Preface to the Russian Edition

The constructions of Volume 1 proceeded on the basis of utilizing a few fundamental spaces: (1) the space K of infinitely differentiable functions having compact supports; (2) the space S of infinitely differentiable functions decreasing at infinity, together with all their derivatives, more rapidly than any power of $1/|x|$; (3) the space Z of analytic functions $\varphi(z)$, satisfying inequalities of the form $z^k\varphi(z) \leqslant C_k e^{a|y|}$. Generalized functions—continuous linear functionals on these spaces—were adequate for the clarification of the fundamental features of the theory and for a number of simple, but important, applications to some questions of analysis, and in particular, to the theory of differential equations.

On the other hand, although we tried there to reduce to a minimum the number of spaces utilized, we did not succeed in bypassing one pair of spaces K and K': by considering generalized functions as continuous linear functionals in K, we inevitably had to consider their Fourier transforms as continuous linear functionals on Z. The advantages of such a viewpoint will be seen particularly clearly in Volume 5, where methods of complex variable function theory will render substantial assistance in algorithmic questions of the theory of generalized functions.

We shall need a considerably more extensive circle of spaces in Volume 3, which is devoted to deeper applications of the theory of generalized functions to differential equations, than those which we encountered periodically in Volume 1, and will meet here and there in Volume 2. Namely, applications of the theory of generalized functions to the Cauchy problem and to the problem of eigenfunction expansions will be elucidated in Volume 3. Here, the fundamental peculiarity of the theory of generalized functions, in that form in which we shall understand it in this book, will be completely apparent; it is that *different classes of problems require different classes of spaces*, and, indeed classes of spaces and not individual spaces.

Thus, uniqueness and existence theorems for the solution of the Cauchy problem for different partial differential equations require different spaces, which however possess some common properties. Problems of eigenfunction expansions for different differential operators also require different spaces which, nevertheless, have a number of common features. And similarly, boundary value problems for elliptic equations require their class of fundamental spaces and spaces of generalized functions.

In the preceding stage of development of functional analysis, which was connected with the theory of integral equations, the common base for the study of the various functional spaces encountered was the theory of linear normed spaces.*

Normed spaces turned out to be inadequate for the needs of the theory of generalized functions. It must not be thought that the situation is such that much more complex constructions would be required. It is directly opposite: among the normed spaces one does not find the simplest spaces, for example the spaces K and S possessing a whole series of essential properties.

In recent years the general theory of linear topological spaces has developed considerably. However, the most general linear topological spaces are rather complicated objects possessing a whole set of "pathological" properties, and are poorly adapted to the needs of the analyst.† The basis of the theory of generalized functions is the theory of the so-called *countably normed spaces (with compatible norms)*, *their unions (inductive limits)*, and also of the *spaces conjugate to the countably normed ones or their unions*. This set of spaces is sufficiently broad on the one hand, and sufficiently convenient for the analyst on the other.

The theory of these spaces is expounded in Chapter I. Let us note that since the countably normed spaces are very close to normed spaces, a number of important theorems is obtained almost automatically by taking them over from the normed spaces into the countably normed spaces.‡ In reading this chapter it should be kept in mind that some of the theorems proved here are actually valid for more general spaces.

In the majority of questions the class of all countably normed spaces turns out to be too broad for the theory of generalized functions. Hence,

* However, even during this period works appeared which anticipated going beyond the limits of this class of spaces, the work of Köthe-Toeplitz and Köthe on spaces of sequences in the 30's, and also the work of Mazur and Orlicz.

† To the analyst it is natural to use estimates, not neighborhoods, which he inevitably reduces to some kind of estimates.

‡ Before reading this chapter it would be useful for the reader not acquainted with the theory of normed spaces to read the first three chapters, say, of the book "Elements of Functional analysis" by L. A. Lyusternik and V. I. Sobolev, Ungar, New York, 1961 or the first volume of the lectures "Elements of the Theory of Functions and Functional Analysis" by A. N. Kolmogorov and S. V. Fomin, Moscow University Press, Moscow, USSR, 1954.

in Chapter I we study the so-called *perfect* spaces (complete countably normed spaces in which the bounded sets are compact). The reader will meet a great number of examples of such spaces in the following chapters.

The reader will also find material referring to the general theory of countably normed spaces in the first three sections of Chapter IV in Volume 3.

The expounded viewpoint certainly excludes the possibility of an *a priori* description of all classes of spaces which may be encountered in connection with various problems of the theory of generalized functions: As we have already said above, each class of problems requires its own class of spaces. Therefore, essentially two classes of spaces are introduced and studied in Chapters II and IV: spaces of the type $K\{M_p\}$ in Chapter II; spaces of the type S and similar spaces of type W in Chapter IV. The spaces of type S and W essentially satisfy the demands of Chapters II and III of Volume 3 (the Cauchy problem), and spaces of type $K\{M_p\}$ the requirements of Chapter IV of Volume 3 (the problem of eigenfunction expansions). Chapter II, and, in part, Chapter III, of the present volume are devoted primarily to transferring the results of Chapters I and II of Volume 1, almost without any difficulty, to more general spaces. The spaces $K\{M_p\}$, which are natural illustrations of the general theory, appear here. On the other hand, the results of Chapter I permit the filling in of a whole series of essential gaps, in particular, the proof of the completeness of spaces of generalized functions on K, and the establishment of a number of new results, concerning for example the structure of generalized functions.

The theory of spaces of type S is discussed in the last Chapter IV. These spaces which, as we have said already, are used in Volume 3 possess great internal orderliness, and we hope that even their independent study will give the analyst some satisfaction. The construction and utilization of these spaces is connected with results of the theory of quasi-analytic functions and the Phragmen-Lindelöf theorem. Applications of these spaces to the Cauchy problem in Volume 3 will illustrate the well-known statement of Hadamard on the relation between uniqueness theorems in the Cauchy problem on the one hand, and the theory of quasi-analytic functions and the general theory of functions of a complex variable, on the other. Spaces of type S yield natural limits for a sufficiently flexible Fourier transform theory because these spaces go over into each other under Fourier transformation; hence, Chapter IV is a natural continuation of Chapter III, devoted to Fourier transforms. Moreover, various operators of the form $f(d/dx)$, where $f(t)$ is an entire function, can be constructed in spaces of type S, and are also applicable to generalized functions. The Fourier transforms of generalized functions, considered as continuous linear functionals on spaces of type S and W, as well as the construction of operators of the form $f(d/dx)$, applicable to the generalized functions, are indeed the fundamental tools which we shall use in Volume 3 for studying the Cauchy problem.

In order not to overburden the exposition here, we have referred a summary of the results referring to spaces of type W to an appendix; proofs of these results are collected in Chapter I of Volume 3. The reader interested only in problems of eigenfunction expansions may turn to Chapter IV of Volume 3 directly after having completed Chapters I and II of the present volume.

The authors take this opportunity to express their heartfelt gratitude to all their colleagues who assisted in writing this volume. To D. A. Raikov we owe a number of essential improvements in the first chapter. B. Ya. Levin constructed the proof of some necessary theorems from the theory of entire functions (Chapter IV) at our request. G. N. Zolotarev indicated some simplifications in the exposition of Chapters II and III. The section on the Hilbert transform (Chapter III) was written according to an idea of N. Ya. Vilenkin. Finally, a multitude of improvements has been inserted in accordance with suggestions of M. S. Agranovich, who edited the entire text of this volume.

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