

INTRODUCTION

An n -dimensional formal group law over a ring A is an n -tuple of power series $F(X, Y)$ in $X_1, \dots, X_n; Y_1, \dots, Y_n$ with coefficients in A such that

$$F(X, 0) = X, \quad F(0, Y) = Y, \quad F(X, F(Y, Z)) = F(F(X, Y), Z)$$

(where X and Y are short for the vectors $(X_1, \dots, X_n), (Y_1, \dots, Y_n)$). If moreover $F(X, Y) = F(Y, X)$, the formal group law is said to be commutative. Three most important examples are $\hat{G}_a(X, Y) = X + Y$, $\hat{G}_m(X, Y) = X + Y + XY$ (both one dimensional) and the infinite dimensional formal group law $\hat{W}_{p^\infty}(X, Y)$ defined by the addition polynomials $\Sigma_0(X; Y), \Sigma_1(X; Y), \dots$ over \mathbf{Z} of the Witt vectors, which in turn are defined by

$$w_{p^n}(\Sigma_0, \dots, \Sigma_n) = w_{p^n}(X) + w_{p^n}(Y)$$

where

$$w_{p^n}(X) = X_0^{p^n} + pX_1^{p^n-1} + \dots + p^nX_n$$

One way to view formal group laws is as recipes for manufacturing ordinary groups (by substituting, say, topologically nilpotent elements for the X_i and Y_i).

There are at least three ways in which formal group laws arise naturally:

(a) Let G be an n -dimensional analytic Lie group. Let $e \in G$ be the identity element of G . Take analytic coordinates in a neighborhood V of e such that e has coordinates $(0, 0, \dots, 0)$. Let $x, y \in V$ have coordinates x_1, \dots, x_n and y_1, \dots, y_n , respectively. If x and y are close enough to e , we have $z = xy \in V$. Let z_1, \dots, z_n be the coordinates of z . Now since G is analytic, the z_i are analytic in the $x_1, \dots, x_n; y_1, \dots, y_n$; and taking a power series development around $(0, \dots, 0)$, we have for x, y close enough to e n power series

$$z_i = f_i(x_1, \dots, x_n; y_1, \dots, y_n)$$

These n power series define a formal group law $\hat{G}(x, y)$ in the sense of the definition above. They constitute so to speak the infinitesimal group structure of order ∞ at e of G . In particular the Lie algebra \mathfrak{g} of G is recoverable from \hat{G} , and thus \hat{G} is an intermediate object between \mathfrak{g} and G .

Now much the same construction can be performed for a smooth algebraic group G defined, say, over a field k of characteristic $p > 0$. In this case the Lie

algebra of G carries very little information about G , and it was as a possibly good substitute for Lie theory in the case of characteristic $p > 0$ that the theory of formal groups found its first vigorous development in the hands of Dieudonné.

In particular, in the case of abelian varieties A , the associated formal group law \hat{A} has since been found to carry much information on the arithmetic of A .

(b) Let $L(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ be a Dirichlet series with coefficients in \mathbf{Z} . One associates to $L(s)$ the power series $f_L(X) = \sum_{n=1}^{\infty} n^{-1}a(n)X^n \in \mathbf{Q}[[X]]$ and $F_L(X, Y) = f_L^{-1}(f_L(X) + f_L(Y))$ where $f_L^{-1}(X)$ is defined by $f_L^{-1}(f_L(X)) = X$. Then the coefficients of $F_L(X, Y)$ are in $\mathbf{Z}_{(p)}$ precisely when $L(s)$ has an Euler factor for the prime p in the sense that

$$L(s) = (1 + e_1 p^{-s} + e_2 p^{1-2s} + e_3 p^{2-3s} + \cdots)^{-1} \sum_{n=1}^{\infty} b(n)n^{-s}$$

with $b(n) \equiv 0 \pmod{p^r}$ if $p^r \mid n$, $e_i \in \mathbf{Z}_p$. These two ways in which formal group laws arise in nature are not independent. Indeed, it is precisely the connection between (a) and (b) which, e.g., in the case of elliptic curves E over \mathbf{Q} , gives some beautiful results concerning the zeta function of E .

(c) Let h^* be a multiplicative extraordinary cohomology theory which has first Chern classes in a suitable technical sense. Then (because \mathbf{CP}^∞ is classifying for line bundles) there is a universal formula

$$c_1(\xi \otimes \eta) = \sum_{i,j} a_{ij} c_1(\xi)^i c_1(\eta)^j$$

which gives the first Chern class of a tensor product of two complex line bundles in terms of the first Chern classes of the factors. The power series $F_h(X, Y) = \sum a_{ij} X^i Y^j$ is then a one dimensional formal group law over $h(pt)$, the ring of coefficients of h^* ; and, as it turns out, $F_h(X, Y)$ carries a good deal of information about h^* .

These three classes of examples make it reasonable to study formal group laws more deeply (even if one did not know about other applications, for example to local class field theory and global class field theory for function fields).

Now in any case for the class of examples arising from analytic Lie groups, the formal group laws are intermediate between Lie groups and Lie algebras. So there ought to be "formal Lie theory," that is, Lie theory without convergence. And indeed, specializing to the one dimensional case, one has: let $F(X, Y)$ be a one dimensional commutative formal group law over a \mathbf{Q} -algebra R , then there is a unique power series $f(X) \in R[[X]]$ such that $f(X) = X + \cdots$ and $f(F(X, Y)) = f(X) + f(Y)$. This $f(X)$ is called the logarithm of $F(X, Y)$. So if $F(X, Y)$ is, e.g., a formal group law over \mathbf{Z} , then over \mathbf{Q} there exists a power series $f(X) \equiv X \pmod{(\text{degree } 2)}$ such that $F(X, Y) = f^{-1}(f(X) + f(Y))$. Thus

the problem of finding all one dimensional formal group laws over \mathbf{Z} becomes, What power series $f(X)$ over \mathbf{Q} are such that $f^{-1}(f(X) + f(Y))$ has integral coefficients? In (b) above we have seen an example of this. Roughly the condition is that $f(X)$ must exhibit the kind of regularity exemplified by the splitting off of an Euler factor in the sense indicated in (b).

The precise answer is given by what I call the functional equation lemma, which is, without a doubt the most important tool in this book. The precise statement of the functional equation lemma takes more space than one should use in an introduction, so let us try to see by means of examples what kind of lemma it is.

(d) Let $f(X), g(X) \in \mathbf{Q}[[X]]$ be two power series in one variable X such that $f(X) \equiv g(X) \equiv X \pmod{\text{degree } 2}$ and $f(X) - p^{-1}f(X^p) \in \mathbf{Z}_{(p)}[[X]]$, $g(X) - p^{-1}g(X^p) \in \mathbf{Z}_{(p)}[[X]]$. Then $F(X, Y) = f^{-1}(f(X) + f(Y))$ and $g^{-1}(f(X))$ have their coefficients in $\mathbf{Z}_{(p)}$ (not just \mathbf{Q}). Thus, for example, Hasse's lemma that $\exp(X + p^{-1}X^p + p^{-2}X^{p^2} + \dots)$ has its coefficients in $\mathbf{Z}_{(p)}$ is an application of the functional equation lemma. This is of course related to the statement made under (b). The Euler factor in this case is $1 - p^{-s}$.

(e) Let $f(X) \in \mathbf{Q}[T][[X]]$ be the power series

$$f(X) = X + p^{-1}TX^{p^h} + p^{-2}TT^{p^h}X^{p^{2h}} + p^{-3}TT^{p^h}T^{p^{2h}}X^{p^{3h}} + \dots$$

then $f^{-1}(f(X) + f(Y))$ has its coefficients in $\mathbf{Z}[T]$. This actually gives us quite a few different formal group laws over \mathbf{Z} by substituting $h = 1, 2, \dots$ and, e.g., $T = 1$. (These are, incidentally, the formal group laws associated to the so-called extraordinary K -theories.)

(f) Consider the Witt polynomials $w_{p^n}(X) = X_0^{p^n} + pX_1^{p^{n-1}} + \dots + p^nX_n$. It is obvious that they satisfy $w_{p^n}(X) \equiv w_{p^{n-1}}(X^p) \pmod{p^n}$. And, given this, the functional equation lemma says that the polynomials $\Sigma_0, \Sigma_1, \dots$ determined by $w_{p^n}(\Sigma_0, \dots, \Sigma_n) = w_{p^n}(X) + w_{p^n}(Y)$, $n = 0, 1, 2, \dots$, have coefficients in \mathbf{Z} .

(g) Let $L(s)$ be a Dirichlet series with Euler factor $1 + e_1p^{-s} + e_2p^{1-2s} + \dots$ as in (b). Let $h(X)$ be any power series with coefficients in \mathbf{Z} and let $f_L(h(X)) = g(X) = \sum_{n=1}^{\infty} n^{-1}d(n)X^n$. Then

$$\sum d(n)n^{-s} = (1 + e_1p^{-s} + e_2p^{1-2s} + \dots)^{-1} \sum c(n)n^{-s}$$

with $c(n) \equiv 0 \pmod{p^r}$ if $p^r \mid n$. That is, the same Euler factor splits off. And this is how we shall prove the Atkin–Swinnerton–Dyer conjectures in Section 33.2.

(h) Let R be a ring in which all prime numbers $\neq p$ are invertible and suppose that R is torsion free. Let $F(X, Y)$ be a formal group law over R and let $f(X) \in R \otimes \mathbf{Q}[[X]]$ be the logarithm of $F(X, Y)$. Write $f(X) = \sum a_n X^n$ and let $\hat{f}(X) = \sum a_{p^m} X^{p^m}$. Then via the functional equation lemma one finds that $\hat{F}(X, Y) = \hat{f}^{-1}(\hat{f}(X) + \hat{f}(Y))$ has its coefficients in R . (Note that the relation

between $f(X)$ and $\hat{f}(X)$ is the same as between the ordinary logarithm, $-\log(1-X) = X + 2^{-1}X^2 + 3^{-1}X^3 + \cdots$ and Hasse's p -logarithm $H(X) = X + p^{-1}X^p + p^{-2}X^{p^2} + \cdots$.) Now this so-called p -typification operation can be applied in topology to split off from complex cobordism cohomology $MU_{(p)}^*$ (localized at (p)) a factor BP^* (Brown–Peterson cohomology) which so to speak involves only the prime number p .

The formal group laws of MU^* and BP^* have the following logarithms

$$\log_{MU}(X) = \sum_{n=0}^{\infty} \frac{[\mathbf{CP}^n]}{n+1} X^{n+1}, \quad \log_{BP}(X) = \sum_{n=0}^{\infty} \frac{[\mathbf{CP}^{p^n-1}]}{p^n} X^{p^n}$$

where $[\mathbf{CP}^m]$ is the class of complex projective space of complex dimension m . Now $F_{MU}(X, Y)$ turns out to be a universal one dimensional formal group law, and it follows that $F_{BP}(X, Y)$ is universal for formal group laws whose logarithms involve only the X^{p^n} and no other powers of X .

Now let $f_V(X)$ over $\mathbf{Q}[V_1, V_2, \dots]$ be the power series

$$(*) \quad f_V(X) = \sum_{n=0}^{\infty} a_n(V) X^{p^n}$$

$$a_0(V) = 1, \quad pa_n(V) = a_{n-1}(V)V_1^{p^{n-1}} + \cdots + a_1(V)V_{n-1}^p + a_0(V)V_n$$

and let $F_V(X, Y) = f_V^{-1}(f_V(X) + f_V(Y))$. Then, again, the functional equation lemma gives us that $F_V(X, Y)$ has its coefficients in $\mathbf{Z}[V]$. And one finds another formal group law which is universal for formal group laws whose logarithms involve only the powers X^{p^n} of X . It follows that $BP(pt)$ and $\mathbf{Z}_{(p)}[V]$ are identifiable in such a way that $a_n(V)$ corresponds to $p^{-n}[\mathbf{CP}^{p^n-1}] = m_n \in BP(pt)$. Then because we have formulas for the V_n in terms of the $a_n(V)$, we find polynomial generators v_1, v_2, \dots of $BP(pt)$ related to m_n by

$$pm_n = m_{n-1}v_1^{p^{n-1}} + \cdots + m_1v_{n-1}^p + v_n$$

These generators v_1, v_2, \dots have proved to be useful for calculations on a number of occasions (e.g., to prove that certain elements in the stable homotopy groups of the spheres are nonzero).

(i) A further contribution of the functional equation lemma to our understanding of formal group laws is that it practically dictates how the logarithm of a universal formal group law should look. It must be (in a certain sense) of the form

$$f_U(X) = \sum a_n(u) X^n$$

$$a_n(u) = \sum_{(i_1, \dots, i_s)} d(i_1, \dots, i_s) U_{i_1} U_{i_2}^{i_1} \cdots U_{i_s}^{i_1 \cdots i_{s-1}}$$

where the sum is over all sequences (i_1, \dots, i_s) , $i_j \in \mathbf{N}$, such that $i_1 \cdots i_s = n$ and where the $d(i_1, \dots, i_s)$ are certain coefficients which can be specified recursively.

In this connection let me remark that to the human eye at least all the

regularity in a universal formal group law sits in its logarithm not in the formal group law itself—maybe understandably, as the differential $f'(X) dX$ of the logarithm $f(X)$ can easily be interpreted as the unique (up to a scalar factor) invariant differential on the formal group law $F(X, Y)$.

To illustrate this remark I have written at the end of this introduction the first few terms of the “3-typical” universal formal group law $F_V(X, Y)$, whose logarithm (cf. (*) above) is certainly eminently regular and also the first few terms of the universal formal group law $F_U(X, Y)$. (The calculations were done by computer to degree 23 for $F_V(X, Y)$ and degree 11 for $F_U(X, Y)$.)

In this introduction I have not tried to give a short description of the contents of the book. For that, the curious reader is invited to glance at the table of contents which is reasonably detailed. Instead, I have tried to give the flavor of some of the more important constructions and results, and I have tried to give some small indication of how diverse and sometimes surprising the applications of the theory of formal groups are.

Nobody who falls down stairs like that can be all bad. (R. A. Lafferty, *Fourth Mansions*)

The first few terms of the one dimensional universal formal group law $F_U(X, Y)$

$$\begin{aligned}
 F_U(X, Y) = & X + Y + XY(-U_2) + (XY^2 + X^2Y)(-U_3 + U_2^2) \\
 & + (XY^3 + X^3Y)(-2U_4 + 2U_2U_3 + 2U_2^3) \\
 & + X^2Y^2(-3U_4 + 4U_2U_3 - 4U_2^3) \\
 & + (XY^4 + X^4Y)(-U_5 + 4U_2U_4 - 3U_2^2U_3 + 3U_2^3 + U_3^2) \\
 & + (X^2Y^3 + X^3Y^2) \\
 & \times (-2U_5 + 11U_2U_4 - 11U_2^2U_3 + 10U_2^3 + 3U_3^2) \\
 & + (XY^5 + X^5Y) \\
 & \times (-6U_6 + 2U_5U_2 - 6U_2U_3^2 \\
 & \qquad \qquad \qquad + 4U_3U_4 - 6U_2^2U_4 + 2U_2^3U_3 - 4U_2^5) \\
 & + (X^2Y^4 + X^4Y^2) \\
 & \times (-15U_6 + 7U_5U_2 - 22U_2U_3^2 \\
 & \qquad \qquad \qquad + 15U_3U_4 - 28U_2^2U_4 + 21U_2^3U_3 - 21U_2^5) \\
 & + X^3Y^3 \\
 & \times (-20U_6 + 10U_5U_2 - 33U_2U_3^2 + 22U_3U_4 \\
 & \qquad \qquad \qquad - 43U_2^2U_4 + 37U_2^3U_3 - 34U_2^5) \\
 & + \dots
 \end{aligned}$$

By the time one reaches degree 11 the coefficient of X^5X^6 involves 42 different monomials in the U 's with coefficients like 78447.

The first few terms of the one dimensional universal p -typical formal group law $F_v(X, Y)$ (for the prime $p = 3$).

$$\begin{aligned}
F_v(X, Y) = & X + Y + (XY^2 + Y^2X)(-V_1) \\
& + (XY^4 + X^4Y)(V_1^2) + (X^2Y^3 + X^3Y^2)(3V_1^2) \\
& + (XY^6 + X^6Y)(-V_1^3) + (X^2Y^5 + X^5Y^2)(-6V_1^3) \\
& + (X^3Y^4 + X^4Y^3)(-13V_1^3) \\
& + (XY^8 + X^8Y)(-3V_2) + (X^2Y^7 + X^7Y^2)(-12V_2 + 6V_1^4) \\
& + (X^3Y^6 + X^6Y^3)(-28V_2 + 27V_1^4) \\
& + (X^4Y^5 + X^5Y^4)(-42V_2 + 52V_1^4) \\
& + (XY^{10} + X^{10}Y)(6V_1V_2 + V_1^5) + (X^2Y^9 + X^9Y^2)(45V_1V_2) \\
& + (X^3Y^8 + X^8Y^3)(163V_1V_2 - 27V_1^5) \\
& + (X^4Y^7 + X^7Y^4)(362V_1V_2 - 106V_1^5) \\
& + (X^5Y^6 + X^6Y^5)(532V_1V_2 - 192V_1^5) \\
& + \cdots \\
& + (X^{10}Y^{13} + X^{13}Y^{10}) \\
& \times (-105024048V_1^3V_2^2 + 95416130V_1^7V_2 + 21339672V_1^{11}) \\
& + \cdots
\end{aligned}$$