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*Sections marked with a single asterisk are not actually needed in conjunction with any of the unmarked sections. These sections may be safely omitted if time permits only a short course.

**Sections marked with a double asterisk are not actually needed in conjunction with any other section. They are included as interesting and useful applications of the theory.