Foreword to the First Russian Edition

Generalized functions have of late been commanding constantly expanding interest in several different branches of mathematics. In somewhat nonrigorous form, they have already long been used in essence by physicists.

Important to the development of the theory have been the works of Hadamard dealing with divergent integrals occurring in elementary solutions of wave equations, as well as some work of M. Riesz. We shall not discuss here the even earlier mathematical work which could also be said to contain some groundwork for the future development of this theory.

The first to use generalized functions in the explicit and presently accepted form was S. L. Sobolev in 1936 in studying the uniqueness of solutions of the Cauchy problem for linear hyperbolic equations.

From another point of view Bochner's theory of the Fourier transforms of functions increasing as some power of their argument can also bring one to the theory of generalized functions. These Fourier transforms, in Bochner's work the formal derivatives of continuous functions, are in essence generalized functions.

In 1950-1951 there appeared Laurent Schwartz's monograph *Théorie des Distributions*. In this book Schwartz systematizes the theory of generalized functions, basing it on the theory of linear topological spaces, relates all the earlier approaches, and obtains many important and far reaching results. Unusually soon after the appearance of *Théorie des Distributions*, in fact literally within two or three years, generalized functions attained an extremely wide popularity. It is sufficient just to point out the great increase in the number of mathematical works containing the delta function.

In the volumes of the present series we will give a systematic development of the theory of generalized functions and of problems in analysis connected with it. On the one hand our aims do not include the colation of all material related in some way to generalized functions, and on the other hand many of the problems we shall consider can be treated without invoking them. However, the concept is a convenient link connecting many aspects of analysis, functional analysis, the theory of differential equations, the representation theory of locally compact Lie groups, and the theory of probability and statistics. It is perhaps for this reason that the title generalized functions is most appropriate for this series of volumes on functional analysis.

Let us briefly recount the contents of the first four volumes of the series.

The first volume is devoted essentially to algorithmic questions of the theory. Its first two chapters represent an elementary introduction to generalized functions. In this volume the reader will encounter many applications of generalized functions to various problems of analysis. Here and there throughout the book theorems are presented whose proofs will be found in the second volume. Volume I makes wide use of Schwartz's book and of the article on homogeneous functions by Gel'fand and Z. Ya. Shapiro (*Uspekhi Matem. Nauk*, 1955). Shapiro has also written some of the paragraphs of the present volume.

The second volume develops the concepts introduced in the first, uses topological considerations to prove theorems left unproved in the latter, and constructs and studies a large number of specific generalized function spaces. The basis for all this is one of the most elementary and, for analysts, one of the most useful fields of the theory of linear topological spaces (developed in Chapter I of the second volume), namely the theory of countably normed spaces.

The third volume is devoted to some applications of generalized functions to the theory of differential equations, in particular to constructing the uniqueness and consistency classes for solutions to the Cauchy problem in partial differential equations, and to expansions in eigenfunctions of differential operators. Here we make systematic use of the results obtained in the second volume.

In the fourth and fifth volumes we consider problems in probability theory related to generalized functions (generalized random processes) and the theory of the representation of Lie groups. The unifying concept here is that of harmonic analysis (the analog of Fourier integral theory) of generalized functions, in particular questions related to the representations of positive definite functions. In these volumes we present the kernel theorem of Schwartz.

Volumes I to III are written by G. E. Shilov and myself, while Volumes IV and V are written by N. Ya. Vilenkin and me.

The section entitled *Notes and References to the Literature* contains some historical remarks, citations of sources, and bibliography. In the text, however, no source references are made; references to the textbook literature are given in footnotes.

Of course all of this hardly exhausts the possibilities of application for generalized functions. The necessity for going deeper into the relations to differential equations is quite apparent (for instance with respect to boundary value problems, equations with variable coefficients, and many problems in quasi-linear equations). Further, the theory of generalized functions is the most convenient foundation on which to construct a general theory of the representation of Lie groups and, in particular, the general theory of spherical and generalized automorphic functions. We hope later to be able to shed some light also on these questions.

The authors of the first volume express their gratitude to their colleagues and students who have in one way or another taken part in the creation of this volume, in particular to V. A. Borovik, N. Ya. Vilenkin, M. I. Graev, and Z. Ya. Shapiro. The authors also express their gratitude to M. S. Agranovich, who edited the entire manuscript and introduced many improvements.

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Foreword to the Second Russian Edition

The material in this second edition of the first volume has been somewhat rearranged in order to make it easier to read. The first two chapters, Definition and Simplest Properties of Generalized Functions, and Fourier Transforms of Generalized Functions, can be recommended as an initial introduction; they contain the standard minimum which must be known by all mathematicians and physicists who have to deal with generalized functions.

The plan for further reading depends on the interests of the reader. Readers interested in the algorithmic aspects of the discussion can turn to Chapter III of this volume, which is devoted to special classes of generalized functions, namely to delta functions on surfaces of various dimension, generalized functions related to higher dimensional quadratic forms (of arbitrary signature), homogeneous functions, and functions equivalent to homogeneous functions. Such readers may also turn to Appendix B, which treats homogeneous generalized functions in the complex domain. To the reader interested in the general theory, we recommend that after reading the first two chapters of this volume he turn to the first three chapters of Volume II. Those contain, among other things, the necessary information from the theory of linear topological spaces. He may then turn to Chapter I of Volume IV, which describes nuclear spaces and measures in them. Those readers who wish to learn about the applications of generalized functions to the theory of partial differential equations may turn to Chapters II and III of Volume III, after first looking at the chapters on spaces of type S and W (Chapter IV of Volume II and Chapter I of Volume III). Spectral theory and its applications will be found in Chapter IV of Volume III and Chapter I of Volume IV; prerequisite for these are the first two chapters of Volume II. Other programs for reading are also possible; for instance questions concerned with the application of Fourier transforms of generalized functions are discussed after Chapter II of Volume I and Chapters III and IV of Volume II, the first three chapters of Volume III, and some of the chapters of Volume V. The application of generalized functions to the theory of group representations and of Fourier transforms over a group is described in Volume V, for which one need only have read the first two chapters of Volume I.

For convenience in using the first volume, we have placed at the end of it a résumé of the basic definitions and formulas and a table of Fourier transforms of generalized functions.

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