Preface to the revised edition

In the first edition of this book the main attention was focused on the methods of solving the inverse problem of spectral analysis and on the conditions (necessary and sufficient) which the spectral data must satisfy in order to make it possible to reconstruct the potential of the corresponding Sturm-Liouville operator. These conditions imply that the spectral data (e.g. spectral function or scattering data) must be known for all values of spectral parameter which belong to the spectrum of the operator.

But from the physical meaning of the inverse problem it is obvious that the values of spectral data on the whole spectrum are impossible to obtain from any observations. For example, in the inverse problem of quantum scattering theory the energy of the particles acts as the spectral parameter, and in order to find the values of scattering data on the whole spectrum one has to conduct an experiment with the particles of infinitely large energy. But for big enough values of energy the scattering process is not any more described by Schrödinger equation with potential $q(x)$. Therefore, even allowing, ideally, the possibility to experiment with particles of arbitrarily large energies, we would obtain, starting from a certain energy, data relevant to process, which has certainly nothing to do with the equation that we want to reconstruct. Hence, a principal question is as follows: what information about the potential $q(x)$ can be obtained, if the spectral function or scattering data are known (generally speaking, approximately) only on a finite interval of values of the spectral parameter?

The new Chapter 5, devoted to solving this problem, was added to this edition. The convenient formulae are obtained, which allow to estimate the precision with which the eigenfunctions and potentials of Schrödinger operator can be restored when the scattering data or spectral function are known only on a finite interval of values of spectral parameter.

V. Marchenko

PREFACE

The development of many important directions of mathematics and physics owes a major debt to the concepts and methods which evolved during the investigation of such simple objects as the Sturm-Liouville equation $y'' + q(x)y = zy$ and the allied Sturm-Liouville operator $L = -d^2/dx^2 + q(x)$ (lately L and $q(x)$ are often termed the one-dimensional Schrödinger operator and the potential). These provided a constant source of new ideas and problems in the spectral theory of operators and kindred areas of analysis. This sourse goes back to the first studies of D. Bernoulli and L. Euler on the solution of the equation describing the vibrations of a string, and still remains productive after more than two hundred years. This is confirmed by the recent discovery, made by C. Gardner, J. Green, M. Kruskal, and R. Miura [6], of an unexpected connection between the spectral theory of Sturm-Liouville operators and certain nonlinear partial differential evolution equations.

The methods used (and often invented) during the study of the Sturm-Liouville equation have been constantly enriched. In the 40's a new investigation tool joined the arsenal - that of transformation operators. The latter first appeared in the theory of generalized translation operators of J. Delsarte and B. M. Levitan (see [16]). Transformation operators for arbitrary Sturm-Liouville equations were constructed by A. Ya. Povzner [24], who used them to derive the eigenfunction expansion for a Sturm-Liouville equation with a decreasing potential (it seem that his work is the first in which transformation operators were used in spectral theory). V. A. Marchenko enlisted transformation operators to investigate both inverse problems of spectral analysis [17] and the asymptotic behavior of the spectral function of singular Sturm-Liouville operators [18].

The role of transformation operators in spectral theory became even more important following several discoveries. Specifically, I. M. Gelfand and

B. M. Levitan [8] found that these operators can be used to provide a complete solution to the problem of recovering a Sturm-Liouville equation from its spectral function; B. M. Levitan [15] proved the equiconvergence theorem in its general form; B. Ya. Levin [14] introduced a new type of transformation operators which preserve the asymptotics of the solutions at infinity; and V. A. Marchenko [19] used them to solve the inverse scattering problem.

The main goal of this monograph is to show what can be achieved with the aid of transformation operators in spectral theory, as well as in its recently revealed untraditional applications. We made such an attempt in our book [20], which was published in 1972 and was based on lectures delivered at Khar'kov University. In the years that followed, transformation operators have been applied to an increasing number of problems, and we felt that a more complete discussion of the results in this area was needed. In the present book, aside from traditional topics that are treated roughly in the same way as in our previous monograph, we include new applications of transformation operators and problems connected with the use of spectral theory in the study of nonlinear equations.

In the first chapter transformation operators are used to investigate the boundary value problem generated on a finite interval by the Sturm-Liouville operator with arbitrary nondegenerate boundary conditions. One proves the completeness of the system of eigenfunctions and generalized eigenfunctions. Moreover, one obtains asymptotic formulas for $\lambda \to \infty$ for the solutions of the Sturm-Liouville equation, and then use them to derive asymptotic formulas for the eigenvalues of the boundary value problems under consideration. All these formulas have been known for a long time. However, it turns out that with the aid of transformation operators one can express the principal parts of their remainders explicitly in terms of the Fourier coefficients of the potential $q(x)$. For example, one can establish the exact relationship between the smoothness of a periodic potential and the rate of decay of the lengths of the lacunae in the spectrum of the corresponding Hill operator. The concluding part of the chapter is devoted to the derivation of the Gelfand-Levitan trace formulas [9], which are becoming more and more important.

In the second chapter we discuss the singular boundary value problems generated on the half line $0 \leqslant x < \infty$ by the Sturm-Liouville operator having an arbitrary complex-valued potential $q(x)$ and subject to the boundary conditions $y'(0) - hy(0) = 0$. The notion of a generalized (distribution) spectral function is introduced, and its existence is established for this class of boundary value problems. The Riesz theorem on the form of linear positive functionals shows that the distribution spectral functions are measures whenever $q(x)$ and h are real. In this case the formulas for the expansion in eigenfunctions and the Parseval equality, generated by a distribution spectral function, lead to classical results of H. Weyl. In Section 3 we derive the Gelfand-Levitan integral equation [8]. This enables us to recover the operator from its distribution spectral function. We also find conditions necessary and sufficient for a distribution to be the spectral function of a Sturm-Liouville operator. In the last section the asymptotic formula of Marchenko [18] for the spectral functions of symmetric boundary value problems is obtained (in the sharpened form due to B. M. Levitan [15]) and Levitan's equiconvergence theorem [15] is proved.

The third chapter is devoted to inverse problems in scattering theory and the inverse problem for the Hill equation. The Levin transformation operators [14] are introduced and then used to study the properties of the solutions to a Sturm-Liouville equation whose potential satisfies the constraint $\int_0^\infty x|q(x)|dx < \infty$. Next, we derive Marchenko's integral equation [19], which enables us to recover the potential from the scattering data, and we establish the characteristic properties of these data. In addition, we discuss the results of V. A. Marchenko and I. V. Ostrovskii [22]: we find conditions necessary and sufficient for a given sequence of intervals to equal the set of stability zones of a Hill equation, and show that the set of potentials having a finite number of such zones (known as "finite-zone" potentials) is dense. We also prove a theorem of Gasymov and Levitan [7] which includes a complete solution of the inverse problem in G. Borg's formulation [2]. The last section of Chapter 3 is devoted to the inverse problem of scattering theory for the Sturm-Liouville operator on the full real line. There we prove Faddeev's theorem [5] which gives the characteristic properties of scattering data.

In the last chapter we show how spectral theory can be used to integrate certain nonlinear partial differential equations - a fact which was discovered by C. Gardner, J. Green, M. Kruskal, and R. Miura [6]. Following the publication of Lax's work [12] and of the paper [27] by V. E. Zakharov and A. B. Shabat, based on the ideas of [12], it became clear that this new integration method can be applied to a large number of nonlinear equations which occur in mathematical physics (see the survey paper [3]). Here we discuss in detail only the Korteweg-de Vries equation $\dot{v} = 6vv' - v'''$, and use the exercises to guide the reader's way towards possible generalizations. In Section 1 we give a general presentation of the new integration method which differs somewhat from Lax's scheme, and which permits us to include auxiliary linear operators which depend arbitrarily upon the spectral parameter z. Next, we solve the Cauchy problem for the Korteweg-de Vries equation in the class of rapidly decreasing potentials using the method developed in [6]. The periodic problem for this equation was attacked first in 1974 using different methods, in studies by S. P. Novikov [23], P. Lax [13], and V. A. Marchenko [21]. In Section 3 we discuss the method invented in [21], while Problems 2 and 3 give the proofs of two theorems of Novikov [23], and thereby exhibit a connection between these methods. For a detailed discussion of the results obtained by following the ideas of [23] we refer the interested reader to the survey paper [3].

In 1961 N. I. Akhiezer [1] discovered a relation between the inverse problems for certain Sturm-Liouville operators having a finite number of lacunae in the spectrum and the Jacobi inversion problem for Abelian integrals. Developing Akhiezer's ideas, A. R. Its and V. B. Matveev [10] found an explicit formula for the finite-zone potentials in terms of Riemann's Θ-function. Combining this formula with results of B. A. Dubrovin and S. P. Novikov [4] one finds a simple expression for the finite-zone periodic and almost-periodic solutions of the Korteweg-de Vries equation. We derive this formula in the last section of Chapter 4.

The exercises in the monograph are presented with enough hints so that one can recover the full proofs. This should enable the reader to see possible refinements and generalizations of the material treated in the main text. In particular, the problems include results of M. Crum, M. G. Krein, and V. F. Korop (on degenerate transformation operators and equations with singularities), of M. G. Gasymov, B. M. Levitan, and I. S. Sargsyan (on Dirac systems of equations), of F. S. Rofe-Beketov (on operator Sturm-Liouville equations), and of V. S. Buslaev, M. I. Lomonosov, and L. D. Faddeev (on a continual analog of the trace formula).

Finally, we wish to emphasize that the author did not intend to exhaust all the aspects and methods of spectral theory, and this is why many of its facets are not discussed here. In particular, we do not touch upon deficiency indices, or the character of the spectrum, or the theory of extensions of operators. Nor have we include fundamental results of H. Weyl, E. Titchmarsh, M. V. Keldysh, M. G. Krein, and M. A. Naimark, the majority of which are treated in well-known monographs on the spectral theory of operators. We also omit an analysis of the stability of the inverse problem of spectral theory. This topic is dealt with in detail in the monograph [20].

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