e^{i\gamma})$ has a finite limit, and the mapping is conformal at $e^{i\gamma}$ if this limit is different from zero.

In many cases it is more convenient to use half planes. For instance, if $f = u + iv$ maps the right half plane into itself, we are able to conclude that

$$\lim_{z\to\infty} \frac{f(z)}{z} = \lim_{z\to\infty} \frac{u(z)}{x} = c = \inf \frac{u(z)}{x}, \tag{1-17}$$

the limits being restricted to $|\arg z| \le \pi/2 - \epsilon$, $\epsilon > 0$. Indeed, if the theorem is applied to $f_1 = (f-1)/(f+1)$ as a function of $z_1 = (z-1)/(z+1)$, we have $\beta = \sup x/u = 1/c$ and

$$\lim_{z_1\to 1} \frac{1-z_1}{1-f_1} = \lim_{z\to\infty} \frac{1+f}{1+z} = c.$$

This easily implies (1-17). Note that c is finite and ≥ 0.

The proof of Theorem 1-5 that we have given is due to Carathéodory [10]. We have chosen this proof because of its clear indication that the theorem is in fact a limiting case of Schwarz's lemma. There is another proof, based on the Herglotz representation of an analytic function with positive real part, which is perhaps even simpler. We recall the Poisson-Schwarz representation used in the proof of Theorem 1-2. For positive U it can be rewritten in the form

$$F(z) = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(\theta) + iC,$$

where μ denotes a finite positive measure on the unit circle. In this form, as observed by Herglotz, it is valid for arbitrary analytic functions with a positive real part.

Apply the formula to $F = (1 + f)/(1 - f)$, where $|f(z)| < 1$ in Δ. Let $c \geq 0$ denote $\mu(\{0\})$, i.e., the part of μ concentrated at the point 1, and denote the rest of the measure by μ_0 so that we can write

$$\frac{1+f}{1-f} = c\,\frac{1+z}{1-z} + \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu_0(\theta) + iC. \tag{1-18}$$

For the real parts we thus have

$$\frac{1 - |f|^2}{|1 - f|^2} = c\,\frac{1 - |z|^2}{|1 - z|^2} + \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2}\, d\mu_0(\theta), \tag{1-19}$$

from which it is already clear that

$$\frac{1 - |f|^2}{|1 - f|^2} \geq c\,\frac{1 - |z|^2}{|1 - z|^2}.$$

We rewrite (1-19) as

$$\frac{1 - |f|^2}{|1 - f|^2} : \frac{1 - |z|^2}{|1 - z|^2} = c + I(z)$$

with

$$I(z) = \int_0^{2\pi} \frac{|1 - z|^2}{|e^{i\theta} - z|^2}\, d\mu_0(\theta).$$

We claim that $I(z) \to 0$ as $z \to 1$ in an angle. For this purpose we choose δ so small that the μ_0 measure of the interval $(-\delta, \delta)$ is less than a given $\epsilon > 0$. Divide $I(z)$ in two parts:

$$I = I_0 + I_1 = \int_{-\delta}^{\delta} + \int_{\delta}^{2\pi - \delta}.$$

If $|1 - z| \leq M(1 - |z|)$, it is immediate that $|I_0| \leq M^2\epsilon$. It is obvious that $I_1 \to 0$, and we conclude that $I(z) \to 0$ in an angle. This proves that $c = 1/\beta$ in the earlier notation

If the same reasoning is applied directly to (1-18), we find that $(1 - z)(1 + f)(1 - f)^{-1} \to 2c$ in an angle, and this is equivalent to $(1 - z)/(1 - f) \to c$. This completes the alternate proof of Theorem 1-5.

As an application we shall prove a theorem known as Löwner's lemma. As before, f will be an analytic mapping of Δ into itself, but this time we add the assumption that $|f(z)| \to 1$ as z approaches an open arc γ on $|z| = 1$. Then f has an analytic extension to γ by virtue of the reflection principle, and $f'(\zeta) \neq 0$ for $\zeta \in \gamma$. Indeed, if $f'(\zeta)$ were zero the value $f(\zeta)$ would be assumed with multiplicity greater than 1, and this is incompatible with $|f(\zeta)| = 1$ and $|f(z)| < 1$ for $|z| < 1$. It is also true that $\arg f(\zeta)$ increases with $\arg \zeta$ so that f defines a locally one-to-one mapping of γ on an arc γ'.

Theorem 1-6 If in these circumstances $f(0) = 0$, then the length of γ' is at least equal to the length of γ.

PROOF We apply Theorem 1-5 to $F(z) = f(\zeta z)/f(\zeta)$, $\zeta \in \gamma$. The angular derivative at $z = 1$ is

$$\lim_{r \to 1} \frac{1 - F(r)}{1 - r} = F'(1) = \frac{\zeta f'(\zeta)}{f(\zeta)} = |f'(\zeta)|,$$

for $\arg f'(\zeta) = \arg [f(\zeta)/\zeta]$. But $|1 - F(r)| \geq 1 - |F(r)| \geq 1 - r$ by Schwarz's lemma. Hence $|f'(\zeta)| \geq 1$, and the theorem follows.

1-5 ULTRAHYPERBOLIC METRICS

Quite generally, a Riemannian metric given by the fundamental form

$$ds^2 = \rho^2(dx^2 + dy^2), \tag{1-20}$$

or $ds = \rho|dz|$, $\rho > 0$, is conformal with the euclidean metric. The quantity

$$K(\rho) = -\rho^{-2} \Delta \log \rho$$

is known as the *curvature* (or gaussian curvature) of the metric (1-20). The reader will verify that the metrics (1-5) and (1-6) have constant curvature -1 [the factor 2 in (1-5) was chosen with this in mind].

In this text, which deals primarily with complex variables, the geometric definition of curvature is unimportant, and we use the name only as a convenience. It is essential, however, that $K(\rho)$ is invariant under conformal mappings.

Consider a conformal mapping $w = f(z)$ and define $\tilde{\rho}(w)$ so that $\rho|dz| = \tilde{\rho}|dw|$ or, more explicitly, $\rho(z) = \tilde{\rho}[f(z)]|f'(z)|$. Because $\log |f'(z)|$ is harmonic, it follows that $\Delta \log \rho(z) = \Delta \log \tilde{\rho}(w)$, both laplacians being with respect to z. Change of variable in the laplacian follows the rule $\Delta_z \log \tilde{\rho} = |f'(z)|^2 \Delta_w \log \tilde{\rho}$, and we find that $K(\rho) = K(\tilde{\rho})$.

From now on the hyperbolic metric in Δ will be denoted by $\lambda|dz|$; that is to say, we set

$$\lambda(z) = \frac{2}{1 - |z|^2}.$$

We wish to compare $\lambda|dz|$ with other metrics $\rho|dz|$.

Lemma 1-1 If ρ satisfies $K(\rho) \leq 1$ everywhere in Δ, then $\lambda(z) \geq \rho(z)$ for all $z \in \Delta$.

PROOF We assume first that ρ has a continuous and strictly positive extension to the closed disk. From $\Delta \log \lambda = \lambda^2$, $\Delta \log \rho \geq \rho^2$ we have $\Delta(\log \lambda - \log \rho) \leq \lambda^2 - \rho^2$. The function $\log \lambda - \log \rho$ tends to $+\infty$ when $|z| \to 1$. It therefore has a minimum in the unit disk. At the point of minimum $\Delta(\log \lambda - \log \rho) \geq 0$ and hence $\lambda^2 \geq \rho^2$, proving that $\lambda \geq \rho$ everywhere.

To prove the lemma in the general case we replace $\rho(z)$ by $r\rho(rz)$, $0 < r < 1$. This metric has the same curvature, and the smoothness condition is fulfilled. Hence $\lambda(z) \geq r\rho(rz)$, and $\lambda(z) \geq \rho(z)$ follows by continuity.

The definition of curvature requires $\Delta \log \rho$ to exist, so we have to assume that ρ is strictly positive and of class C^2. These restrictions are inessential and cause difficulties in the applications. They can be removed in a way that is reminiscent of the definition of subharmonic functions.

Definition 1-1 A metric $\rho|dz|$, $\rho \geq 0$ is said to be ultrahyperbolic in a region Ω if it has the following properties:

(*i*) ρ is upper semicontinuous.
(*ii*) At every $z_0 \in \Omega$ with $\rho(z_0) > 0$ there exists a "supporting metric" ρ_0, defined and of class C^2 in a neighborhood V of z_0, such that $\Delta \log \rho_0 \geq \rho_0^2$ and $\rho \geq \rho_0$ in V, while $\rho(z_0) = \rho_0(z_0)$.

Because $\log \lambda - \log \rho$ is lower semicontinuous, the existence of a minimum is still assured. The minimum will also be a local minimum of $\log \lambda - \log \lambda_0$, and the rest of the reasoning applies as before. The inequality $\lambda(z) \geq \rho(z)$ holds as soon as ρ is ultrahyperbolic.

We are now ready to prove a stronger version of Schwarz's lemma.

Theorem 1-7 Let f be an analytic mapping of Δ into a region Ω on which there is given an ultrahyperbolic metric ρ. Then $\rho[f(z)]|f'(z)| \leq 2(1 - |z|^2)^{-1}$.

The proof consists in the trivial observation that $\rho[f(z)]|f'(z)|$ is ultrahyperbolic on Δ. Observe that the zeros of $f'(z)$ are singularities of this metric.

REMARK The notion of an ultrahyperbolic metric makes sense, and the theorem remains valid if Ω is replaced by a Riemann surface. In this book only the last two chapters deal systematically with Riemann surfaces, but we shall not hesitate to make occasional references to Riemann surfaces when the need arises. Thus in our next section we shall meet an application of Theorem 1-7 in which Ω is in fact a Riemann surface, but the adaptation will be quite obvious.

1-6 BLOCH'S THEOREM

Let $w = f(z)$ be analytic in Δ and norm lized by $|f'(0)| = 1$. We may regard f as a one-to-one mapping of Δ onto a Riemann surface W_f spread over the w plane. It is intuitively clear what is meant by an unramified disk contained in W_f. As a formal definition we declare that an unramified disk is an open disk Δ' together with an open set $D \subset \Delta$ such that f restricted to D defines a one-to-one mapping of D onto Δ'. Let B_f denote the least upper bound of the radii of all such disks Δ'. Bloch made the important observation that B_f cannot be arbitrarily small. In other words, the greatest lower bound of B_f for all normalized f is a positive number B, now known as Bloch's constant. Its value is not known, but we shall prove Theorem 1-8:

Theorem 1-8 $B \geq \sqrt{3}/4$.

PROOF Somewhat informally we regard $w = f(z)$ both as a point on W_f and as a complex number. Let $R(w)$ be the radius of the largest unramified disk of center w contained in W_f [at a branch-point $R(w) = 0$]. We introduce a metric $\tilde{\rho}|dw|$ on W_f defined by

$$\tilde{\rho}(w) = \frac{A}{R(w)^{\frac{1}{2}}[A^2 - R(w)]}$$

where A is a constant $> B_f^{\frac{1}{2}}$. This induces a metric $\rho(z) = \tilde{\rho}[f(z)]|f'(z)|$ in Δ. We wish to show that $\rho(z)$ is ultrahyperbolic for a suitable choice of A.

Suppose that the value $w_0 = f(z_0)$ is assumed with multiplicity $n > 1$. For w close to w_0 (or rather z close to z_0), $R(w) = |w - w_0|$, which is of the order $|z - z_0|^n$. Since $|f'(z)|$ is of order $|z - z_0|^{n-1}$, it follows that $\rho(z)$ is of order $|z - z_0|^{n/2-1}$. If $n > 2$, it follows that ρ is continuous and

$\rho(z_0) = 0$. We recall that there is no need to look for a supporting metric at points where ρ is zero.

In case $n = 2$ we have

$$\rho(z) = \frac{A|f'(z)|}{|f(z) - f(z_0)|^{\frac{1}{2}}[A^2 - |f(z) - f(z_0)|]}$$

near z_0. This metric is actually regular at z_0, and it satisfies $\Delta \log \rho = \rho^2$ as seen either by straightforward computation or from the fact that $\rho|dz| = 2|dt|/(1 - |t|^2)$ with $t = A^{-1}[f(z) - f(z_0)]^{\frac{1}{2}}$.

It remains to find a supporting metric at a point $w_0 = f(z_0)$ with $f'(z_0) \neq 0$. Denote the disk $\{w; |w - w_0| < R(w_0)\}$ by $\Delta'(w_0)$ and by $D(z_0)$ the component of its inverse image that contains z_0. The boundary of $D(z_0)$ must contain either a point $a \in \Delta$ with $f'(a) = 0$, or a point a on the unit circle, for otherwise $\Delta'(w_0)$ would not be maximal. In the first case the boundary of $\Delta'(w_0)$ passes through the branch-point $b = f(a)$. In the second case $f(a)$ is not defined, but we make the harmless assumption that f can be extended continuously to the closed unit disk. The point $b = f(a)$ is then on the boundary of $\Delta'(w_0)$ and may also be regarded as a boundary point of the Riemann surface W_f.

Choose $z_1 \in D(z_0)$, $w_1 = f(z_1) \in \Delta'(w_0)$. It is geometrically clear that $R(w_1) \leq |w_1 - b|$. For a more formal reasoning we consider $\Delta'(w_1)$ and $D(z_1)$. Let c be the line segment from w to b. If b were in $\Delta'(w_1)$, all of c except the last point would be in $\Delta'(w_0) \cap \Delta'(w_1)$. But the inverse functions f^{-1} with values in $D(z_0)$ and $D(z_1)$ agree on this set, and it would follow by continuity that $a \in D(z_1)$. This is manifestly impossible. We conclude that b is not in $\Delta'(w_1)$, and hence that $R(w_1) \leq |w_1 - b|$.

Now we compare $\rho(z)$ with

$$\rho_0(z) = \frac{A|f'(z)|}{|f(z) - b|^{\frac{1}{2}}[A^2 - |f(z) - b|]}$$

when z is close to z_0. This metric has constant curvature -1 and

$$\rho_0(z_0) = \rho(z_0).$$

Moreover, the inequality $\rho(z) \geq \rho_0(z)$ holds near z_0 if the function $t^{\frac{1}{2}}(A^2 - t)$ remains increasing for $0 \leq t \leq R(w_0)$. The derivative changes sign at $t = A^2/3$. We conclude that $\rho(z)$ is ultrahyperbolic if $A^2 > 3B_f$.

All that remains is to apply Lemma 1-1 with $z = 0$. We obtain $A \leq 2R[f(0)]^{\frac{1}{2}}\{A^2 - R[f(0)]\} \leq 2B_f^{\frac{1}{2}}(A^2 - B_f)$. The inequality $B_f \geq \sqrt{3}/4 > 0.433$ follows on letting A tend to $(3B_f)^{\frac{1}{2}}$.

It is conjectured that the correct value of B is approximately 0.472. This value is assumed for a function that maps Δ on a Riemann surface with branch points of order 2 over all vertices in a net of equilateral triangles.

1-7 THE POINCARÉ METRIC OF A REGION

The hyperbolic metric of a disk $|z| < R$ is given by

$$\lambda_R(z) = \frac{2R}{R^2 - |z|^2}. \tag{1-21}$$

If ρ is ultrahyperbolic in $|z| < R$, we must have $\rho \leq \lambda_R$. In particular, if ρ were ultrahyperbolic in the whole plane we would have $\rho = 0$. Hence there is no ultrahyperbolic metric in the whole plane.

The same is true of the punctured plane $\{z;\, z \neq 0\}$. Indeed, if $\rho(z)$ were ultrahyperbolic in the punctured plane, then $\rho(e^z)|e^z|$ would be ultrahyperbolic in the full plane. These are the only cases in which an ultrahyperbolic metric fails to exist.

Theorem 1-9 In a plane region Ω whose complement has at least two points, there exists a unique maximal ultrahyperbolic metric, and this metric has constant curvature -1.

The maximal metric is called the *Poincaré metric* of Ω, and we denote it by λ_Ω. It is maximal in the sense that every ultrahyperbolic metric ρ satisfies $\rho \leq \lambda_\Omega$ throughout Ω. The uniqueness is trivial.

The existence proof is nonelementary and will be postponed to Chap. 10. The reader will note, however, that the applications we are going to make do not really depend on the existence of the Poincaré metric. At present its main purpose is to allow a convenient terminology.

Theorem 1-10 If $\Omega \subset \Omega'$, then $\lambda_{\Omega'} \leq \lambda_\Omega$.

This is obvious, for the restriction of $\lambda_{\Omega'}$ to Ω is ultrahyperbolic in Ω.

Theorem 1-11 Let $\delta(z)$ denote the distance from $z \in \Omega$ to the boundary of Ω. Then $\lambda_\Omega(z) \leq 2/\delta(z)$.

Ω contains the disk with center z and radius $\delta(z)$. The estimate follows from Theorem 1-10 together with (1-21). It is the best possible, for equality holds when Ω is a disk and z its center.

It is a much harder problem to find lower bounds.

1-8 AN ELEMENTARY LOWER BOUND

Let $\Omega_{a,b}$ be the complement of the two-point set $\{a,b\}$ and denote its Poincaré metric by $\lambda_{a,b}$. If a and b are in the complement of Ω, then

$\Omega \subset \Omega_{a,b}$ and $\lambda_\Omega \geq \lambda_{a,b}$. A lower bound for $\lambda_{a,b}$ is therefore a lower bound for λ_Ω. Because

$$\lambda_{a,b}(z) = |b - a|^{-1}\lambda_{0,1}\frac{(z - a)}{(b - a)}$$

it is sufficient to consider $\lambda_{0,1}$. There are known analytic expressions for $\lambda_{0,1}$, but they are not of great use. What we require is a good elementary lower bound.

The region $\Omega_{0,1}$ is mapped on itself by $1 - z$ and by $1/z$. Therefore $\lambda_{0,1}(z) = \lambda_{0,1}(1 - z) = |z|^{-2}\lambda_{0,1}(1/z)$. It follows that we need consider only $\lambda_{0,1}$ in one of the regions $\Omega_1, \Omega_2, \Omega_3$ marked in Fig. 1-1.

We begin by determining a better upper bound than the one given by Theorem 1-11. $\Omega_{0,1}$ contains the punctured disk $0 < |z| < 1$. The Poincaré metric of the punctured disk is found by mapping its universal covering, an infinitely many-sheeted disk, on the half plane $\operatorname{Re} w < 0$ by means of $w = \log z$. The metric is $|dw|/|\operatorname{Re} w| = |dz|/|z| \log (1/|z|)$, and we obtain

$$\lambda_{0,1}(z) \leq \left(|z| \log \frac{1}{|z|}\right)^{-1} \tag{1-22}$$

for $|z| < 1$. This estimate shows what order of magnitude to expect.

Let $\zeta(z)$ be the function that maps the complement of $[1, +\infty]$ conformally on the unit disk, origins corresponding to each other and symmetry with respect to the real axis being preserved.

Theorem 1-12 For $|z| \leq 1$, $|z| \leq |z - 1|$, i.e., for $z \in \Omega_1$,

$$\lambda_{0,1}(z) \geq \left|\frac{\zeta'(z)}{\zeta(z)}\right| [4 - \log |\zeta(z)|]^{-1}. \tag{1-23}$$

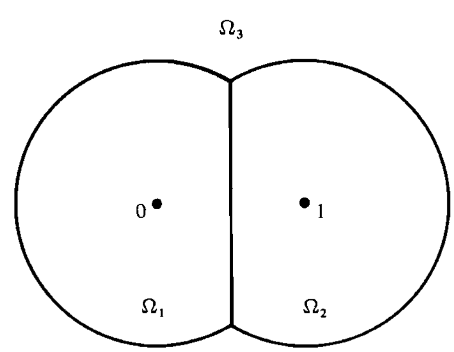

FIGURE 1-1

For $z \to 0$, (1-22) and (1-23) imply

$$\log \lambda_{0,1}(z) = -\log |z| - \log\log \frac{1}{|z|} + O(1). \tag{1-24}$$

PROOF It is immediate that the metric defined by

$$\rho(z) = \left| \frac{\zeta'(z)}{\zeta(z)} \right| [4 - \log |\zeta(z)|]^{-1} \tag{1-25}$$

has curvature -1, for it is obtained from the Poincaré metric of the punctured disk $0 < |\zeta| < e^4$. We use (1-25) only in Ω_1 and extend ρ to Ω_2 and Ω_3 by means of the symmetry relations $\rho(1 - z) = \rho(z)$ and $\rho(1/z) = |z|^2\rho(z)$. The extended metric is obviously continuous. We need to verify that ρ has a supporting metric on the lines that separate $\Omega_1, \Omega_2, \Omega_3$. Because of the symmetry it is sufficient to consider the line segment between Ω_1 and Ω_2. It is readily seen that the original ρ, as given by (1-25) in Ω_1 and part of Ω_2, constitutes a supporting metric provided $\partial\rho/\partial x < 0$ on the separating line segment.

The mapping function is given explicitly by

$$\zeta(z) = \frac{\sqrt{1 - z} - 1}{\sqrt{1 - z} + 1}$$

with $\operatorname{Re} \sqrt{1 - z} > 0$. In

$$\frac{\partial \log \rho}{\partial x} = \operatorname{Re} \left(\frac{d}{dz} \log \frac{\zeta'}{\zeta} \right) + \operatorname{Re} \frac{\zeta'}{\zeta} (4 - \log |\zeta|)^{-1}$$

we substitute

$$\frac{\zeta'}{\zeta} = \frac{1}{z\sqrt{1 - z}}$$

$$\frac{d}{dz} \log \frac{\zeta'}{\zeta} = \frac{3z - 2}{2z(1 - z)}.$$

On taking into account that $1 - z = \bar{z}$ on the line segment, we find

$$\frac{\partial \log \rho}{\partial x} = -\frac{1}{4|z|^2} + \frac{\operatorname{Re} \sqrt{z}}{|z|^2} (4 - \log |\zeta|)^{-1},$$

and this is negative because $|\zeta| < 1$ and $\operatorname{Re} \sqrt{z} < 1$.

We conclude that (1-23) holds. The passage to (1-24) is a trivial verification.

1-9 THE PICARD THEOREMS

We use Theorems 1-7 and 1-12 to prove a classic theorem known as the Picard-Schottky theorem. The emphasis is on the elementary nature of the proof and the explicit estimates obtained.

Theorem 1-13 Suppose that $f(z)$ is analytic and different from 0 and 1 for $|z| < 1$. Then

$$\log|f(z)| \leq [7 + \overset{+}{\log}|f(0)|]\frac{1+|z|}{1-|z|}. \tag{1-26}$$

REMARK As usual, $\overset{+}{\log}|f(0)|$ is the greater of $\log|f(0)|$ and 0. The constant in the bound is not the best possible, but the order of magnitude of the right-hand side is right.

PROOF Because $1/f$ satisfies the same conditions as f it is irrelevant whether we derive an upper or a lower bound for $\log|f|$. The way we have formulated Theorem 1-12, it is slightly more convenient to look for a lower bound.

By assumption f maps Δ into $\Omega_{0,1}$. By Theorem 1-7 we therefore have

$$\lambda_{0,1}[f(z)]|f'(z)| \leq \frac{2}{1-|z|^2}.$$

We obtain by integration

$$\int_{f(0)}^{f(z)} \lambda_{0,1}(w)|dw| \leq \log\frac{1+|z|}{1-|z|}, \tag{1-27}$$

where the integral is taken along the image of the line segment from 0 to z. We use the notation Ω_1 of the previous section and assume first that the whole path of integration lies in Ω_1. The estimate (1-23) can be applied and gives

$$\int_{f(0)}^{f(z)} (4 - \log|\zeta(w)|)^{-1}|d\log\zeta(w)| \leq \log\frac{1+|z|}{1-|z|}. \tag{1-28}$$

On noting that $|d\log\zeta| \geq -d\log|\zeta|$ we find

$$\frac{4 - \log|[f(z)]|}{4 - \log|[f(0)]|} \leq \frac{1+|z|}{1-|z|}. \tag{1-29}$$

From the explicit expression

$$|\zeta(w)| = \frac{|w|}{|1+\sqrt{1-w}|^2}$$

we derive $(1+\sqrt{2})^{-2}|w| \le |\zeta(w)| \le |w|$, the lower bound being quite crude. With these estimates, and since $\log(1+\sqrt{2}) < 1$, we obtain from (1-29)

$$-\log|f(z)| < [6 - \log|f(0)|]\frac{1+|z|}{1-|z|}. \tag{1-30}$$

Now let us drop the assumption that the path in (1-27) stays in Ω_1. If $f(z) \in \Omega_1$, (1-28) is still true if we start the integral from w_0, the last point on the boundary of Ω_1. Since $|w_0| \ge \frac{1}{2}$, the inequality (1-30) is replaced by

$$-\log|f(z)| < (6 + \log 2)\frac{1+|z|}{1-|z|} \tag{1-31}$$

which is also trivially true in case $f(z)$ is not in Ω_1. The inequalities (1-30) and (1-31) can be combined to give

$$-\log|f(z)| < \left[6 + \log 2 + \overset{+}{\log}\frac{1}{|f(0)|}\right]\frac{1+|z|}{1-|z|},$$

and (1-26) is a weaker version with f replaced by $1/f$. The theorem is proved.

Corollary The little Picard theorem If f is meromorphic in the whole plane and omits three values, then f is constant.

PROOF If f omits a,b,c then $F = [(c-b)/(c-a)][(f-a)/(f-b)]$ is holomorphic and omits 0,1. Apply Theorem 1-13 to $F(Rz)$ with $R > 0$. It follows that $|F(Re^{i\theta}/2)|$ lies under a finite bound, independent of R and θ. Hence $|F(z)|$ is bounded, and F must be a constant by Liouville's theorem.

Theorem 1-14 The big Picard theorem If f is meromorphic and omits three values in a punctured disk $0 < |z| < \delta$, then it has a meromorphic extension to the full disk.

PROOF We may assume that $\delta = 1$ and that f omits $0, 1, \infty$. Comparison of $\lambda_{0,1}$ with the Poincaré metric of the punctured disk yields

$$\lambda_{0,1}[f(z)]|f'(z)| \le \left(|z| \log\frac{1}{|z|}\right)^{-1}.$$

We integrate along a radius from $z_0 = r_0e^{i\theta}$ to $z = re^{i\theta}$, $r < r_0 < 1$. If $f(z) \in \Omega_1$, we obtain as in the preceding proof

$$\log\{4 - \log|f(z)|\} \le \log\log\frac{1}{|z|} + A,$$

where A is an irrelevant constant. This implies

$$-\log |f(z)| \leq C \log \frac{1}{|z|}$$

with some other constant, showing that $1/|f|$ is bounded by a power of $1/|z|$. Hence the isolated singularity at the origin is not essential.

NOTES The Schwarz lemma and its classic proof are due to Carathéodory [10]; Schwarz proved it only for one-to-one mappings [58, p. 109]. Although Poincaré had used noneuclidean geometry for function theoretic purposes, Pick [50, 51] seems to be the first to have fully realized the invariant character of Schwarz's lemma. Theorem 1-2 has been included mainly for historical reasons.

Theorem 1-5 was first proved by Carathéodory [11] but independently and almost simultaneously by Landau and Valiron [35]. All three were unaware that the theorem is an easy consequence of Herglotz's integral representation of positive harmonic functions. We have given preference to Carathéodory's proof because of its geometric character.

Ultrahyperbolic metrics (without the name) were introduced by Ahlfors [1]. They have recently found many new applications in the theory of several complex variables.

There are many proofs of Bloch's theorem, that of Landau [34] probably being the simplest. The original theorem is in Bloch [8]. Heins has improved on the author's bound by showing that $B > \sqrt{3}/4$ (Heins [28]). See also Pommerenke [52].

Stronger forms of (1-26) can be found in Jenkins [32], but his proof uses the modular function. Our proof of the Picard theorems is elementary not only because it avoids the modular function, but also because it does not use the monodromy theorem.

EXERCISES

1 Derive formulas for the noneuclidean center and radius of a circle contained in the unit disk or the half plane.

2 Show that two circular arcs in the unit disk with common end points on the unit circle are noneuclidean parallels in the sense that the points on one arc are at constant distance from the other.

3 Let $z = z(t)$ be an arc of class C^3. Show that the rate of change of its curvature can be expressed through

$$|z'(t)|^{-1} \operatorname{Im} \left[\frac{z'''(t)}{z'(t)} - \frac{3}{2} \left(\frac{z''(t)}{z'(t)} \right)^2 \right].$$

4 Formulate and prove the analog of Theorem 1-5 for functions with positive real part on the right half plane.

5 Verify that the spherical metric

$$ds = \frac{2|dz|}{1 + |z|^2}$$

has constant curvature 1.

6 If f is analytic in the unit disk Δ and normalized by $|f'(0)| = 1$, let L_f be the least upper bound of the radii of all disks covered by the image $f(\Delta)$. Imitating the proof of Bloch's theorem, show that the greatest lower bound of L_f is a constant $L \geq \frac{1}{2}$.